Analytical and numerical investigations of time-periodic mechanical systems

J. Awrejcewicz and G. Kudra

Department of Automation and Biomechanics, Lodz University of Technology, 92-524, Łódź, Poland, jan.awrejcewicz@p.lodz.pl, grzegorz.kudra@p.lodz.pl

Abstract

In this work the plane system of coupled identical triple pendulums is investigated numerically. The first link of each pendulum is excited by a common harmonic signal. The coupling between subsystems is realised by viscous and elastic connections between the first links of neighbouring sets of triple pendulums. In this preliminary research we have identified and presented examples of rich dynamics exhibited by the investigated system, including many different kinds of synchrony and opening the route to more deep and general view of synchronization phenomenon.

INTRODUCTION

Although investigations of pendulum possess a very long history in mechanics, it is still subject of interest of scientists from all the world [1, 4]. But a single degree-of-freedom models are only the first step to understand a real behaviour of either natural or engineering systems, since many physical objects are modelled by a few degrees of freedom. In mechanics, but even in physics, an attempt to investigate coupled pendulums is recently observed [2, 3]. On the other hand, the synchronization phenomenon is one of the most known, spectacular and important phenomena of nonlinear dynamics [7]. In the last years, the interest in synchronization problems is especially observable [5, 6, 8], because of its importance in complex physical, biological or even social systems. The present work joins all the above mentioned research directions and the preliminary investigations of the complex system of coupled pendulums are presented.

MATHEMATICAL MODEL

In the paper the plane system of N coupled and identical triple pendulums is analysed. The i-th pendulum is presented in Fig.1, where $\psi_{i,j}$ ($i=1,2,...N$, $j=1,2,3$) denote angles defining position of the system. It is assumed that the mass centres of the links lie on the lines including the joints. The first link of each pendulum is excited by common signal $\varphi(t)$, realised by relative rotation of additional body connected to the first link. The pendulums are coupled by viscous and elastic connections between the first links of neighbouring sets of triple pendulums. Assuming, that $\varphi(t)=\omega t$, the system is governed by the following set of non-dimensional differential equations

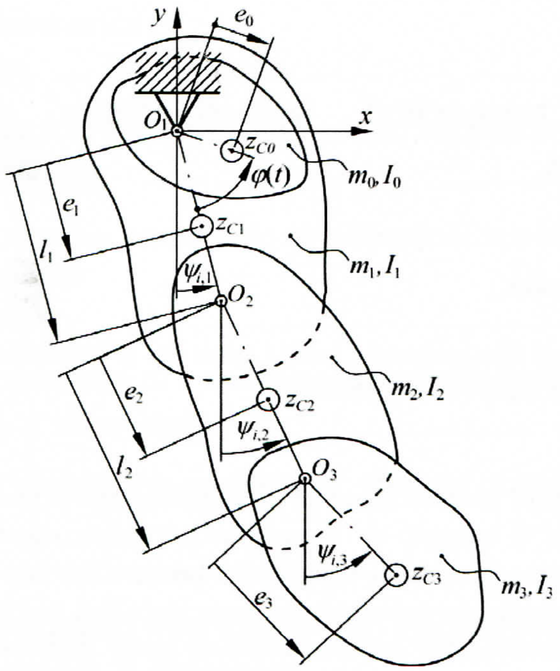

Figure 1: The *i*-th triple pendulum.

$$\mathbf{M}(\boldsymbol{\psi}_i)\ddot{\boldsymbol{\psi}}_i + \mathbf{N}(\boldsymbol{\psi}_i)\dot{\boldsymbol{\psi}}_i^2 + \mathbf{C}\dot{\boldsymbol{\psi}}_i + \mathbf{p}(\boldsymbol{\psi}_i, t) = \mathbf{f}(\boldsymbol{\psi}_{i-1}, \boldsymbol{\psi}_i, \boldsymbol{\psi}_{i+1}, \dot{\boldsymbol{\psi}}_{i-1}, \dot{\boldsymbol{\psi}}_i, \dot{\boldsymbol{\psi}}_{i+1}), \quad i=1,2,..,N \quad (1)$$

where

$$\boldsymbol{\psi}_i = [\psi_{i,1}, \psi_{i,2}, \psi_{i,3}]^T, \quad \ddot{\boldsymbol{\psi}}_i = [\ddot{\psi}_{i,1}, \ddot{\psi}_{i,2}, \ddot{\psi}_{i,3}]^T, \quad \dot{\boldsymbol{\psi}}_i^2 = [\dot{\psi}_{i,1}^2, \dot{\psi}_{i,2}^2, \dot{\psi}_{i,3}^2]^T, \quad \dot{\boldsymbol{\psi}}_i = [\dot{\psi}_{i,1}, \dot{\psi}_{i,2}, \dot{\psi}_{i,3}]^T,$$

$$\mathbf{M}(\boldsymbol{\psi}_i) = \begin{bmatrix} 1 & \nu_{12}\cos(\psi_{i,1}-\psi_{i,2}) & \nu_{13}\cos(\psi_{i,1}-\psi_{i,3}) \\ \nu_{12}\cos(\psi_{i,1}-\psi_{i,2}) & \beta_2 & \nu_{23}\cos(\psi_{i,2}-\psi_{i,3}) \\ \nu_{13}\cos(\psi_{i,1}-\psi_{i,3}) & \nu_{23}\cos(\psi_{i,2}-\psi_{i,3}) & \beta_3 \end{bmatrix},$$

$$\mathbf{N}(\boldsymbol{\psi}_i) = \begin{bmatrix} 0 & \nu_{12}\sin(\psi_{i,1}-\psi_{i,2}) & \nu_{13}\sin(\psi_{i,1}-\psi_{i,3}) \\ -\nu_{12}\sin(\psi_{i,1}-\psi_{i,2}) & 0 & \nu_{23}\sin(\psi_{i,2}-\psi_{i,3}) \\ -\nu_{13}\sin(\psi_{i,1}-\psi_{i,3}) & -\nu_{23}\sin(\psi_{i,2}-\psi_{i,3}) & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_2 & c_3 \end{bmatrix}, \quad \mathbf{p}(\boldsymbol{\psi}_i, t) = \begin{Bmatrix} \sin\psi_{i,1} + \mu_0\sin(\psi_{i,1}+\omega t) \\ \mu_2\sin\psi_{i,2} \\ \mu_2\dfrac{\nu_{12}}{\nu_{13}}\sin\psi_{i,3} \end{Bmatrix},$$

$$\mathbf{f} = k_s \begin{Bmatrix} \psi_{i-1} - 2\psi_i + \psi_{i+1} + c_{rs}(\dot{\psi}_{i-1} - 2\dot{\psi}_i + \dot{\psi}_{i+1}) \\ 0 \\ 0 \end{Bmatrix}, \quad (2)$$

and where $\psi_{0,1} = \psi_{1,1}$, $\dot{\psi}_{0,1} = \dot{\psi}_{1,1}$, $\psi_{N+1,1} = \psi_{N,1}$ and $\dot{\psi}_{N+1,1} = \dot{\psi}_{N,1}$. The parameter c_{rs} stands for relative damping of coupling, while k_s is the overall coupling coefficient.

NUMERICAL SIMULATIONS

The following parameters are fixed during numerical simulations presented in this section: $\beta_2 = 0.5517$, $\beta_3 = 0.1379$, $\mu_0 = 0.25$, $\mu_2 = 0.75$, $\nu_{12} = 0.6207$, $\nu_{13} = 0.2068$, $\nu_{23} = 0.2068$, $c_1 = c_2 = c_3 = 0.01438$ and $c_{rs} = 1$. Fig. 2 exhibits bifurcational diagram for one (N=1) or for uncoupled triple pendulums ($k_s = 0$), with the excitation angular frequency ω as a bifurcational parameter. Then, for $\omega = 0.72$ (chaotic behaviour of uncoupled systems), we present bifurcational diagrams (Fig. 3) of dynamical behaviour of three coupled pendulums (N=3), with coupling coefficient k_s as a control parameter.

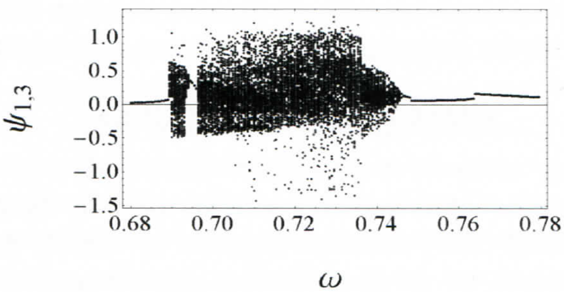

Figure 2: Bifurcational diagram for one (N=1) triple pendulum.

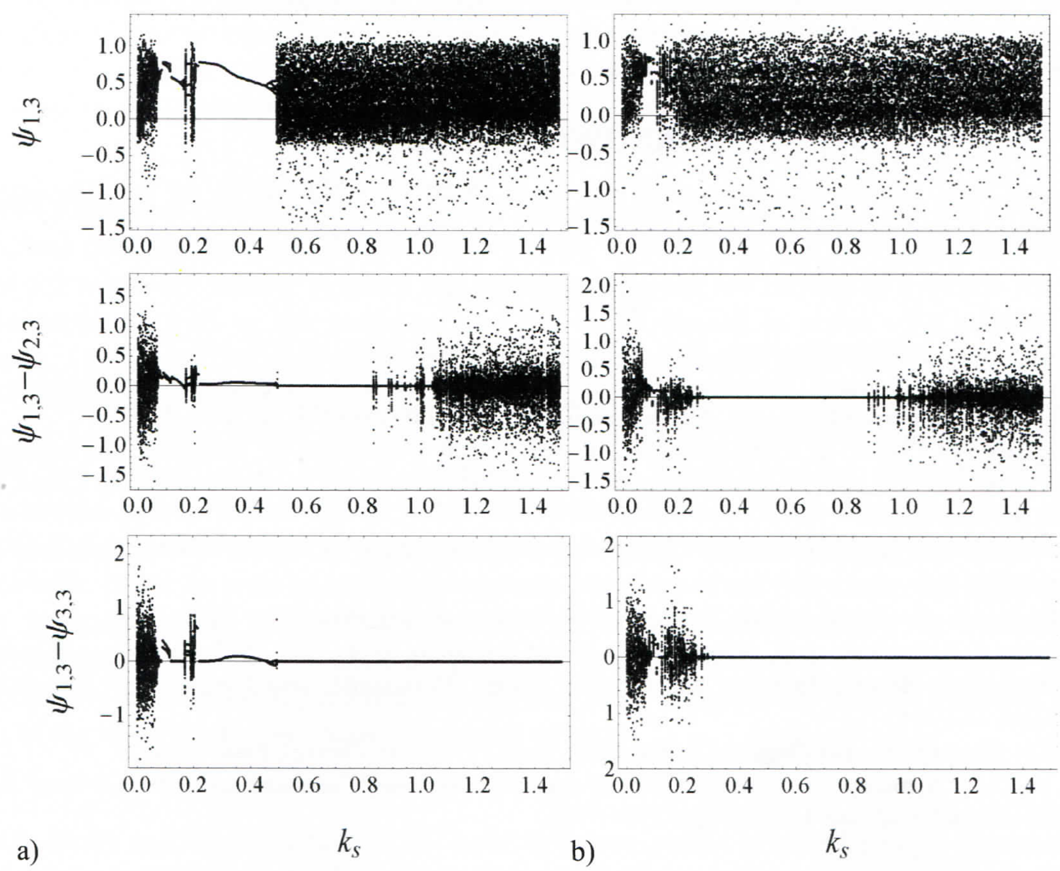

Figure 3: Bifurcational diagrams for three (N=3) coupled triple pendulums.

The first Poincaré section (for $k_s = 0$) of each bifurcational diagram is performed by the use of the following set of initial conditions

$$\psi_{i,j}(0) = 0 \,,\; \dot{\psi}_{i,j}(0) = 10^{-5} i \quad \text{where } i=1,2,..,N, \;\; j=1,2,3, \tag{3}$$

so the pendulums start from very close, but different states. During the jump to the next Poincaré section (the change of control parameter) (Fig. 3a), the system state preserves continuity or is restarted to the initial conditions (3) (Fig. 3b). Fig. 3 exhibits rich spectrum of synchronization phenomena exhibited by the investigated system. In particular, we have observed the intervals of chaotic and periodic behaviour of the system, or even regions of coexistence of chaotic and periodic attractors. We have also found the intervals of exact synchronization between chaotic behaviour of all three pendulums and the zones of exact synchronization between irregular motion of the first pendulum and the third one, while the second pendulum moves non-synchronically on chaotic attractor. We can also observe other kinds of synchronization, usually between periodic motions of the pendulums.

CONCLUDING REMARKS

In the work the preliminary research results of the system of coupled triple pendulums are presented. We have identified and shown examples of rich dynamics exhibited by the investigated system, including many different kinds of synchrony and opening the route to more deep and general view of synchronization phenomenon. Since there is a direct mechanical interpretation of the proposed model, the experimental verification is potentially possible. There are many possibilities of further research of the system, e.g. investigations of larger number of coupled subsystems of pendulums consisting of larger or smaller number of links.

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