Mechanics Research Communications 105 (2020) 103505

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

On the Bolotin's reduced beam model versus various boundary conditions

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ARTICLE INFO

Article history:
Received 2 October 2019
Revised 8 January 2020
Accepted 8 March 2020
Available online 13 March 2020

Keywords:
Beam
Vibrations
PDE
Boundary conditions
Asymptotics

ABSTRACT

This paper is devoted to the construction of asymptotically correct simplified models of nonlinear beam equations for various boundary conditions. V.V. Bolotin mentioned that in some cases (e.g., if compressed load is near the buckling value), the so-called "nonlinear inertia" must be taken into account. The effect of nonlinear inertia on the oscillations of the clamped-free beam is investigated in many papers. Bolotin used some physical assumption and did not compare the order of nonlinear terms in original equations. Below we propose our method for deriving those, which we will named "Bolotin's equations". This approach is based on fractional analysis of original boundary value problems.

1. Introduction

Beams are commonly used as structural elements in macro-, micro- or nanoscales [1,2]. Consequently, models for their analysis are currently met in any field of civil and industrial engineering. Beams are frequently used in many practical applications, for example in buildings, bridges, mining supports, railroads, biomechanics etc.

Following the trend to downscale electronic devices, mechanical devices are also entering the micro- and even nanometer regime [3]. To read out the motion of a beam, it has to be coupled to an electronic circuit. These systems are the so-called microelectromechanical systems (MEMS) and they find commercial applications in accelerometers, gyroscopes, mass sensing, pressure sensing, band-pass filters and scanning probe microscopy. The increasing demand for realistic simulations leads to a higher level of detail during the modeling phase, especially in nanomechanics and biology (for example, for describing mechanical behavior of DNA). The resulting complicated non-linear PDEs can be solved by discrete methods (finite elements, finite differences, etc.). But the time required to solve high-dimensional discretized models remains a bottleneck towards efficient and optimal design of structures. To simplify the original equations, model order reduction methods are widely used [4,5]. This approach is based on partial discretization followed by an analysis of the high-dimensional system. An alternating approach is fractional analysis [6], based on the detection of small parameters using non-dimensionalization and normalization with following asymptotic splitting.

Our paper is devoted to the construction of asymptotically correct simplified models of non-linear beam equations for various boundary conditions. The paper is organized as follows. First, we employ the traditional Kirchhoff's approximation. In Section 3, we obtain Bolotin's equations for clamped–clamped beam. Section 4 deals with generalization for different boundary conditions. Section 5 presents an example of non-linear normal mode construction. Section 6 is devoted to study correct dynamical equations of a buckled beam. Finally, Section 7 presents concluding remarks.

2. Kirchhoff's approximation

Kirchhoff [7] proposed simple approximate equations of non-linear beam vibration, which became very popular [8,9]. Let us briefly discuss this approximation. Consider the governing equations of non-linear beam vibration in the following form

$$\rho F \frac{\partial^2 W}{\partial t^2} + \frac{\partial^2 M}{\partial x^2} - \frac{\partial}{\partial x}\left(T \frac{\partial W}{\partial x}\right) = 0, \quad (1)$$

$$\rho F \frac{\partial^2 U}{\partial t^2} - \frac{\partial T}{\partial x} = 0, \quad (2)$$

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https://doi.org/10.1016/j.mechrescom.2020.103505
0093-6413/

where: $M = EI\kappa$, $T = EF\varepsilon$, $\kappa = \frac{\partial^2 W}{\partial x^2}$, $\varepsilon = \frac{\partial U}{\partial x} + 0.5(\frac{\partial W}{\partial x})^2$; E is the Young's modulus; F, I are the area and the static moment of transversal beam cross section, respectively; κ is the curvature; U, W are the longitudinal and normal beam displacements; ρ is the density of beam material; t is the time, and x is the spatial coordinate.

Below, we consider two cases of boundary conditions in the axial direction: prescribed end shortening or dead (forced) loading:

$$U = U^{(0)} \text{ at } x = 0,\ U = U^{(L)} \text{ at } x = L, \tag{3}$$

or

$$T = T^{(0)} \text{ at } x = 0,\ \ T = T^{(L)} \text{ at } x = L. \tag{4}$$

The Kirchhoff hypothesis is that the axial inertial term in Eq. (2) can be neglected. Then, one obtains $\frac{\partial T}{\partial x} = 0$, i.e.

$$\varepsilon = \frac{\partial U}{\partial x} + 0.5\left(\frac{\partial W}{\partial x}\right)^2 = N(t). \tag{5}$$

Upon integration of relation (5) with taking into account boundary conditions (3) we have

$$N = \frac{U_b}{L} + \frac{1}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx,\ U_b = U^{(L)} - U^{(0)}. \tag{6}$$

Using Eqs. (1) and (6) one obtains

$$\rho F \frac{\partial^2 W}{\partial t^2} + EI\frac{\partial^4 W}{\partial x^4} + \\ + \frac{EF}{L}\left[U_b - \frac{1}{2}\left(\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx\right)\right]\frac{\partial^2 W}{\partial x^2} = 0. \tag{7}$$

Eq. (7) describes the approximate Kirchhoff model [8–10]. The same equation is obtained by applying a correct asymptotic procedure [11].

It is worth noting that in Kirchhoff's book Eq. (7) is not presented. Kirchhoff [7], in spite of neglecting the longitudinal inertial term in Eq. (2), has also omitted the second term in Eq. (1), and the original 'Kirchhoff's equation' has the form

$$\rho F \frac{\partial^2 W}{\partial t^2} - \frac{EF}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx \frac{\partial^2 W}{\partial x^2} = 0.$$

Maybe that is why Eq. (7) is sometimes called the "Mettler equation" referring to the work [12]. For axial boundary conditions (4), Eq. (1) is linearized, and takes the following form

$$\rho F \frac{\partial^2 W}{\partial t^2} + EI\frac{\partial^4 W}{\partial x^4} + T_b\frac{\partial^2 W}{\partial x^2} = 0,\ T_b = T^{(L)} - T^{(0)} \tag{7a}$$

3. Bolotin's approximation for a clamped–clamped beam

Bolotin [13] mentioned that in some cases (e.g., if compressed load is near the buckling value), the so-called "non-linear inertia" must be taken into account. The effect of non-linear inertia on the vibrations of the clamped-free beam (with a non-linear curvature expression different from ours) is investigated in [14,15].

Bolotin used some physical assumption and did not compare order of linear and non-linear terms in original equations. Below we propose our way for obtaining the mentioned equations, which we will name "Bolotin's equations". We will show that Bolotin's approach leads to a system of two equations, first of them is Kirchhoff-type equation with taking into account non-linear curvature, and the second one takes into account non-linear inertia and non-linear curvature of the system. These equations stand for the first and the second approximations of non-quasilinear asymptotics, using quantity $\delta = I/(FL^2)$ as a small perturbation parameter.

Lacarbonara and Yabuno [16] obtained the geometrically exact equations governing nonlinear beam motion and their simplification for moderately large amplitude motion. In this case they used Mac Laurin series expansions. We use also these equations but with restricted expansions up to the second polynomial order

$$\rho F \frac{\partial^2 W}{\partial t^2} + \frac{\partial^2 M}{\partial x^2} - \frac{\partial}{\partial x}\left(T\frac{\partial W}{\partial x}\right) = 0, \tag{8}$$

$$\rho F \frac{\partial^2 U}{\partial t^2} - \frac{\partial T}{\partial x} = 0, \tag{9}$$

where:

$$M = EI\kappa,\ T = EF\varepsilon,$$

$$\kappa = \frac{\partial^2 W}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial W}{\partial x}\right) - \left(\frac{\partial W}{\partial x}\right)^2\frac{\partial^2 W}{\partial x^2} \\ + \frac{\partial}{\partial x}\left(\left(\frac{\partial U}{\partial x}\right)^2\frac{\partial W}{\partial x}\right),$$

$$\varepsilon = \frac{\partial U}{\partial x} + 0.5\left(\frac{\partial W}{\partial x}\right)^2 - 0.5\frac{\partial U}{\partial x}\left(\frac{\partial W}{\partial x}\right)^2.$$

In addition, we use the expression for curvature proposed in [17–20]. This curvature is known as material, normalized, flexural or mechanical curvature

Let us assume that

$$W = w_0 + \delta w_1 + \cdots, \\ U = \delta u_0 + \delta^2 u_1 + \cdots. \tag{10}$$

Physically, it means that we deal with the low part of the frequency spectrum, i.e. with prevalent bending oscillations and investigate the influence of axial displacement and inertia on those vibrations.

Further, we will use the Fourier series in a spatial variable with the general term $f_m(t)\sin(m\pi x/L)$ for solving our problem. So we must account of the following estimation

$$m < \min\left[\sqrt{\frac{FL^2}{I}}, \frac{L}{\sqrt{F}}\right]. \tag{11}$$

For larger values of m, one must take into account a correction term yielded by 3D elasticity.

Using Ansatz (10) and employing the first approximation, the following Kirchhoff-type equations are obtained

$$\frac{\partial T_0}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial u_0}{\partial x} + 0.5\left(\frac{\partial w_0}{\partial x}\right)^2\right] = 0, \tag{12}$$

$$\rho F \frac{\partial^2 w_0}{\partial t^2} + EI\frac{\partial^4 w_0}{\partial x^4} - T_0\frac{\partial^2 w_0}{\partial x^2} = 0,\quad T_0 = \frac{EF}{2L}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx. \tag{13}$$

Eq. (12) with an account of boundary conditions

$$u_0 = 0 \text{ at } x = 0, L \tag{14}$$

yields

$$u_0 = -\frac{1}{2}\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx + \frac{x}{2L}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx. \tag{15}$$

If we suppose that beam ends are simply supported with respect to normal displacements, then we have

$$w_0 = 0,\quad \frac{\partial^2 w_0}{\partial x^2} = 0 \text{ at } x = 0, L. \tag{16}$$

For construction of the second approximation we suppose

$$T = T_0 + \delta T_1 + \cdots$$

and hence

$$\frac{\partial T_1}{\partial x} = \rho F \frac{\partial^2 u_0}{\partial t^2} = = \frac{\rho F}{2}\frac{\partial^2}{\partial t^2}\left[-\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx + \frac{x}{L}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right]. \tag{17}$$

Integration of Eq. (17) allows to find explicitly

$$T_1 = \rho F \frac{\partial^2}{\partial t^2}\Phi(x,t) + \varphi(t), \tag{18}$$

where

$$\Phi(x,t) = -\frac{1}{2}\int_0^x \left(\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx + \frac{x^2}{4L}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx, \tag{19}$$

For defining the function $\varphi(t)$, the following expression is used

$$T_1 = EF\left[\frac{\partial u_1}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_1}{\partial x}\right] \tag{20}$$

and the following boundary conditions are taken

$$u_1 = 0 \text{ at } x = 0, L. \tag{21}$$

Then, one obtains

$$u_1 = \frac{\rho}{E}\frac{\partial^2}{\partial t^2}\int_0^x \Phi(x,t)dx - \int_0^x \frac{\partial w_0}{\partial x}\frac{\partial w_1}{\partial x}dx + \frac{x}{EF}\varphi(t), \tag{22}$$

where

$$\varphi(t) = -\frac{\rho F}{L}\frac{\partial^2}{\partial t^2}\int_0^L \Phi(x,t)dx + \frac{EF}{L}\int_0^L \frac{\partial w_0}{\partial x}\frac{\partial w_1}{\partial x}dx. \tag{23}$$

Finally, the equation of the second approximation can be written as follows

$$\rho F \frac{\partial^2}{\partial t^2}\left[w_1 - \frac{\partial}{\partial x}\left[\Phi - \frac{1}{L}\int_0^L \Phi dx\right]\frac{\partial w_0}{\partial x}\right] + EI\frac{\partial^4 w_1}{\partial x^4} - -\frac{EF}{2L}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx \frac{\partial^2 w_1}{\partial x^2} - \frac{EF}{L}\int_0^L \frac{\partial w_0}{\partial x}\frac{\partial w_1}{\partial x}dx\frac{\partial^2 w_0}{\partial x^2} = = -\frac{EI}{2}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 w_0}{\partial x^2}\left[\left(\frac{\partial w_0}{\partial x}\right)^2 - \frac{1}{L}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right]\right]. \tag{24}$$

Boundary conditions for Eq. (24) are as follows

$$w_1 = 0, \quad \frac{\partial^2 w_1}{\partial x^2} = 0 \text{ at } x = 0, L. \tag{25}$$

In paper [14], it has been shown that non-linear inertia has the most substantial impact on the dispersion relation of a beam. Let us neglect in Eq. (24) terms caused by nonlinear curvature. Then, we add Eqs. (13) and (24) and suppose that $W \approx w_0 + w_1$. Bolotin's equations can be approximately written as follows:

$$\rho F \frac{\partial^2}{\partial t^2}\left[W - \frac{\partial}{\partial x}\left[\Phi - \frac{1}{L}\int_0^L \Phi dx\right]\frac{\partial W}{\partial x}\right] + +EI\frac{\partial^4 W}{\partial x^4} - \frac{EF}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx\frac{\partial^2 W}{\partial x^2} = 0, \tag{26}$$

$$\Phi(x,t) = -\frac{1}{2}\int_0^x \left(\int_0^x \left(\frac{\partial W}{\partial x}\right)^2 dx\right)dx + \frac{x^2}{4L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx. \tag{27}$$

4. Bolotin's approximation for various boundary conditions

Consider the construction of Bolotin's approximation, limiting ourselves to a linear approximation for the curvature. First, consider the beam with the free axis conditions in the axial direction, i.e. we take

$$T = 0 \text{ at } x = 0, L. \tag{28}$$

Then, from Eq. (12) and boundary conditions (28), one obtains $T_0 = 0$ and

$$u_0 = -\frac{1}{2}\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx + C(t). \tag{29}$$

Function $C(t)$ will be found later. For the additional longitudinal force, owing to the effects of inertia, one obtains

$$\frac{\partial T_1}{\partial x} = \rho F \frac{\partial^2 u_0}{\partial t^2} = \frac{\rho F}{2}\frac{\partial^2}{\partial t^2}\left[-\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx + C(t)x\right]. \tag{30}$$

Integrating Eq. (30) taking into account boundary conditions (28), one obtains

$$T_1(x,t) = \rho F \frac{\partial^2 \Phi}{\partial t^2} = \tag{31}$$

$$= \rho F \frac{\partial^2}{\partial t^2}\left[-\frac{1}{2}\int_0^x \left(\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx + \frac{x^2}{4L}\int_0^L \left(\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx\right]. \tag{31}$$

The equation exhibiting the Bolotin's approximation can be written as follows

$$\rho F \frac{\partial^2}{\partial t^2}\left[w_0 - \frac{\partial}{\partial x}\left(\Phi \frac{\partial w_0}{\partial x}\right)\right] + EI\frac{\partial^4 w_0}{\partial x^4} = 0. \tag{32}$$

For a beam with clamped-free boundary conditions in the axial direction, we have

$$U = 0 \text{ at } x = 0, \; T = 0 \text{ at } x = L. \tag{33}$$

and hence the equation of Bolotin's approximation has the form (32) with

$$\Phi = -\frac{1}{2}\left[\int_0^x \left(\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx - \int_0^L \left(\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx\right]. \tag{34}$$

5. Non-linear normal modes (NNMs) for a clamped–clamped beam

Concept of NNMs for discrete systems plays important role in nonlinear dynamics of lumped mass mechanical systems (see for details [21,22]). Kirchhoff model allows for an exact separation of spatial and time variables for some type of boundary conditions. Wah [23] was the first who used this possibility and constructed NNMs for continuous system. Though Bolotin's equation does not allow for exact separation of spatial and time variables, but NNMs can be constructed approximately.

Let us introduce the following parameters:

$$h = \sqrt{\frac{I}{F}}, \; \varepsilon = \frac{h}{L}, \; \varepsilon << 1, \; w = \frac{W}{h}, \; \tau = \frac{t}{L^2}\sqrt{\frac{EI}{\rho F}}, \; \xi = x/L. \tag{35}$$

Then Eq. (26),(27) can be rewritten as follows

$$w_{\tau\tau} + w_{\xi\xi\xi\xi} - \frac{1}{2}w_{\xi\xi}\int_0^1 (w_\xi)^2 d\xi -$$

$$-\frac{\varepsilon^2}{2}\frac{\partial^2}{\partial\tau^2}\left[-\int_0^\xi (w_\xi)^2 d\xi + \xi\int_0^1 (w_\xi)^2 d\xi\right]w_\xi = 0. \quad (36)$$

We suppose simply supported ends of the following form

$$w = w_{\xi\xi} = 0 \text{ for } \xi = 0, 1. \quad (37)$$

Let us apply one-term approximation, which satisfies the boundary conditions (37):

$$w \approx w_i(\tau)\sin(i\pi\xi). \quad (38)$$

We must take into account natural restriction $i << \varepsilon^{-1}$ caused by conditions of applicability of the beam theory.

Substituting Ansatz (38) to PDE (36) and using the Kantorovich procedure [24] one obtains the following ODEs with regard to $w_i(\tau)$:

$$\left[1 + \frac{3(\pi i)^2\varepsilon^2}{16}w_i^2\right]w_{i\tau\tau} + \frac{3(\pi i)^2\varepsilon^2}{8}w_{i\tau}^2 w_i +$$
$$+(\pi i)^4 w_i + \frac{(\pi i)^4}{4}w_i^3 = 0. \quad (39)$$

Observe that the oscillator (39) can be treated as a system with variable mass [25], since we have

$$\frac{d(M_1 w_{i\tau})}{d\tau} + (\pi i)^4 w_i + \frac{(\pi i)^4}{4}w_i^3 = 0, \quad (40)$$

where $M_1 = 1 + \frac{3(\pi i)^2\varepsilon^2}{16}w_i^2$.

We choose initial conditions in the following form

$$w = 1, \quad w_\tau = 0 \text{ for } \tau = 0. \quad (41)$$

In order to solve ODEs (39), the successive approximations method is employed. From the very beginning, let us suppose $\varepsilon^2 = 0$ and obtain expressions for $w_{i\tau\tau}$ and $w_{i\tau}^2$, i.e. we have

$$w_{i\tau\tau} = -(\pi i)^4 w_i - \frac{(\pi i)^4}{4}w_i^3 = 0, \quad (42)$$

$$w_{i\tau}^2 = 2C - (\pi i)^4 w_i^2 - \frac{(\pi i)^4}{8}w_i^4 = 0, \quad (43)$$

where for initial conditions (41), C takes the following value

$$2C = \frac{9}{8}(\pi i)^4. \quad (44)$$

Then, the considered oscillator (39) with variable mass can be approximately replaced by an oscillator with cubic and quintic non-linear terms of the following form

$$w_{i\tau\tau} + (\pi i)^4\left[1 + \frac{27(\pi i)^2}{64}\varepsilon^2\right]w_i +$$
$$+\frac{(\pi i)^4}{4}\left[1 - \frac{9(\pi i)^2}{4}\varepsilon^2\right]w_i^3 - \frac{3(\pi i)^6}{32}\varepsilon^2 w_i^5 = 0. \quad (45)$$

One can see that an account non-linear inertia slightly modified linear and cubic terms and leads to appearing of the quintic term.

The Cauchy problem (45),(41) has the following exact solution [26] of the following form:

$$w_i(t) = \frac{cn(q_i t, m_i)}{\sqrt{cn^2(q_i t, m_i) + \left(\frac{6q_1^{(i)}}{q_2^{(i)}}\right)^{1/2} sn^2(q_i t, m_i) dn^2(q_i t, m_i)}}, \quad (46)$$

where:

$q_i = (\frac{1}{6}q_1^{(i)}q_2^{(i)})^{1/4}$, $m_i = \frac{1}{2} - \frac{q_3^{(i)}}{4}(\frac{3}{2q_1^{(i)}q_2^{(i)}})^{1/2}$, $q_1^{(i)} = a_1^{(i)} + a_3^{(i)} + a_5^{(i)}$, $q_2^{(i)} = 6a_1^{(i)} + 3a_3^{(i)} + 2a_5^{(i)}$, $q_3^{(i)} = 4a_1^{(i)} + 3a_3^{(i)} + 2a_5^{(i)}$, $a_1^{(i)} = (\pi i)^4[1 + \frac{27(\pi i)^2}{64}\varepsilon^2]$, $a_3^{(i)} = \frac{(\pi i)^4}{4}[1 - \frac{9(\pi i)^2}{4}\varepsilon]^2$, $a_5^{(i)} = -\frac{3(\pi i)^6\varepsilon^2}{32}$, $cn(,)$, $sn(,)$, $dn(,)$ are the basic Jacobi elliptic functions.

The period of vibration is yielded by the following formula

$$T = 4\left(\frac{6}{q_1^{(i)}q_2^{(i)}}\right)^{1/4} K(m_i), \quad (47)$$

where $K()$ is the complete elliptic integral of the first kind.

It should be mentioned that the expressions (38),(46),(47) approximately describe NNMs of non-linear beam vibrations with taking into account non-linear inertia term.

6. Correct nonlinear dynamic equation of buckled beam

Consider a construction of the correct equations for nonlinear beam vibrations under boundary conditions, when employed axial loads are close to the buckling value. It should be emphasized that in a general case, in order to fit appropriately the experimental results, the boundary conditions can have more complex form [27]. If this circumstance is not taken into account, a comparison of theoretical and experimental results raises questions.

Starting from papers [28,29], for analysis of the postcritical behavior of a beam, usually an equation of the form (7) is used, where the term $(EF/L)U_b$ is changed to T_b (see, e.g., [30–32]). In what follows construct the correct model, restricting ourselves to the case of linear curvature. In this case, the equation of the first approximation has the form (7a). Fair ratio

$$\frac{\partial u_0}{\partial x} + 0.5\left(\frac{\partial w_0}{\partial x}\right)^2 = \frac{T_b}{EF}. \quad (48)$$

For the construction of the second order approximation we suppose that the following asymptotic expansion holds

$$T = T_b + \delta T_1 + \cdots \quad (49)$$

Taking into account of the first approximation of Eq. (2) yields

$$\frac{\partial T_1}{\partial x} = \rho F\frac{\partial^2 u_0}{\partial t^2}. \quad (50)$$

Eq. (48) gives

$$u_0 = \frac{T_b}{EF}x - \frac{1}{2}\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx + C(t), \quad (51)$$

and then

$$\frac{\partial T_1}{\partial x} = \rho F\frac{\partial^2 u_0}{\partial t^2} = -\frac{\rho F}{2}\frac{\partial^2}{\partial t^2}\int_0^x \left(\frac{\partial w_0}{\partial x}\right)^2 dx + C_1(\tau). \quad (52)$$

Integrating expression (52) with respect to x and taking into account the boundary conditions

$$T_1 = 0 \text{ at } x = 0, L \quad (53)$$

one gets

$$T_1 = -\rho F\frac{\partial^2\Psi}{\partial t^2},$$
$$\Psi = \frac{1}{2}\left[\int_0^x\left(\int_0^x\left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx + \frac{x}{L}\int_0^L\left(\int_0^x\left(\frac{\partial w_0}{\partial x}\right)^2 dx\right)dx\right]. \quad (54)$$

Finally, the equation for the second order approximation can be written as follows

$$\rho F\frac{\partial^2}{\partial t^2}\left[w_0 - \frac{\partial}{\partial x}(\Psi\frac{\partial w_0}{\partial x})\right] + EI\frac{\partial^4 w_0}{\partial x^4} + T_b\frac{\partial^2 w_0}{\partial x^2} = 0. \quad (55)$$

7. Concluding remarks

1D non-linear thin-walled structures (rods, beams, arches, rings, etc.) are commonly used as structural elements in macro-, micro- or nanoscales. Consequently, models for their analysis are currently met in any field of civil and industrial engineering. The resulting complicated non-linear PDEs can be solved by discrete methods (FEM, FD, etc.). However, the time required to solve high-dimensional discretized models remains a bottleneck towards the efficient and optimal design of structures. In this regard, a reasonable simplification of the initial boundary value problems with the ability to control the accuracy of the obtained limit systems is relevant. Asymptotic approaches, i.e. singular and regular asymptotics, give the most natural way to solve the stated problem appropriately. The original non-linear boundary value problems contain two basic small parameters characterizing both system stiffness and non-linearity. The interplay between these parameters defines various simplified non-linear boundary value problems for spatially 1D thin-walled structures. In this paper, reliability and validity of the Kirchhoff's and Bolotin's approximations have been addressed, analyzed and discussed.

In addition, the correct equations governing nonlinear behavior of a buckled beam are constructed. Analysis of the nonlinear response of a beam in presence of axial loads close to the buckling value is an important topic for further research. It seems that some results regarding internal resonances and chaotic behavior of a buckled beam need to be revisited.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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