Two Approaches in the Analytical Investigation of the Spring Pendulum

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Abstract

Dynamics of the nonlinear spring pendulum is analysed using two asymptotic approaches. The multiple scale method is commonly applied with using two time scales. The purpose of the research is to justify the introduction of an additional third scale. Results of the analysis clearly show that introducing the third scale improve correctness of the approximate analytical solution. The obtained results allow for qualitative and quantitative analysis of the behavior of the studied system with a high accuracy. Calculations are made both in the neighbourhood of the resonance and also far from it.

Keywords: nonlinear dynamics, asymptotic method, spring pendulum

1. Introduction

The asymptotic method of multiple scales (MS) is quite popular in recent years. Although the majority of authors use only two scales, in recent years many papers have appeared in which three time scales have been applied. The question arises whether and how the choice of the number of time scales affects the quality of solutions. The motivation of this paper is to compare solutions obtained with using two and three time scales in MS method. A relatively simple system what is a spring pendulum was chosen to investigate this problem. Despite its simplicity, the pendulum is characterized by non-linearity of geometrical nature and additionally by non-linear properties of the spring itself what seems to be a conductive circumstance to investigate the problem. Research is perform both for the close neighbourhood and far from the main resonances.

There exists many papers devoted to analysis of pendulum-like systems [1-7]. In the engineering machinery there are numerous elements of similar character. Moreover pendula serve as a good example of the intuitive nonlinear dynamical systems, that can help to discover and explain many of exciting and important non-linear dynamical phenomena.

2. Formulation of the problem

Plane motion of a point of mass *m* mounted on a spring-damper suspension is investigated in the paper. The scheme of the system is presented in Figure 1. The spring is assumed to be massless and having the nonlinearity of the cubic type. L_0 denotes the spring length in the non-stretched state. There are purely viscous dampers in the system. The resistance force **R** depends on the point velocity **v** according to the relation $\mathbf{R} = -C_2\mathbf{v}$. The system is loaded by a torque the magnitude of which changes harmonically and by the force **F** which also is harmonic. The total spring elongation $X(t)$ and the angle $\Phi(t)$ are assumed as the generalized coordinates.

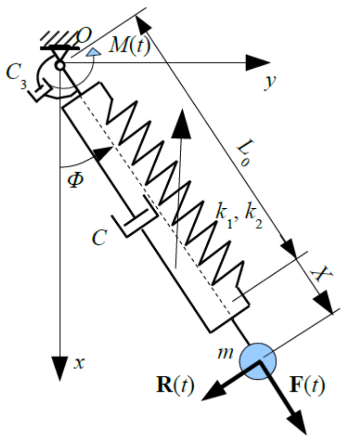

Figure 1. Spring pendulum

The dimensionless form of the motion equations of the system is as follows

$$\ddot{\xi} + \xi + \alpha\left(3\xi_e^2\,\xi + 3\xi_e\,\xi^2 + \xi^3\right) + c_1\dot{\xi} - (1+\xi)\dot{\varphi}^2 + w^2(1-\cos\varphi) = f_1\cos(p_1\tau) \tag{1}$$

$$(1+\xi)^2\ddot{\varphi} + w^2(1+\xi)\sin\varphi + c_2(1+\xi)^2\dot{\varphi} + c_3\dot{\varphi} + 2(1+\xi)\dot{\xi}\dot{\varphi} = f_2\cos(p_2\tau) \tag{2}$$

where:

$\xi = \dfrac{X}{L}$, $\alpha = k_2L^2/k_1$, $c_1 = \dfrac{C_2 + C}{\omega m}$, $c_2 = \dfrac{C_2}{\omega m}$, $c_3 = \dfrac{C_3}{\omega m}$, $w = \sqrt{g/L}/\omega$, $\omega = \sqrt{k_1/m}$,

$L = L_0 + X_e$, $\xi_e = X_e/L$, X_e is elongation of the spring at the static equilibrium, $\tau = t\,\omega$ denotes dimensionless time, and f_1, f_2, p_1, p_2 are dimensionless parameters describing external loading.

The static elongation ξ_e satisfies the equilibrium condition

$$\alpha\,\xi_e^3 + \xi_e - w^2 = 0\,. \tag{3}$$

Equations (1) – (2) are supplemented by the initial conditions

$$\xi(0) = \xi_0,\ \dot{\xi}(0) = v_0,\ \varphi(0) = \varphi_0,\ \dot{\varphi}(0) = \omega_0\,, \tag{4}$$

where quantities $\xi_0, v_0, \varphi_0, \omega_0$ are known.

3. Solution method

The multiple-scale method is used to solve the initial value problem (1) – (4). The method requires approximation of the trigonometric functions of the angle φ by a few terms of their power series, so we assume $\sin\varphi \approx \varphi - \varphi^3/6,\ \cos\varphi \approx 1 - \varphi^2/2$.

According to MS method, we introduce the small parameter ε and n time scales, where n takes value 2 or 3. The time scales τ_i are related to the dimensionless time as follows $\tau_i = \varepsilon^i \tau$, $i = 0, \ldots, n-1$.

The differential operators in (1) and (2) take the form

$$\frac{d}{d\tau} = \sum_{i=0}^{n-1} \varepsilon^i \frac{\partial}{\partial \tau_i}\,,$$
$$\frac{d^2}{d\tau^2} = \frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon \frac{\partial^2}{\partial \tau_0 \partial \tau_1} + \varepsilon^2 \left(\frac{\partial^2}{\partial \tau_1^2} + 2 \frac{\partial^2}{\partial \tau_0 \partial \tau_2} \right) + \ldots + O(\varepsilon^n). \tag{5}$$

The solution is sought in the form of the power series of the small parameter ε

$$\xi(\tau;\varepsilon) = \sum_{k=1}^{k=n} \varepsilon^k x_k(\tau_0, .., \tau_{n-1}) + O(\varepsilon^{n+1}), \quad \varphi(\tau;\varepsilon) = \sum_{k=1}^{k=n} \varepsilon^k \phi_k(\tau_0, .., \tau_n) + O(\varepsilon^{n+1}), \tag{6}$$

where n is both the number of time scales and the number of the power series terms taken into account in the approximation.

Assumed smallness of some parameters is expressed using the small parameter [1]. Since two approaches are performed in parallel to find the solution to the problem (1) – (4), two sets of assumptions are formulated:

- for two time scales: $f_1 = \tilde{f}_1\varepsilon^2$, $f_2 = \tilde{f}_2\varepsilon^2$, $c_1 = \tilde{c}_1\varepsilon$, $c_2 = \tilde{c}_2\varepsilon$, $c_3 = \tilde{c}_3\varepsilon$, $\alpha = \tilde{\alpha}\,\varepsilon$,
- for three time scales: $f_1 = \tilde{f}_1\varepsilon^3$, $f_2 = \tilde{f}_2\varepsilon^3$, $c_1 = \tilde{c}_1\varepsilon^2$, $c_2 = \tilde{c}_2\varepsilon^2$, $c_3 = \tilde{c}_3\varepsilon^2$, $\alpha = \tilde{\alpha}\,\varepsilon^2$.

The first step of the MS method is substitution of (5) – (6) and one set of the above assumptions into the equations of motion (1) – (2) which yields equations containing the small parameter ε in various powers. These equations should be satisfied for any value of ε, hence after rearrangement each of them according to the powers of ε, a system of successive recurrence equations emerges. The solution of the lower order approximation equations are then introduced into the equations of the higher order approximation, and in this way the solution of the whole problem (1) – (4) in the form of (6) is achieved. At every step in the procedure, the secular terms have to be removed. This demanding leads to the so-called solvability conditions. The procedure is described in details in references [1-3].

The main goal of the paper is comparison between the two asymptotic solutions. Let us discuss separately solution far from resonance and in the neighbourhood of the main resonance.

4. Vibration far from resonance

The MS method allows to obtain analytical form of the approximate solution of the problem (1) – (4).

CASE 1 – *two time scales*

The solution obtained using two scales has the following form

$$\xi = a_{10}e^{-c_1\tau/2}\cos\left(\tau + \frac{3}{2}\alpha\,\xi_e^2 + \psi_{10}\right) - \frac{f_1\cos(p_1\tau)}{p_1^2-1} + \frac{1}{4}a_{20}^2 e^{-(c_2+c_3)\tau}\left(1 + \frac{3\cos\left(2(w\tau+\psi_{20})\right)}{4w^2-1}\right), \tag{7}$$

$$\varphi = a_{20}e^{-(c_2+c_3)\tau/2}\cos\left(w\tau+\psi_{20}\right) - \frac{f_2\cos(p_2\tau)}{p_2^2-w^2} - a_{10}a_{20}e^{-(c_1+c_2+c_3)\tau/2}w\left(\left(2+3w-2w^2\right)\cos\left((1-w+\frac{3}{2}\alpha\xi_e^2)\tau+\psi_{10}-\psi_{20}\right) + \left(3w-2+2w^2\right)\cos\left((1+w+\frac{3}{2}\alpha\xi_e^2)\tau+\psi_{10}+\psi_{20}\right)\right)\Big/\left(8w^2-2\right), \tag{8}$$

where $a_{10}, a_{20}, \psi_{10}, \psi_{20}$ are known initial values of the amplitudes and the phases, and they are compatible with the initial values occurring in conditions (4).

CASE 2 – *three time scales*

The solution in this case has the form

$$\xi = \frac{1}{4}w^2a_2^2 - \frac{f_1\cos(p_1\tau)}{p_1^2-1} + a_1\cos(\tau+\psi_1) + \frac{3w^2a_2^2\cos(2(w\tau+\psi_2))}{4(4w^2 1)} + \frac{3(w-1)wa_1a_2^2\cos(\tau-2w\tau+\psi_1-2\psi_2)}{16(2w-1)} - \frac{3(w+1)wa_1a_2^2\cos(\tau+2w\tau+\psi_1+2\psi_2)}{16(2w+1)}, \tag{9}$$

$$\varphi = \frac{f_2\cos(p_2\tau)}{w^2-p_2^2} + a_2\cos(w\tau+\psi_2) + \frac{w(6-5w+w^2)a_1^2a_2\cos((w-2)\tau-2\psi_1+\psi_2)}{16(2w-1)}$$
$$-\frac{(w+2)wa_1a_2\cos(\tau+w\tau+\psi_1+\psi_2)}{2(2w+1)} + \frac{(w-2)wa_1a_2\cos(\tau-w\tau+\psi_1-\psi_2)}{2(2w-1)} + \tag{10}$$
$$\frac{w(6-5w+w^2)a_1^2a_2\cos((w-2)\tau-2\psi_1+\psi_2)}{16(2w-1)} + \frac{(1-49w^2a_2^3\cos(3(w\tau+\psi_2)))}{192(4w^2-1)},$$

where a_1, a_2, ψ_1, ψ_2 are functions which satisfy the following solvability conditions

$$\dot{a}_1 = -\frac{c_1}{2}a_1, \qquad \dot{\psi}_1 = \frac{3}{2}\alpha\,\xi_e^2 + \frac{3w^2(w^2-1)a_2^2}{16w^2-4},$$
$$\dot{a}_2 = -\frac{c_2+c_3}{2w}a_2, \quad \dot{\psi}_2 = \frac{3w(w^2-1)a_1^2}{16w^2-4} + \frac{w(1+7w^2-8w^4)a_2^2}{16(4w^2-1)}. \tag{11}$$

The system of differential equations (11) is completed by the following initial conditions

$$a_1(0) = a_{10},\ a_2(0) = a_{20},\ \psi_1(0) = \psi_{10},\ \psi_2(0) = \psi_{20}, \tag{12}$$

where initial values of the amplitudes and the phases $a_{10}, a_{20}, \psi_{10}, \psi_{20}$ are known and compatible with the initial values $\xi_0, v_0, \varphi_0, \omega_0$ occurring in conditions (4).

The time histories of the generalized co-ordinates according to (7) – (10) are presented in Figs. 2 and 3. The assumed parameters are as follows: $f_1 = 0.006$, $f_2 = 0.001$, $c_1 = 0.001$, $c_2 = 0.0002$, $c_3 = 0.0003$, $\alpha = 0.02$, $w = 0.21$, $p_1 = 0.65$, $p_2 = 0.78$.

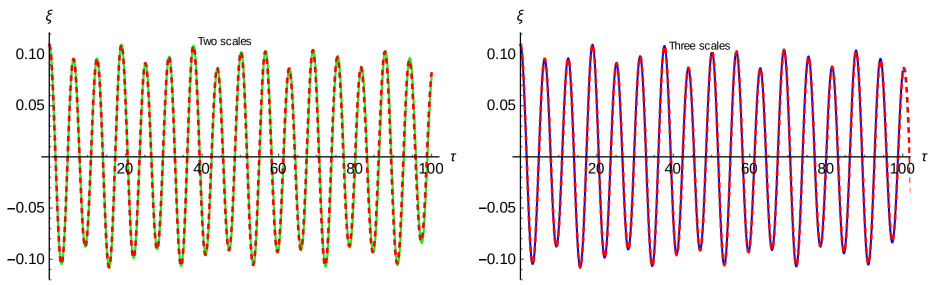

Figure 2. Time histories of longitudinal vibrations; dashed curve – numerical solution, continuous curve – analytical solution

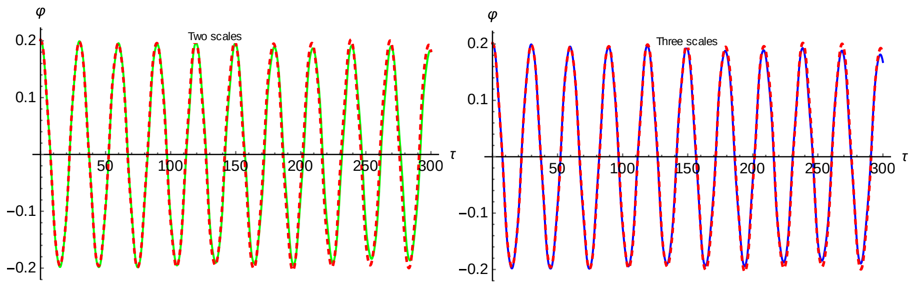

Figure 3. Time histories of swing vibrations; dashed curve – numerical solution, continuous curve – analytical solution

The graphs presented in Figs. 2 and 3 show that both results regarding two and three time scales results are very similar. A measure of the accuracy of the approximate solution can be the absolute error of fulfilment of the governing equations (1) – (2). The error calculated for the solutions (7) – (8) and (9) – (10) is presented in Fig. 4.

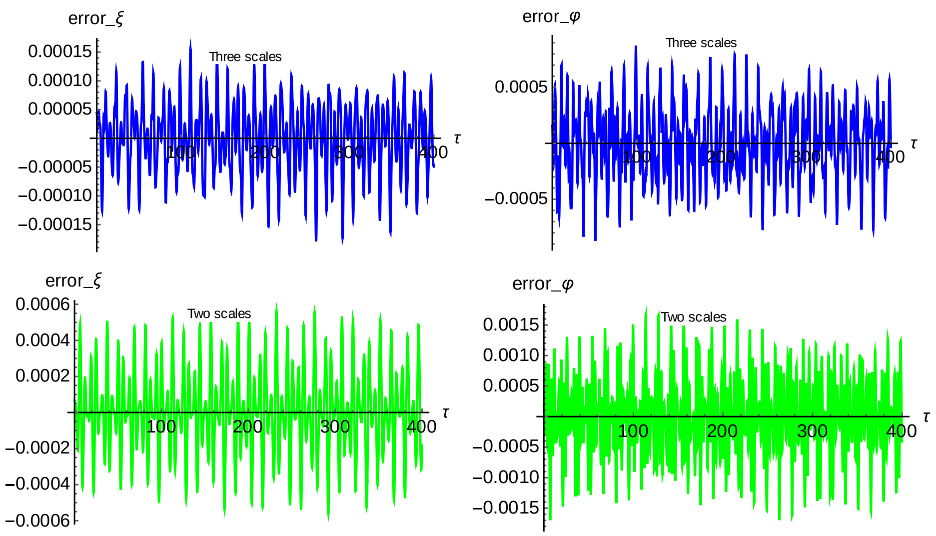

Figure 4. Absolute error

Though the error is very small for both approaches but it is approximately two times lower when three time scales are used than in the case with two time scales.

5. Vibration in the neighbourhood of resonance

We are focused on the case when two main resonances appear simultaneously in the system what is caused by the circumstances that $p_1 \approx 1$ and $p_2 \approx w$.

The detuning parameters σ_1 and σ_2 are necessary to deal with this case. They are introduced in the way

$$p_1 = 1 + \sigma_1, \quad p_2 = w + \sigma_2, \tag{13}$$

where $\sigma_1 = \varepsilon^{n-1}\tilde{\sigma}_1, \quad \sigma_2 = \varepsilon^{n-1}\tilde{\sigma}_2.$

Employing the analogous procedure to the one described in Section 3, the asymptotic solution to the problem (1) – (4) with additional conditions (13) is obtained. However, due to assumptions (13), the solvability conditions have more complicated form than previously, what makes impossible to solve them analytically. The solvability conditions describe the slow modulation of the amplitudes and phases and the knowledge of which is necessary to obtain the final form of solution. They are also of significantly meaning in the study of resonance. For the reasons mentioned, we are focused on the slow modulation equations. In dependence on the number of time scales the system of the first order differential equations governing the slow modulation in main resonances case have the following form.

CASE 3 – two time scales; vibration near resonance

$$\begin{aligned}
\dot{\psi}_1 &= \frac{3}{2}\alpha\,\xi_e^2 - \frac{f_1\cos(\sigma_1\tau - \psi_1)}{2a_1},\\
\dot{a}_1 &= -\frac{c_1}{2}a_1 + \frac{f_1}{2}\sin(\sigma_1\tau - \psi_1),\\
\dot{\psi}_2 &= -\frac{f_2\cos(\sigma_2\tau - \psi_2)}{2wa_2},\\
\dot{a}_2 &= -\frac{c_2 + c_3}{2w}a_2 + \frac{f_2}{2w}\sin(\sigma_2\tau - \psi_2),
\end{aligned} \tag{14}$$

CASE 4 – three time scales; vibration near resonance

$$\begin{aligned}
\dot{\psi}_1 &= \frac{3}{2}\alpha\,\xi_e^2 + \frac{3w^2(w^2-1)a_2^2}{16w^2-4} - \frac{f_1\cos(\sigma_1\tau - \psi_1)}{2a_1},\\
\dot{a}_1 &= -\frac{c_1}{2}a_1 + \frac{f_1}{2}\sin(\sigma_1\tau - \psi_1),\\
\dot{\psi}_2 &= \frac{3w(w^2-1)a_1^2}{16w^2-4} + \frac{w(1+7w^2-8w^4)a_2^2}{16(4w^2-1)} - \frac{f_2\cos(\sigma_2\tau - \psi_2)}{2wa_2},\\
\dot{a}_2 &= -\frac{c_2 + c_3}{2w}a_2 + \frac{f_2}{2w}\sin(\sigma_2\tau - \psi_2).
\end{aligned} \tag{15}$$

The initial conditions for equations (14) and for equations (15) have the same form

$$a_1(0) = a_{10},\ a_2(0) = a_{20},\ \psi_1(0) = \psi_{10},\ \psi_2(0) = \psi_{20}. \tag{16}$$

where $a_{10}, a_{20}, \psi_{10}, \psi_{20}$, which are compatible with $\xi_0, v_0, \varphi_0, \omega_0$, are known.

Solving equations (14) or (15), depending on the chosen number of time scales, allows us to obtain the approximate solution to the problem (1) – (4) involving conditions (13) for two main resonances. Time histories of the generalized coordinates obtained analytically compared to the numerical solution are presented in Fig. 5 and Fig. 6. The parameters assumed in simulation are as follows: $f_1 = 0.001$, $f_2 = 0.0001$, $c_1 = 0.001$, $c_2 = 0.001$, $c_3 = 0.0$, $\alpha = 0.01$, $w = 0.12$, $\sigma_1 = 0.005$, $\sigma_2 = 0.01$.

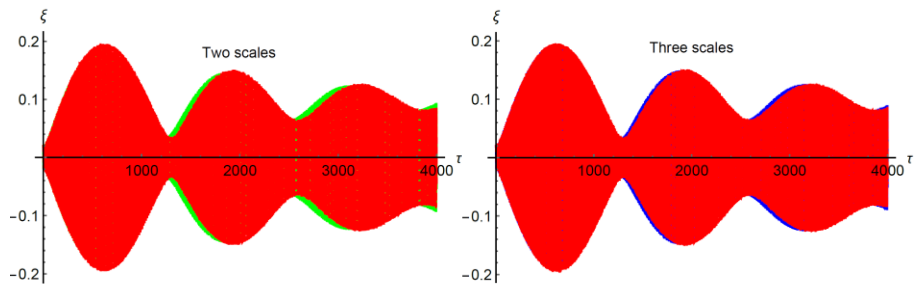

Figure 5. Time histories of longitudinal vibrations; dashed (red) curve – numerical solution, continuous curve – analytical solution

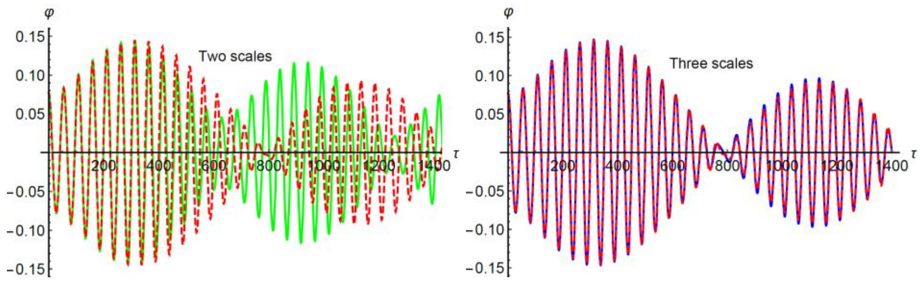

Figure 6. Time histories of swing vibrations; dashed (red) curve – numerical solution, continuous curve – analytical solution

The graphs reported in Fig. 5 and Fig. 6 show that the results obtained using three time scales are much more closer to the numerical solution than these ones for two scales.

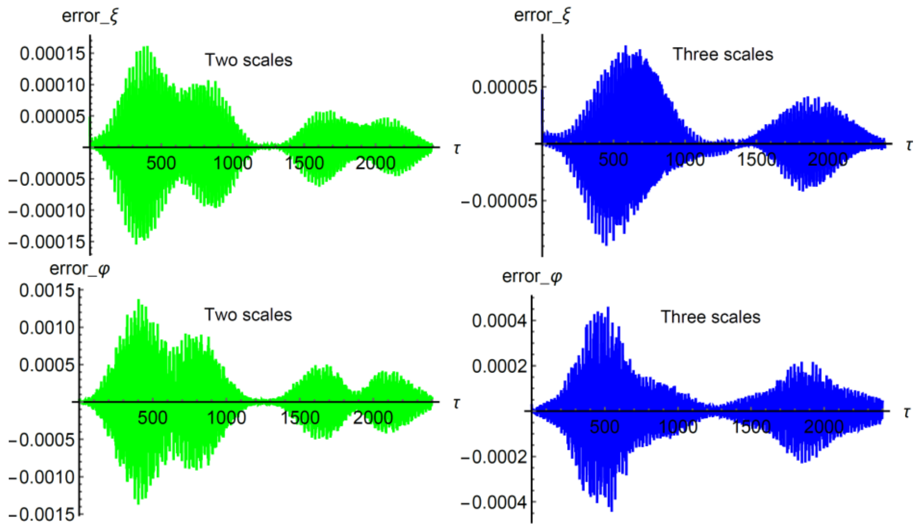

Figure 7. Absolute error of the analytical solutions for resonance vibration

The error as a function of time for the asymptotic solutions obtained using two and three scales is presented in Fig. 7. The graphs show that the error of the solution with three scales is much lower that the error for the solution with two scales.

While studying the resonance very important role play response curves presenting the relationship between the magnitude of vibration amplitude and the frequency of the forcing force. Because this relation applies to steady state vibration, we postulate that derivatives of amplitudes and phases are equal to zero in equations (14) or their counterparts (15) concerning the analysis with three time scales. As a result of this assumption we get system of algebraic equations, the solution of which are amplitudes and phases regarding steady state vibration. Resonance response curves shown in Fig. 8, present dependence of the amplitudes a_1 and a_2 versus the detuning parameter σ_2 (it means physically the frequency dependence of the harmonically changing torque driving the pendulum). The results are obtained for the following fixed parameters: $f_1 = 0.001$, $f_2 = 0.001$, $c_1 = 0.006$, $c_2 = 0.006$, $c_3 = 0.0001$, $\alpha = 0.01$, $w = 0.21$, $\sigma_1 = -0.01$, differ significantly. The graphs obtained using the approach with two time scales suggest there is no coupling between longitudinal and swing vibration in the system. Surprisingly, the numerical results do not confirm this conclusion. Both significant decreasing of the amplitude a_1 as well as the strongly nonlinear compatible with dependence of the amplitude a_2 on the σ_2 are noticed in numerical experiments made for the above values of parameters. Moreover, a high quantitative agreement is observed between the numerical results and results obtained using MS method with three time scales.

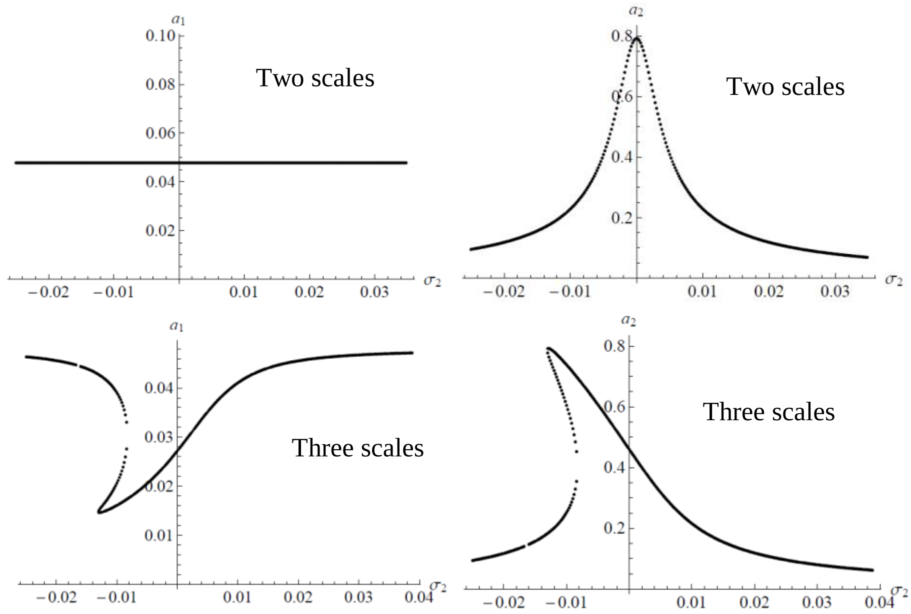

Figure 8. Response curves for two main resonances occurring simultaneously

6. Concluding remarks

The problem of motion of the nonlinear spring pendulum has been solved in two variants of the MS method, i.e. with the help of two and three scales in time domain. Comparison of analytical results with numerical solutions shows that for the case of non-resonant vibration both approaches yield good results. However, the absolute error of solving the equations of motion is significantly lower when three time scales are used. However the situation changes dramatically in the case of resonance. Based on the asymptotic analysis it can be concluded that three time scales are necessary to achieve reasonably small error of the approximate analytical solution.

Numerical solutions of the original motion equations confirm that the spring pendulum behave in double main resonance as a strongly nonlinear and coupled system. The increase of the swing vibration affect the decreasing of the longitudinal vibration what is observable only on the base of approach with using three time scales. Both effects are caused by nonlinear terms that occur only in the slow modulation equations (15). All simplifications proper for the MS method in the approach based on using two time scales turned out too much coarse. In effect, the behaviour of the relatively simple mechanical system is not described sufficiently well.

The study presented in the paper shows that one should be very careful when applying the MS method while studying nonlinear dynamical systems.

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