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Linear and nonlinear free vibration analysis of laminated functionally graded shallow shells with complex plan form and different boundary conditions

Jan Awrejcewicz [a,*], Lidiya Kurpa [b], Tetyana Shmatko [b]

[a] Lodz University of Technology, Department of Automation, Biomechanics and Mechatronics, Stefanowskiego St. 1/15, 90-924, Lodz, Poland
[b] National Technical University "Kharkiv Polytechnic Institute", Department of Applied Mathematics, Kyrpychova 2, 61002, Kharkiv, Ukraine

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ABSTRACT

Linear and geometrically nonlinear vibrations of the three-layered functionally graded shallow shells with a complex form of the base are studied. It is assumed that outer and inner layers are made of functionally graded materials (FGM) or an isotropic material (metal or ceramic). The first-order shear deformation theory of shallow shells (FSDT) is employed. Effective material properties of layers are varied along the thickness according to a power law. To calculate different mechanical characteristics for different types of lamination schemes, analytical expressions are obtained. The linear problem is solved by combining the Ritz and the R-functions method. Linearization of the nonlinear problem is carried out by a novel original approach. Namely, the initial problem is reduced to solving a sequence of linear problems including those vibrations related to linear, special type of the elasticity problem and the nonlinear system of ordinary differential equations. Validation of the proposed method and the developed software has been examined on test problems for shallow shells with rectangular plan form and different boundary conditions. In order to demonstrate possibilities of the proposed approach, new results for laminated FGM shallow shells of the complex form of the base are presented. Effects of different material distributions, lamination schemes, curvatures, boundary conditions, and geometrical parameters on natural frequencies and backbone curves are reported and analyzed.

1. Introduction

Composite and functionally graded materials are extensively used in many fields of modern industries, especially in spacecrafts and nuclear plants. Shallow shells are often employed to fabricate structural elements of modern constructions made of advanced materials. Taking into account that these elements can be loaded dynamically, a study of their dynamical behavior is of very significant practical interest.

To mathematically simulate shells made of functionally graded materials, various shell theories have been developed. In particular, for shallow shells, the classical theory (CST), first-order shear deformation theory (FSDT), and higher-order shear deformation theory (HSDT) are most commonly used [1–4]. Analysis of vibration of laminated and FGM shallow shells has been carried out by many researchers. An extensive literature review concerning free linear and nonlinear vibrations of shells and plates made of traditional and advanced materials can be found in Refs. [3–8]. On the other hand, one can find papers devoted to the analysis of vibrations of the FGM shells [9–17]. It should be noted that in the general case (complex plan form, mixed boundary conditions, etc.), construction of an analytical solution to equations governing the dynamics of FGM plates and shallow shells is not an easy task. That is why many researchers use numerical or semi-analytical methods. Swaminathan et al. [8], Thai and Kim [9,10] presented a comprehensive review of various methods employed to study FGM plates and shells. A vast number of analytical, semi-analytical, and numerical methods has been analyzed. Effective modern methods, such as Ritz method [18–23], differential quadrature method [24,25], and Haar wavelet method [26,27], have been successively employed to study functionally graded panels, plates, and shells. One of new methods employed to study FGM plates and shallow shells is a meshless method. A review of using this method for FGM plates and shells was presented by Liew et al. [28]. This approach has been applied in [29–34].

Currently, the combined application of the FGM and pure metallic and ceramic materials is widely used for the design of many elements of modern constructions. Many important geometrically nonlinear problems of inhomogeneous isotropic and FGM plates and shells have

* Corresponding author.
E-mail address: jan.awrejcewicz@p.lodz.pl (J. Awrejcewicz).

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been solved by Shen [35], F.Alijani and M.Amabili [36–39], and other researchers. However, here and in other publications [40,41], laminated FGM plates and shells with simple plan forms have been considered.

The present paper offers an efficient and moderately universal approach to solve this problem for the laminated functionally graded shallow shells with complex shapes of their plans and different boundary conditions. The proposed method is based on the joint application of the R-functions theory and the variational Ritz method [42–51]. In many published works, the Ritz method has been used to solve vibration problems of plates and shells. The comprehensive review of the application of this method to study linear dynamic, static, and buckling behavior of beams, plates, and shells is presented in Refs. [52,53].

In the paper [48], the authors proposed a method to investigate free vibrations of single-layer FGM shells with complex plan form. In the present study, this approach was developed for three-layered shallow shells like sandwiches. Four types of lamination schemes were considered: 1-1, 1-2, 2-1, 2-2. Shallow shells of Type 1-1 and Type 1-2 correspond to sandwich shallow shells with FGM face sheets and an isotropic (metal or ceramic) core. The shells of Type 2-1 and Type 2-2 correspond to a sandwich shallow shell with the FGM core and ceramics or metal on top and bottom face sheets. It was assumed that FGM layers are made of a mixture of metal and ceramics and effective material properties of layers are varied according to Voigt's rule. Formulation of the problem was carried out using the first-order shear deformation shallow shells theory.

The proposed method was validated by investigating test problems for shallow shells with a rectangular plan form and different boundary conditions. The results were obtained for linear and nonlinear vibration problems for double curved shallow shells with complex form of the base.

The paper is organized in the following way. Basic relations of laminated functionally graded shallow shells and fundamental equations of motion are presented in Section 2, where analytical expressions for force and moment resultant vectors are derived. The method of solution of the linear and nonlinear vibration problems is described in Sections 3, 4. Numerical results of linear and nonlinear vibration problems of laminated FGM shallow shells with complex forms of the base are presented in Section 5. The last Section 6 summarizes the obtained results.

2. Mathematical formulation

Consider a three-layered functionally graded shallow shell with uniform thickness *h*. It is assumed that the FGM layers are made of a mixture of ceramics and metals. A double curved shallow shell can have an arbitrary plan form. The effective material properties of layers vary continuously and smoothly in the thickness direction and obey Voigt's law:

$$E^{(r)} = \left(E_u^{(r)} - E_l^{(r)}\right) V_c^{(r)} + E_l^{(r)}, \tag{1}$$

$$\nu^{(r)} = \left(\nu_u^{(r)} - \nu_l^{(r)}\right) V_c^{(r)} + \nu_l^{(r)} \tag{2}$$

$$\rho^{(r)} = \left(\rho_u^{(r)} - \rho_l^{(r)}\right) V_c^{(r)} + \rho_l^{(r)}, \tag{3}$$

where $E_u^{(r)}, \nu_u^{(r)}, \rho_u^{(r)}$ and $E_l^{(r)}, \nu_l^{(r)}, \rho_l^{(r)}$ are Young's modulus, Poisson's ratio, and mass density of the upper and lower surfaces of the *r*-layer, respectively; $V_c^{(r)}$ is the volume fraction of ceramic. Below, the values $V_c^{(r)}$ are presented for the scheme lamination of Types 1-1 (FGM-metal-FGM); 2-1 (metal-FGM-ceramic); 1-2 (FGM-ceramic-FGM) and 2-2 (ceramic-FGM-metal)

$$Type\ 1-1 \quad \begin{cases} V_c^{(1)} = \left(\dfrac{z+h/2}{h_1+h/2}\right)^{p_1} \\ V_c^{(2)} = 1 \\ V_c^{(3)} = \left(\dfrac{z-h/2}{h_2-h/2}\right)^{p_3} \end{cases} \qquad Type\ 2-1 \quad \begin{cases} V_c^{(1)} = 1 \\ V_c^{(2)} = \left(\dfrac{z-h_2}{h_1-h_2}\right)^{p_2} \\ V_c^{(3)} = 0 \end{cases} \tag{4}$$

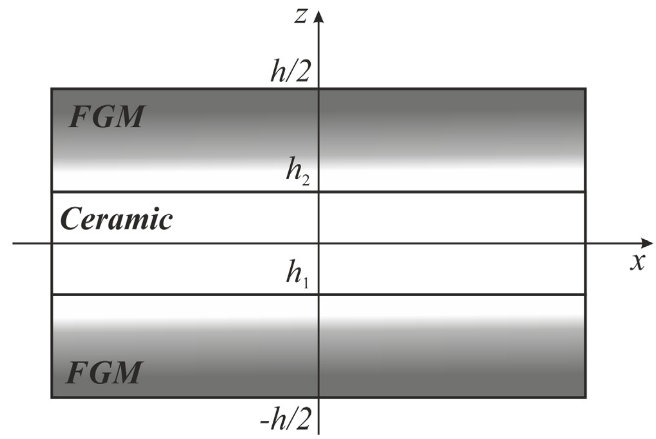

Fig. 1. Example of the shell type 1-1.

$$Type\ 1-2 \quad \begin{cases} V_c^{(1)} = \left(\dfrac{z-h_1}{-h_1-h/2}\right)^{p_1} \\ V_c^{(2)} = 0 \\ V_c^{(3)} = \left(\dfrac{z-h_2}{h/2-h_2}\right)^{p_3} \end{cases} \qquad Type\ 2-2 \quad \begin{cases} V_c^{(1)} = 0 \\ V_c^{(2)} = \left(\dfrac{z-h_1}{h_2-h_1}\right)^{p_2} \\ V_c^{(3)} = 1 \end{cases} \tag{5}$$

Note that the value $V_c^{(1)}$ is valid for $z \in [-h/2, h_1]$, $V_c^{(2)}$ is valid for $z \in [h_1, h_2]$, and $V_c^{(3)}$ is valid for $z \in [h_2, h/2]$, (Fig. 1).

It should be emphasized that the values p_1, p_2, p_3 are the power-law FGM exponents of the corresponding layer. The thickness of the layers can be varied. The ratio of the thicknesses of layers from the bottom to the top is denoted by the combination of three numbers. For example, "1-2-1" means that the ratio of the thickness of layers is defined as $h^{(1)} : h^{(2)} : h^{(3)} = 1 : 2 : 1$, where $h^{(1)} = h_1 + h/2$, $h^{(2)} = h_2 - h_1$, $h^{(3)} = h/2 - h_2$ (see Fig. 1).

According to the first-order shear deformation theory of shallow shells, the components of displacement u_1, u_2, u_3 at a point (x, y, z) are expressed as functions of the middle surface displacements u, v, and w in the Ox, Oy, and Oz directions [1,2,6], i.e. we have

$$u_1 = u + z\psi_x, \quad u_2 = v + z\psi_y, \quad u_3 = w. \tag{6}$$

Strain components $\varepsilon = \{\varepsilon_{11};\ \varepsilon_{22};\ \varepsilon_{12}\}^T$, $\chi = \{\chi_{11};\ \chi_{22};\ \chi_{12}\}^T$ at an arbitrary point of the shallow shell are:

$$\varepsilon_{11} = \varepsilon_{11}^L + \varepsilon_{11}^N, \quad \varepsilon_{22} = \varepsilon_{22}^L + \varepsilon_{22}^N, \quad \varepsilon_{12} = \varepsilon_{12}^L + \varepsilon_{12}^N, \tag{7}$$

$$\varepsilon_{11}^L = u_{,x} + w/R_x \quad \varepsilon_{22}^L = v_{,y} + w/R_y \quad \varepsilon_{12}^L = u_{,y} + v_{,x}, \tag{8}$$

$$\varepsilon_{11}^N = \frac{1}{2}(w_{,x})^2 \varepsilon_{22}^N = \frac{1}{2}(w_{,y})^2, \quad \varepsilon_{12}^N = w_{,y}w_{,x}, \tag{9}$$

$$\varepsilon_{13} = w_{,x} + \psi_x, \quad \varepsilon_{23} = w_{,y} + \psi_y, \tag{10}$$

$$\chi_{11} = \psi_{x,x}, \quad \chi_{22} = \psi_{y,y} \quad \chi_{12} = \psi_{x,y} + \psi_{y,x}, \tag{11}$$

where ψ_x, ψ_y are the independent rotations of the transverse normal to middle surface about the Oy and Ox axes, respectively. In formulas (8)–(11), the subscripts following the commas denote partial differentiation with respect to the corresponding coordinates.

In-plane force resultant vector $N = (N_{11}, N_{22}, N_{12})^T$, bending and twisting moments resultant vector $M = (M_{11}, M_{22}, M_{12})^T$, and transverse shear force resultant $Q = (Q_x, Q_y)^T$ are calculated by integrating along the Oz-axis and defined as follows

$$[N] = [A]\{\varepsilon\} + [B]\{\chi\}, \quad [M] = [B]\{\varepsilon\} + [D]\{\chi\}. \tag{12}$$

Elements A_{ij}, B_{ij}, D_{ij} of the matrices *A, B* and *D* of (12) are calculated by the following formulas

$$A_{ij} = \sum_{r=1}^{3} \int_{z_r}^{z_{r+1}} Q_{ij}^{(r)} dz, \qquad B_{ij} = \sum_{r=1}^{3} \int_{z_r}^{z_{r+1}} Q_{ij}^{(r)} z dz,$$

$$D_{ij} = \sum_{r=1}^{3} \int_{z_r}^{z_{r+1}} Q_{ij}^{(r)} z^2 dz, \tag{13}$$

where: $z_1 = -h/2, z_2 = h_1, z_3 = h_2, z_4 = h/2$. Values $Q_{ij}^{(r)}$ $(i, j = 1, 2, 6)$ are defined by the following expression:

$$Q_{11}^{(r)} = Q_{22}^{(r)} = \frac{E^{(r)}}{1-\left(\nu^{(r)}\right)^2}, \quad Q_{12}^{(r)} = \frac{\nu^{(r)}E^{(r)}}{1-\left(\nu^{(r)}\right)^2}, \quad Q_{66}^{(r)} = \frac{E^{(r)}}{2\left(1+\nu^{(r)}\right)}.$$

Transverse shear force resultants Q_x, Q_y are estimated from the following formulas

$$Q_x = K_s^2 A_{33}\varepsilon_{13}, \qquad Q_y = K_s^2 A_{33}\varepsilon_{23},$$

where K_s^2 denotes the shear correction factor. In this paper, it is taken as 5/6.

Let us consider materials with temperature independent of Poisson's ratio, and the same value for ceramics and metal ($\nu_m = \nu_c$). In this case, the coefficients A_{ij}, B_{ij}, D_{ij} can be calculated in a direct way. Note that analytical expressions for these coefficients, in the case of Types 1-2 and 2-2, have been already derived in Ref. [47]. Namely, in the present paper, we obtained analogous formulas for the Types 1-1 and 2-1.

Type 1-1:

$$A_{11} = v_0\left(E_{cm}\left(\frac{as1}{p_1+1} - \frac{as2}{p_3+1}\right) + E_m(h - as21) + E_c as21\right)$$

$$B_{11} = v_0 E_{cm}\left(\frac{as1}{p_1+2}\left(h_1 - \frac{h}{2(p_1+1)}\right) - \frac{as2}{p_3+2}\left(h_2 + \frac{h}{2(p_3+1)}\right) + \frac{h_2^2-h_1^2}{2}\right),$$

$$D_{11} = v_0\left(E_{cm}\left(as1\left(\frac{(as1)^2}{p_1+3} - \frac{as1}{p_1+2}h - \frac{h^2}{4(p_1+1)}\right) - as2\left(\frac{(as2)^2}{p_3+3} - \frac{as2}{p_3+2}h + \frac{h^2}{4(p_3+1)}\right) + \frac{h_2^3-h_1^3}{3}\right) + \frac{E_m}{12}h^3\right)$$

Type 2-1:

$$A_{11} = v_0\left(E_{cm}\left(h_1 + \frac{as21}{p_2+1}\right) + \frac{h}{2}\left(E_c + E_m\right)\right),$$

$$B_{11} = v_0 E_{cm}\left(\frac{as21}{p_2+2}\left(h_1 + \frac{h_2}{p_2+1}\right) + \frac{1}{2}\left(h_1^2 - \frac{h^2}{4}\right)\right),$$

$$D_{11} = v_0\left(E_{cm}\left(\frac{h_1^3}{3} + as21\left(\frac{h_2^2}{p_2+1} + \frac{2as21}{p_2+2}h_2 + \frac{(as21)^2}{p_2+3}\right)\right) + \frac{(E_m+E_c)}{24}h^3\right)$$

where

$$as1 = \left(\frac{h}{2} + h_1\right), \; as2 = h_2 - \frac{h}{2}, \quad as21 = h_2 - h_1,$$

$$bs1 = \frac{1}{2as1}, \quad bs2 = \frac{1}{2as2}, \quad v_0 = \frac{1}{\left(1-\nu^2\right)}, \quad E_{cm} = E_c - E_m.$$

Observe that for all types of lamination schemes, the values $A_{12}, A_{66}, .B_{12}, B_{66}, D_{12}, D_{66}$ are defined as follows

$$R_{12} = \nu R_{11}, \quad R_{22} = R_{11}, \quad R_{66} = \frac{1-\nu}{2}R_{11},$$

where R is a common designation of *A, B, D*.

The governing differential equations of equilibrium for free vibration of a shear deformable shallow shell can be expressed as [1,18]

$$\frac{\partial N_{11}}{\partial x} + \frac{\partial N_{12}}{\partial y} - \frac{Q_x}{R_x} = I_0\frac{\partial^2 u}{\partial t^2} + I_1\frac{\partial^2 \psi_x}{\partial t^2},$$

$$\frac{\partial N_{22}}{\partial y} + \frac{\partial N_{12}}{\partial x} - \frac{Q_y}{R_y} = I_0\frac{\partial^2 v}{\partial t^2} + I_1\frac{\partial^2 \psi_y}{\partial t^2},$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{N_{11}}{R_x} + \frac{N_{22}}{R_y} + N_{11}\frac{\partial^2 w}{\partial x^2} + 2N_{12}\frac{\partial^2 w}{\partial x \partial y} + N_{22}\frac{\partial^2 w}{\partial y^2} = I_0\frac{\partial^2 w}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x} + \frac{\partial M_{12}}{\partial y} - Q_x = I_2\frac{\partial^2 \psi_x}{\partial t^2} + I_1\frac{\partial^2 u}{\partial t^2}, \tag{14}$$

$$\frac{\partial M_{22}}{\partial y} + \frac{\partial M_{12}}{\partial x} - Q_y = I_2\frac{\partial^2 \psi_y}{\partial t^2} + I_1\frac{\partial^2 v}{\partial t^2},$$

where

$$\left(I_0, I_1, I_2\right) = \sum_{r=1}^{3}\int_{z_r}^{z_{r+1}}\left(\rho^{(r)}\right)\left(1, z, z^2\right)dz \tag{15}$$

and $\rho^{(r)}$ stands for a mass density of the rth layer.

Analytical expressions for these coefficients for shells with $\nu_m = \nu_c$ are presented below:

Type 1-1:

$$I_0 = \rho_{cm}\left(\frac{as1}{p_1+1} - \frac{as2}{p_3+1}\right) + \rho_m(h - as21) + \rho_c as21, \rho_{cm} = \rho_c - \rho_m,$$

$$I_1 = \rho_{cm}\left(\frac{as1}{p_1+2}\left(h_1 - \frac{h}{2(p_1+1)}\right) - \frac{as2}{p_3+2}\left(h_2 + \frac{h}{2(p_3+1)}\right) + \frac{h_2^2-h_1^2}{2}\right),$$

$$I_2 = \rho_{cm}\left(as1\left(\frac{(as1)^2}{p_1+3} - \frac{as1}{p_1+2}h - \frac{h^2}{4(p_1+1)}\right) - as2\left(\frac{(as2)^2}{p_3+3} - \frac{as2}{p_3+2}h + \frac{h^2}{4(p_3+1)}\right) + \frac{h_2^3-h_1^3}{3}\right) + \frac{\rho_m}{12}h^3$$

Type 2-1:

$$I_0 = \left(\rho_{cm}\left(h_1 + \frac{as21}{p_2+1}\right) + \frac{h}{2}\left(\rho_c + \rho_m\right)\right),$$

$$I_1 = \rho_{cm}\left(\frac{as21}{p_2+2}\left(h_1 + \frac{h_2}{p_2+1}\right) + \frac{1}{2}\left(h_1^2 - \frac{h^2}{4}\right)\right)$$

$$I_2 = \rho_{cm}\left(\frac{h_1^3}{3} + as21\left(\frac{h_1^2}{p_2+1} + \frac{2as21}{p_2+2}h_1 + \frac{as21^2}{p_2+3}\right)\right) + \frac{\rho_m+\rho_c}{24}h^3$$

3. Solution of linear vibration problem

The total strain energy U and kinetic energy T are given by

$$U = \frac{1}{2}\iint_{\Omega}\left(N_{11}^L\varepsilon_{11}^L + N_{22}^L\varepsilon_{22}^L + N_{1212}^L\varepsilon_{12}^L + M_{11}^L\chi_{11} + M_{22}^L\chi_{22} + M_{12}^L\chi_{12} + Q_x\varepsilon_{13} + Q_y\varepsilon_{23}\right)dxdy, \tag{16}$$

$$T = \frac{1}{2}\iint_{\Omega} I_0\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) + 2I_1\left(\dot{u}\dot{\psi}_x + \dot{v}\dot{\psi}_y\right) I_2\left(\dot{\psi}_x^2 + \dot{\psi}_y^2\right)dxdy. \tag{17}$$

In turn, the total energy functional for a laminated FGM shallow shell is as follows

$$J = T - U \tag{18}$$

Assuming that the shell vibrates periodically, the vector of unknown functions can be presented as

$$\begin{aligned}&\vec{U}(u(x,y,t), v(x,y,t), w(x,y,t), \psi_x(x,y,t), \psi_y(x,y,t)) = \\ &= \vec{U}(u(x,y), v(x,y), w(x,y), \psi_x(x,y), \psi_y(x,y))\sin\lambda t,\end{aligned} \tag{19}$$

where λ is the vibration frequency. Using (18), (19) and Hamilton's principle, we get the variational statement of the problem

$$\delta J = 0, \tag{20}$$

where

$$J = U(u, v, w, \psi_x, \psi_y) - \lambda^2 T\left(u, v, w, \psi_x, \psi_y\right). \quad (21)$$

The variation (20) of the functional (21) is carried out on the set of functions that satisfy the given boundary conditions.

The expressions for U and T in formula (21) depend on (x,y), and the kinetic energy takes the following form

$$T = \frac{1}{2}\iint_\Omega I_0\left(u^2+v^2+w^2\right) + 2I_1\left(u\psi_x + v\psi_y\right) + I_2\left(\psi_x^2+\psi_y^2\right)dxdy.$$

In this study, the minimization of functional (21) is performed by using the Ritz method, which is a powerful tool in the analysis of vibration of shells and plates. The accuracy and stability of the Ritz method depend essentially on the choice of admissible functions. Note, that the admissible functions should satisfy the following conditions:

1. They should be continuous, differentiable up to the degree needed, and should stand for a complete system.

2. They have to satisfy at least the geometric boundary conditions.

Suppose that admissible functions $\{u_i\}, \{v_i\}, \{w_i\}, \{\psi_{xi}\}, \{\psi_{yi}\}$ were constructed. Then, according to the Ritz method, unknown functions $u(x,y), v(x,y), w(x,y), \psi_x(x,y), \psi_y(x,y)$ are presented as follows

$$u = \sum_{i=1}^{N_1} a_i u_i,\ v = \sum_{i=N_1+1}^{N_2} a_i v_i, \quad w = \sum_{i=N_2+1}^{N_3} a_i w_i,$$
$$\psi_x = \sum_{i=N_3+1}^{N_4} a_i\psi_{xi}, \quad \psi_y = \sum_{i=N_4+1}^{N_5} a_i\psi_{yi}. \quad (22)$$

Coefficients of this expansion $\{a_i\}$, $i = 1,\dots,N_5$ in (22) are yielded by the Ritz system

$$\frac{\partial I}{\partial a_i} = 0, i = 1,\dots,N_5.$$

In order to construct a system of admissible functions satisfying the main (kinematic) boundary conditions, it is usually sufficient to derive an equation of either the boundary of the considered region or a part of the boundary of this region. For example, if a shell is clamped, then the boundary conditions have the following form

$$u = 0, v = 0, w = 0, \psi_x = 0, \psi_y = 0.$$

The corresponding admissible functions may be chosen as:

$$u_i = \omega(x,y)\varphi_i^{(1)}, \quad v_i = \omega(x,y)\varphi_i^{(2)}, \quad w_i = \omega(x,y)\varphi_i^{(3)},$$
$$\psi_{xi} = \omega(x,y)\varphi_i^{(4)}, \quad \psi_{yi} = \omega(x,y)\varphi_i^{(5)},$$

where $\left\{\varphi_i^{(k)}\right\}$, $k = 1,\dots,5$ stand for a complete system (power, Chebyshev's or trigonometric polynomials, splines or others), $\omega(x,y) = 0$ is the equation of the domain border. In the case of a complex shape, the equation of the boundary $\omega(x,y) = 0$ can be constructed by the R-functions theory [43,45,48–51], which is used in present work.

4. Solution of the nonlinear problem

In order to solve the nonlinear problem, let us implement the approach proposed in Ref. [45,47–49]. It is assumed that inertia forces in the middle surface of a shell are ignored in solving the nonlinear problem. In what follows, we introduce unknown functions in the form of expansion with respect to the eigenfunctions $w_1^{(e)}(x,y)$, $u_1^{(e)}(x,y)$, $v_1^{(e)}(x,y)$, $\psi_{x1}^{(e)}(x,y)$, $\psi_{y1}^{(e)}(x,y)$. They correspond to the fundamental vibration form:

$$\begin{cases} u(x,y,t) = y_1(t)u_1^{(e)}(x,y) + y_1^2(t)u_2(x,y), \\ v(x,y,t) = y_1(t)v_1^{(e)}(x,y) + y_1^2(t)v_2(x,y), \\ w(x,y,t) = y_1(t)w_1^{(e)}(x,y), \\ \psi_x(x,y,t) = y_1(t)\psi_{x1}^{(e)}(x,y) + y_1^2(t)\psi_{x2}(x,y), \\ \psi_y(x,y,t) = y_1(t)\psi_{y1}^{(e)}(x,y) + y_1^2(t)\psi_{y2}(x,y), \end{cases} \quad (23)$$

The coefficient of this expansion, i.e. the function $y(t)$, depends on time. The functions $u_2, v_2, \psi_{x2}, \psi_{y2}$ have to satisfy the following system of differential equations

$$\begin{cases} L_{11}u_2 + L_{12}v_2 + L_{14}\psi_{x2} + L_{15}\psi_{y2} = NL_1 w_1^{(e)}(x,y), \\ L_{21}u_2 + L_{22}v_2 + L_{24}\psi_{x2} + L_{25}\psi_{y2} = NL_2 w_1^{(e)}(x,y), \\ L_{41}u_2 + L_{42}v_2 + L_{44}\psi_{x2} + L_{45}\psi_{y2} = NL_4 w_1^{(e)}(x,y), \\ L_{51}u_2 + L_{52}v_2 + L_{54}\psi_{x2} + L_{55}\psi_{y2} = NL_5 w_1^{(e)}(x,y), \end{cases} \quad (24)$$

where

$$\begin{aligned} NL_1(w) &= -L_{11}(w)\frac{\partial w}{\partial x} - L_{12}(w)\frac{\partial w}{\partial y}, \\ NL_2(w) &= -L_{12}(w)\frac{\partial w}{\partial x} - L_{22}(w)\frac{\partial w}{\partial y}, \\ NL_4(w) &= -L_{41}(w)\frac{\partial w}{\partial x} - L_{42}(w)\frac{\partial w}{\partial y}, \\ NL_5(w) &= -L_{51}(w)\frac{\partial w}{\partial x} - L_{52}(w)\frac{\partial w}{\partial y}. \end{aligned} \quad (25)$$

Linear operators L_{ij} $(i,j = 1,\dots,5)$, occurred in Eq. (24), are defined by relations

$$L_{11} = A_{11}\left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right), \quad L_{12} = L_{21} = \frac{1+\nu}{2}A_{11}\frac{\partial^2}{\partial x\partial y},$$

$$L_{14} = L_{41} = B_{11}\left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right),$$
$$L_{15} = L_{51} = L_{24} = L_{42} = \frac{1+\nu}{2}B_{11}\frac{\partial^2}{\partial x\partial y},$$
$$L_{22} = A_{11}\left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right), \quad L_{25} = L_{52} = B_{11}\left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right),$$

$$L_{44} = D_{11}\left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right) - \frac{1-\nu}{2}K_s^2 A_{11},$$
$$L_{45} = L_{54} = \frac{1+\nu}{2}D_{11}\frac{\partial^2}{\partial x\partial y},$$

$$L_{55} = D_{11}\left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right) - \frac{1-\nu}{2}K_s^2 A_{11}.$$

The system (24) is supplemented by corresponding boundary conditions. The solution to this problem is carried out by means of the variational Ritz method and the R-functions method (RFM) [42,45,47–51]. The following algorithm holds: choose $u_2(x,y), v_2(x,y), \psi_{x2}(x,y)$, $\psi_{y2}(x,y)$, substitute expressions (23) in the equation of motion (14), and apply the Bubnov–Galerkin procedure. The following second-order nonlinear differential equation is obtained

$$\ddot{y}(t) + \omega_L^2 y_1(t) + y_1^2(t)\beta + y_1^3(t)\gamma = 0. \quad (26)$$

It should be mentioned that values for coefficients of Eq. (26) have been obtained in an analytical form. They are expressed through the double integrals as follows

$$\beta = \frac{-1}{m_1\left\|w_1^{(e)}\right\|^2}\iint_\Omega \left(L_{31}u_2 + L_{32}v_2 + L_{34}\psi_{x2} + L_{35}\psi_{y2} + \right.$$
$$\left. + N_{11}^{L_1}\frac{\partial^2 w_1^{(e)}}{\partial x^2} + N_{22}^{L_1}\frac{\partial^2 w_1^{(e)}}{\partial y^2} + 2N_{12}^{L_1}\frac{\partial^2 w_1^{(e)}}{\partial x\partial y}\right) w_1^{(e)}dxdy,$$

$$\gamma = \delta\iint_\Omega \left(NL_{33}w_1^{(e)} + N_{11}^{(L_2)}w_{,xx}^{(e)} + N_{22}^{(L_2)}w_{,yy}^{(e)} + 2N_{12}^{(L_2)}w_{,xy}^{(e)}\right) w_1^{(e)}dxdy.$$

Here

$$\delta = -\frac{1}{I_0\left\|w_1^{(e)}\right\|^2} \quad L_{31} = A_{11}(k_1+\nu k_2)\frac{\partial}{\partial x}, \quad L_{32} = A_{11}(\nu k_1 + k_2)\frac{\partial}{\partial y},$$
$$L_{34} = \left(\frac{1-\nu}{2}A_{11}K_s^2 + B_{11}\left(k_1+\nu k_2\right)\right)\frac{\partial}{\partial x},$$
$$L_{35} = \left(A_{11}K_s^2 + B_{11}\left(\nu k_1 + k_2\right)\right)\frac{\partial}{\partial y},$$

$$NL_{33}(w) = -\frac{1}{2}A_{11}\left(\left(w_{,x}^2 + \nu w_{,y}^2\right)w_{,xx} + \left(\nu w_{,x}^2 + w_{,y}^2\right)w_{,yy} + 2(1-\nu)w_{,x}w_{,y}w_{,xy}\right),$$

and $N_{ij}^{L_1}$ $(i,j = 1,2)$ are defined by formula (12).

Finally, $N_{ij}^{(L_2)}$ $(i,j = 1,2)$ are defined as follows:

$$\left\{N^{(L_2)}\right\} = \left\{N_{11}^{(L_2)};\ N_{22}^{(L_2)};\ N_{12}^{(L_2)}\right\}^T = [A]\left\{\varepsilon^{L_2}\right\}^T,$$

$$\left\{\varepsilon^{L_2}\right\} = \left\{\frac{\partial u_2}{\partial x}; \frac{\partial v_2}{\partial y}; \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right\}.$$

The solution to Eq. (26) was found numerically using the classical 4th-order Runge–Kutta method.

It should be mentioned that the proposed algorithm takes only one mode into account. If we use a few modes of the linear vibrations, then the initial problem is reduced to the system of nonlinear ordinary differential equations. To obtain this system, it is necessary to solve the auxiliary problems as it has been shown in Ref. [48]. Obviously, it is better to use larger numbers of eigenfunctions, but this significantly complicates the algorithm and its numerical implementation. The obtained numerical results regarding the analysis of the shell nonlinear vibrations can be viewed as an accurate approximation of the desired solution.

5. Numerical results

Below we present a few case studies and numerical results obtained by the proposed approach and developed software. In order to verify the accuracy of the present results, we consider the solution of several test problems.

5.1. Validation of the presented results (linear vibration)

Problem 1. Natural frequencies of laminated FGM shallow shells of double curvature and a square base were calculated for a wide variety of cases. Different Types (1-1; 1-2; 2-1; 2-2), schemes of lamination, and various boundary conditions were employed. As an example, we present the solution for cylindrical shells with fixed geometrical parameters: $h/a = 0.1$; $b/a = 1$; $a/R_x = 0.2$. The material constituents M_1 and M_2 are assumed to be aluminum and alumina [2,3,5,28]. The material properties of the FGM mixture used in the study are:

$$Al:\ E_c = 70\text{ GPA},\quad \nu_c = 0.3,\quad \rho_c = 270\text{ kg/m}^3,$$

$$Al_2O_3:\ E_c = 380\text{ GPA},\quad \nu_c = 0.3,\quad \rho_c = 3800\text{ kg/m}^3.$$

The boundary conditions are defined as follows:

CCCC — the shell is clamped on sides $x = \pm\frac{a}{2}$, $y = \pm\frac{b}{2}$;

SSSS — the shell is simply supported on sides $x = \pm\frac{a}{2}$, $y = \pm\frac{b}{2}$;

SFSF — the shell is free on sides $x = \pm\frac{a}{2}$ and simply supported on sides $y = \pm\frac{b}{2}$;

SCSC — the shell is simply supported on sides $x = \pm\frac{a}{2}$ and clamped on sides $y = \pm\frac{b}{2}$.

Values of the nondimensional fundamental frequency parameters, defined by formula $\Omega_L^{(1)} = \lambda_1 h\sqrt{\rho_c/E_c}$, of the cylindrical shells of Type 2-2 and for the thickness scheme $1-1-1$ are presented in Table 1.

The above results were obtained using 28 admissible functions to approximate each of the functions u, v, ψ_x, ψ_y and 36 admissible functions to approximate the deflection w. Due to the doubly-symmetric nature of the shell, the integration is performed only above one-quarter domain at numerical implementation of the developed method. It can be observed that the presented results differ from the results reported in [41] by not more than 0.5%. We compared the values of the fundamental frequency for plates, spherical and parabolic shallow shells with data presented in Ref. [41] and obtained an excellent agreement as well.

Table 1
Comparison of nondimensional fundamental frequency $\Omega_L^{(1)} = \lambda_1 h\sqrt{\rho_c/E_c}$ of cylindrical shallow shells with square plan form and various boundary conditions (thickness scheme 1-1-1).

Type of the shell	p	Method	Cylindrical panel $k_1 = 0.2$, $k_2=0$			
			SFSF	SSSS	CCCC	SCSC
2-2	0.6	(41)	0.6242	1.2939	2.2574	1.8195
		RFM	0.6247	1.2954	2.2685	1.8268
	5	(41)	0.6001	1.2404	2.1561	1.7404
		RFM	0.6007	1.2420	2.1675	1.7481
	20	(41)	0.6007	1.2396	2.1497	1.7370
		RFM	0.6012	1.2414	2.1616	1.7450

5.2. Free linear vibration of the functionally graded shells with complex shape of the plan

Problem 2. In order to contribute to new results and to illustrate the versatility and efficiency of the proposed method and the developed software, let us consider the shallow shell with the shape of the plan as presented in Fig. 2 and with the following geometrical parameters

$$k_1 = R_x/2a = 0.2,\quad k_2 = R_y/2a = (0; 0.2; -0.2),$$

$$b/a = 1,\quad a_1/2a = 0.25,\quad b_1/2a = 0.35,\quad h/2a = 0.1.$$

Consider two types of boundary conditions: clamped and simply supported on the entire border. Then, the solution structure for shells with completely clamped borders can be taken in the following way [43,45,47]

$$w = \omega\Phi_1,\quad u = \omega\Phi_2,\quad v = \omega\Phi_3,\quad \psi_x = \omega\Phi_4,\quad \psi_y = \omega\Phi_5.$$

For simply supported boundary conditions, we propose to take the solution structure satisfying kinematic boundary conditions in the following form

$$w = \omega^{(w)}\Phi_1,\quad u = \omega^{(u)}\Phi_2,\quad v = \omega^{(v)}\Phi_3, \psi_x = \omega^{(\psi_x)}\Phi_4,\quad \psi_y = \omega^{(\psi_y)}\Phi_5,$$

where $\Phi_i \in C^2(\Omega \cup \partial\Omega)$, $i = 1,2,\dots,5$ are indefinite components of the structure [42–50], presented as an expansion in a series of a complete system (power polynomials, trigonometric polynomials, splines etc.), $\omega = 0$ is the equation of the whole border of the shell plan form. The functions $\omega^{(u)}$, $\omega^{(v)}$, $\omega^{(w)}$, $\omega^{(\psi_x)}$, $\omega^{(\psi_y)}$ are constructed by the R-functions theory in such a way that they vanish on those parts of the boundary, where the functions u, v, w, ψ_x, ψ_y are zero.

The constructed analytical expressions for these functions are presented using the R-operations $\wedge_0, \vee_0$ (see [43,44,50] for more details)

$$\omega = \left(f_1 \wedge_0 f_2\right) \wedge_0 \left(f_3 \vee_0 f_4\right). \tag{27}$$

Functions f_i, $i = 1,\dots,4$ in relation (27) are defined as follows

$$f_1 = \left(a^2 - x^2\right)/2a \geq 0, f_2 = \left(b^2 - y^2\right)/2b \geq 0,$$

$$f_3 = \left(c^2 - x^2\right)/2c \geq 0, f_4 = \left(d^2 - y^2\right)/2d \geq 0.$$

Below we listed the expressions for functions $\omega^{(u)}$, $\omega^{(v)}$, $\omega^{(w)}$, $\omega^{(\psi_x)}$, $\omega^{(\psi_y)}$ for clamped and simply supported boundary conditions.

Clamped edge: $\omega^{(u)} = \omega^{(v)} = \omega^{(w)} = \omega^{(\psi_x)} = \omega^{(\psi_y)} = \omega$.

Simply supported edge:

$$\omega^{(w)} = \omega,\quad \omega^{(u)} = \omega^{(\psi_x)} = \left(f_3 \vee_0 f_4\right) \vee_0 \left(f_5 \vee_0 f_6\right) \vee_0 \left(f_7 \vee_0 f_8\right) \vee_0 f_2,$$
$$\omega^{(v)} = \omega^{(\psi_y)} = \left(f_3 \vee_0 f_4\right) \vee_0 \left(f_9 \vee_0 f_{10}\right) \vee_0 \left(f_{11} \vee_0 f_{12}\right) \vee_0 f_1,$$

$$f_5 = \left(r_1^2 - (x-c)^2 - \left(y - \left(b - r_1\right)\right)^2\right)/2r_1 \geq 0,$$

$$f_6 = \left(r_1^2 - (x+c)^2 - \left(y - \left(b - r_1\right)\right)^2\right)/2r_1 \geq 0,$$

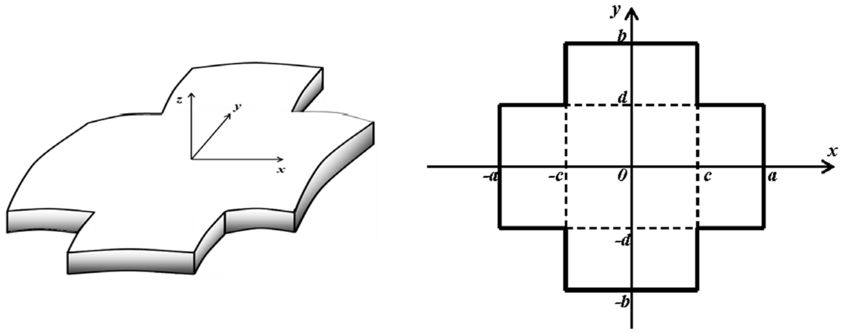

Fig. 2. Shape of the plan of the laminated FGM shallow shell.

Table 2
Nondimensional fundamental frequency $\Omega_L^{(1)} 10^2 = \lambda_1 a^2 h \sqrt{\rho_c / E_c}$ for shells with clamped boundary conditions (Fig. 2).

Type	Shell type	2-1-2	1-1-1	2-2-1	1-2-1	1-8-1
1-1	Pl	8.426	8.761	9.019	9.289	10.54
	Cy	8.563	8.901	9.157	9.430	10.67
	Sp	8.768	9.109	9.364	9.640	10.88
	Par	8.627	8.967	9.223	9.495	10.73
1-2	Pl	10.52	10.24	9.867	9.741	8.106
	Cy	10.62	10.33	9.953	9.820	8.174
	Sp	10.75	10.46	10.08	9.939	8.277
	Par	10.66	10.37	9.990	9.856	8.206
2-2	Pl	8.022	8.109	7.959	8.268	8.654
	Cy	8.147	8.233	8.070	8.390	8.771
	Sp	8.335	8.419	8.239	8.573	8.947
	Par	8.213	8.298	8.131	8.454	8.832

Table 3
Nondimensional fundamental frequency $\Omega_L^{(1)} 10^2 = \lambda_1 a^2 h \sqrt{\rho_c / E_c}$ for shells with simply supported boundary conditions (Fig. 2).

Type	Shell type	2-1-2	1-1-1	2-2-1	1-2-1	1-8-1
1-1	Pl	6.805	7.071	7.273	7.487	8.464
	Cy	6.895	7.163	7.363	7.579	8.553
	Sp	7.098	7.370	7.568	7.788	8.757
	Par	6.926	7.194	7.395	7.611	8.583
1-2	Pl	8.415	8.192	7.886	7.786	6.479
	Cy	8.471	8.244	7.942	7.834	6.521
	Sp	8.606	8.371	8.071	7.951	6.623
	Par	8.490	8.262	7.957	7.850	6.535
2-2	Pl	6.479	6.545	6.411	6.667	6.972
	Cy	6.556	6.621	6.479	6.742	6.742
	Sp	6.741	6.805	6.644	6.922	7.208
	Par	6.593	6.658	6.513	6.777	7.067

$$f_7 = \left(r_1^2 - (x-c)^2 - \left(y + (b - r_1)\right)^2\right)/2r_1 \geq 0,$$

$$f_8 = \left(r_1^2 - (x+c)^2 - \left(y + (b - r_1)\right)^2\right)/2r_1 \geq 0,$$

$$f_9 = \left(r_2^2 - \left(x - (a - r_2)\right)^2 - (y-d)^2\right)/2r_2 \geq 0,$$

$$f_{10} = \left(r_2^2 - \left(x + (a - r_2)\right)^2 - (y-d)^2\right)/2r_2 \geq 0,$$

$$f_{11} = \left(r_2^2 - \left(x - (a - r_2)\right)^2 - (y+d)^2\right)/2r_2 \geq 0,$$

$$f_{12} = \left(r_2^2 - \left(x + (a - r_2)\right)^2 - (y+d)^2\right)/2r_2 \geq 0,$$

$$r_1 = (b-d)/2, r_2 = (a-c)/2.$$

In Tables 2 and 3, fundamental frequency parameters $\Omega_L^{(1)} = \lambda_1 a^2 h \sqrt{\rho_c / E_c}$ for clamped (Table 2) and simply supported (Table 3) plates (Pl), cylindrical (Cy), spherical (Sp), and hyperbolic paraboloidal shells (Par) are presented for Types 1-1; 1-2; 2-2 and different thickness schemes. For all layers, power exponents were taken as $p = 1, p_1 = p_2 = p_3 = p$.

Note that the greatest value of nondimensional fundamental frequencies was obtained for spherical shells and the smallest value for plates. These values essentially depend on the type of the shell and the scheme of thickness. The behaviors of the shells of Type 1-1 and Type 2-2 are similar and the frequencies parameter takes the maximum value for the thickness scheme 1-8-1. For the plate and shallow shells of Type 1-2, the maximum value of the frequency parameter is reached for the scheme 2-1-2, and the minimum frequency is obtained for the thickness scheme 1-8-1.

The effect of the power-law exponent $p = p_1, p_2, p_3$ on the fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 h \sqrt{\rho_c / E_c}$ for the clamped cylindrical shell of Type 1-2 and different thickness schemes is shown in Fig. 3. The curve 1-8-1* corresponds to the cylindrical shell with curvatures $k_1 = 0.5$; $k_2 = 0$.

Let us fix the thickness scheme $1-2-1$ and analyze the effect of the boundary conditions on the behavior of the dimensionless frequency parameters $\Omega_L^{(1)}$. As follows from Fig. 4, the value of fundamental frequency parameters essentially depends on the shell type and the boundary conditions. Obviously, the fundamental frequency parameters decrease with the increase in the power-law exponent for all considered cases. However, for shells of Type 1-1 and Type 1-2, the decrease is more significant than for shells of Type 2-2.

5.3. Nonlinear vibrations

The proposed approach was also used to investigate nonlinear vibration of the shells. In order to check the proposed method and software, let us analyze free nonlinear vibration of the simply-supported spherical shallow shell with a square plan form. Suppose that geometrical parameters of the shell are: $k_1 = R_x/2a = 0.1, k_2 = R_y/2a = 0.1$, $b/a = 1$.

For an isotropic shell, this problem has been studied in paper [54] and for the FGM shell, a similar problem has been investigated in paper [14]. Consider two values of the thickness: $h/2a = 0.01$; $h/2a = 0.1$.

As it has been shown by Kobayashi and Leissa [54] for isotropic shells, as the thickness increases, the "soft spring" response becomes weak and does not appear in the thick shell, in which the thickness ratio is larger than 0.1. Applying the proposed method, we obtained the same behavior of the shell (see Fig. 5, $p = 0$).

For the FGM shell made of a mixture of ceramics and metals (Al/Al_2O_3), the effect of the amplitude on the frequency, studied for

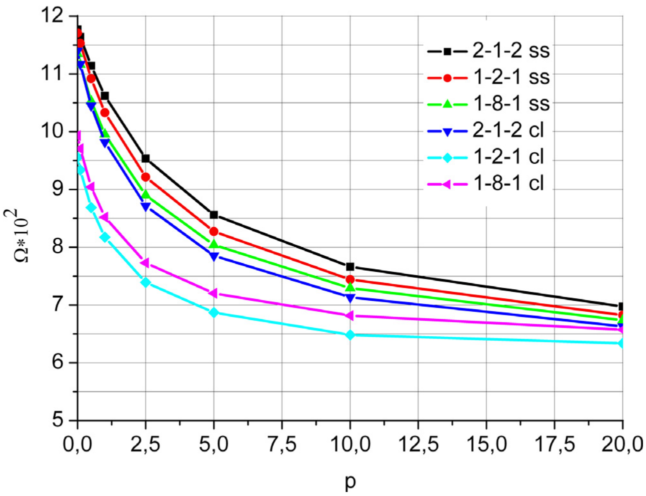

Fig. 3. Effect of the power-law exponent on the fundamental frequency of clamped cylindrical shells of Type 1-2 with different thickness schemes and curvatures $k_1 = 0.2$; $k_2 = 0$.

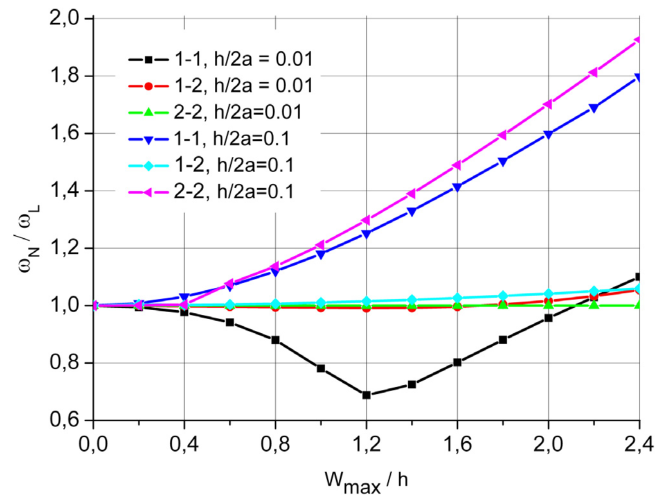

Fig. 6. Effect of amplitude on the frequency of the simply-supported spherical laminated FGM shallow shell $(Al/Al_2O_3, 1-8-1, p=1. k_1 = k_2 = 0.1)$.

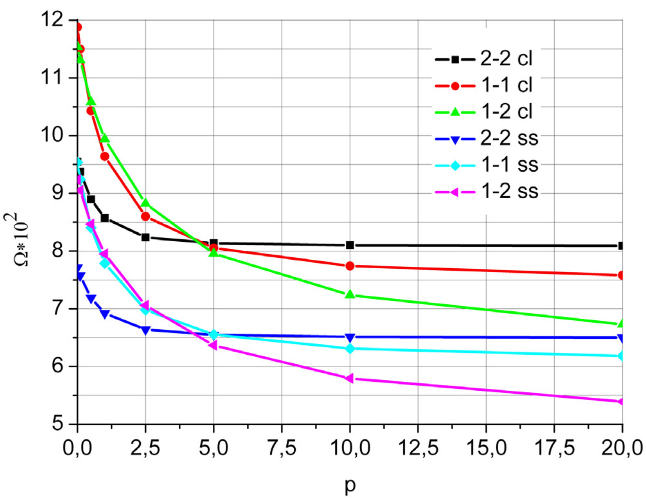

Fig. 4. Nondimensional fundamental frequencies of clamped (cl) and simply supported (ss) spherical shells with the same scheme of thickness 1-2-1 and different type: 1-1, 1-2, 2-2.

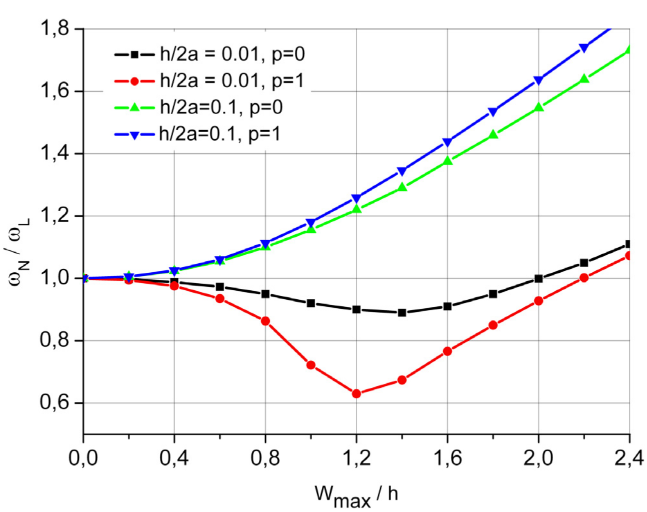

Fig. 5. Effect of the amplitude on the frequency for various thicknesses of the spherical FGM shallow shell (Al/Al_2O_3).

various thicknesses, is analogous to the isotropic case (see Fig. 5, *p=1*). However, in the case of the FGM shell, the soft spring response becomes stronger. It can be explained as follows. For small amplitudes, the initial stiffness of the given FGM shell is primarily membrane. As the amplitude increases, the role of bending becomes more important.

Fig. 6 shows the variation of the frequency ratio as a function of the amplitude-thickness ratio for laminated FGM shallow shells with a square plan form. The general thickness of the shell is varied; the scheme of layers thickness is taken as 1-8-1. Different Types (1-1, 1-2, 2-2) of the shells are considered. The power exponent for all layers is taken the same, *p=1*.

Note that qualitative behavior backbone curves for shells of Types 1-1 is similar to the previous case. There are no differences between the ratios of linear to nonlinear frequencies for Type 1-2 shells for both thickness ratios 0.01 and 0.1 and for the amplitude less than 2.5. However, for Type 2-2 shells, behavior of the reference curves depends essentially on the thickness. If the thickness ratio is 0.01, then the ratio of linear and nonlinear frequencies remains constant when the amplitude ratio increases, but if the thickness ratio is 0.1, then the backbone curve has a hardening type of nonlinearity. Therefore, we can conclude that the behavior of backbone curves of laminated FGM shallow shells depends on many factors: thickness, curvature, type of the core, ratio of layers thickness, boundary conditions, geometry of the plan, and physical properties of the constituent materials. It means that in each specific case, it is necessary to carry out a computer experiment.

The proposed approach was also used to investigate nonlinear vibration of shells with complex plan form (Fig. 2). The dependence between the maximum deflection and the ratio of the nonlinear frequency to the linear one for different types of shells of and fixed thickness scheme $1-8-1$ and for the fixed values of power exponent $(p = 0.5, p_1 = p_2 = p_3 = p)$ was obtained. Figs. 7 and 8 show the variation of the nonlinear frequency amplitude relationships (backbone curves) for clamped cylindrical (Fig. 7) and spherical (Fig. 8) shallow shells of Types 1-1; 1-2; 2-2. The hardening type of nonlinearity is observed for both shells and their different types.

The results presented in Figs. 7 and 8 are similar and. As shown by the computational experiment, backbone curves for simply supported shells are also close to those shown in Figs. 7 and 8.

In order to demonstrate the difference between the behavior of nonlinear frequencies for cylindrical and spherical shells, let us consider the variation of the nonlinear frequencies of the maximum amplitude of vibration for simply supported shells. Fig. 9 shows the relation of the maximum amplitude of vibration and the fundamental nonlinear

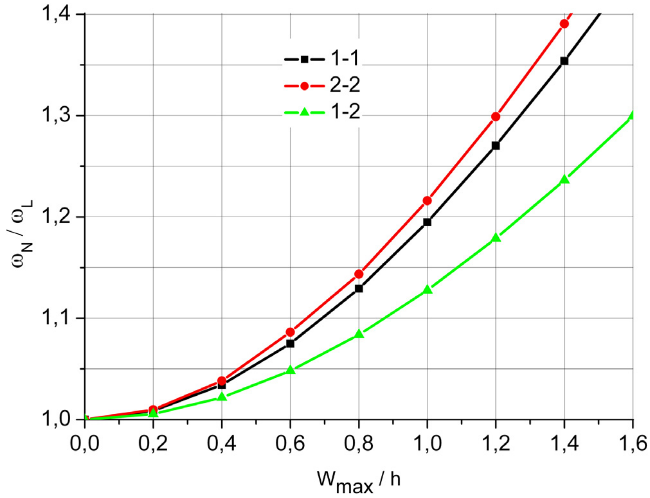

Fig. 7. Backbone curves for clamped cylindrical shells (1-8-1, *p=0.5*).

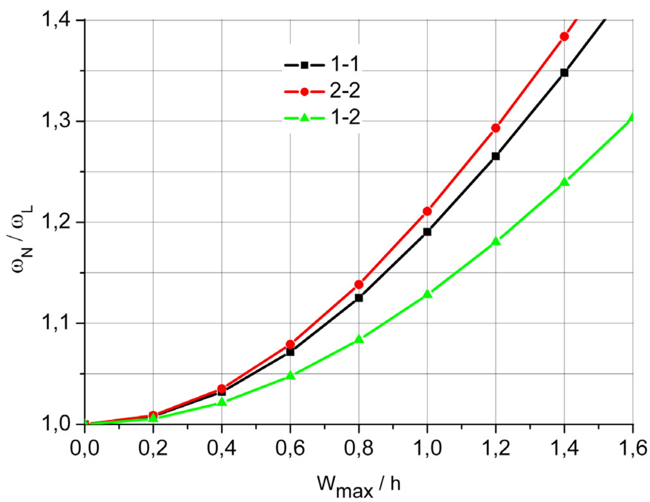

Fig. 8. Backbone curves for clamped spherical shells (1-8-1, $p = 0.5$).

frequency for SS-simply-supported cylindrical and spherical shells of different types.

As follows from Fig. 9, the curvature of shells influences the nonlinear frequencies slightly, while the type of the shell yields significant changes. Shells of the Type 1-1 have the largest values of nonlinear frequencies, and shells of the Type 1-2 have the lowest values of frequencies from all cases considered in this study.

6. Concluding remarks

In this study, we have proposed a method of investigation of free linear and geometrically nonlinear vibrations of laminated functionally graded shallow shells with a complex shape of the plan. Three-layered shells with different types of layers have been considered: the middle layer fabricated from FGM and external layers made from metal or ceramics, and the converse external layers made of FGM, but the core made from metal or ceramic. Taking into account that the properties of materials are varied along thickness, their analytical expressions have been obtained provided that Poisson's ratio is the same for ceramics and metal. The proposed method consists of a solution of the linear vibration problem and special types of elasticity problems. In the study, the problems have been solved by the variational Ritz method and the R-functions theory. In order to solve the nonlinear problem, Galerkin and

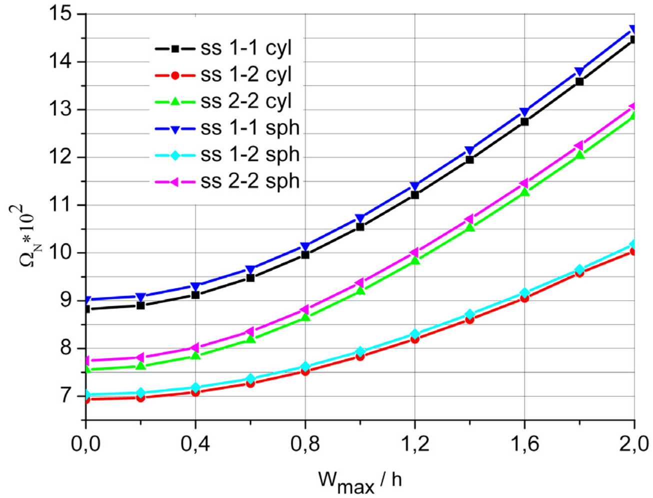

Fig. 9. Backbone curves for simply-supported spherical and cylindrical shells with the lamination scheme (1-8-1, $p = 0.5$).

Runge–Kutta methods have been applied. A comparison of the results obtained for double-curved shallow shells the with square plan form has been used to validate the developed software and the proposed method. For shells with a complex plan form, new solution structures have been built for simply supported boundary condition. Novel numerical results have been obtained for cylindrical, spherical, and hyperbolic paraboloidal shallow shells with complex plan forms, different boundary conditions, and different types of layers. The effect of the gradient-index and thickness schemes has been studied for linear and geometrically nonlinear vibrations of shells with complex shape of the plan, and engineering-oriented results have been presented and discussed.

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