Composite Structures 180 (2017) 351–359

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

Asymptotic models for transport properties of densely packed, high-contrast fibre composites. Part II: Square lattices of rhombic inclusions and hexagonal lattices of circular inclusions

Igor V. Andrianov [a], Jan Awrejcewicz [b,c,*], Galina A. Starushenko [d]

[a] Institut für Allgemeine Mechanik, RWTH Aachen University, Templergraben 64, D-52056 Aachen, Germany
[b] Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz, Poland
[c] Institute of Vehicles, Warsaw University of Technology, 84 Narbutta St., 02-524 Warsaw, Poland
[d] Dnipropetrovs'k Regional Institute of State Management, of National Academy of State Management at the President of Ukraine, Gogolya 29, UA-49631 Dnipropetrovs'k, Ukraine

ARTICLE INFO

Article history:
Received 30 October 2016
Revised 20 June 2017
Accepted 19 July 2017
Available online 27 July 2017

Keywords:
Composite
Transport property
Homogenization
Asymptotic approach
Asymptotically equivalent function
Nonsmooth argument substitution

ABSTRACT

This paper is a continuation of the investigations reported in the Part I. Based on the asymptotic approach and the lubrication theory, the composite with rhombic inclusions are studied. Models of composites with curvilinear rhombic inclusions and thin interfaces on phase boundaries are constructed with the help of non-smooth argument substitution and asymptotically equivalent functions. The effective conductivity is derived for absolutely conducting and non-conducting rhombic inclusions, taking into account thin interface effects.

Generalisation of the classical Dykhne formula is proposed. Finally, asymptotic solutions are obtained for the effective conductivity of a hexagonal lattice with circular large-size and absolutely conducting/non-conducting inclusions.

1. Introduction

This work extends the investigations presented in paper [4]. Based on the combination of the homogenization [7] asymptotic homogenization [1–6,8,11,12,14], multiscale asymptotic approach [15], lubrication theory [3,4,9,12], non-smooth argument substitution [18–20], and the asymptotically equivalent functions method [1], the following problems have been discussed and solved:

(i) asymptotic analysis of models of a composite with curvilinear rhombic inclusions has been carried out using the lubrication theory;
(ii) composites with absolutely conducting/non-conducting large rhombic inclusions and with a thin interface on a phase boundary are studied;
(iii) conditions of the physical equivalence of composite structures are derived and the corresponding asymptotics are reported for composites of different structures and inclusions;
(iv) proposed approaches and obtained results have been generalised to fit the case of the hexagonal lattice of circular inclusions.

The paper is organised as follows. Models of composites with curvilinear rhombic inclusions are described in Section 2. Composite structures with curvilinear rhombic inclusions and with the interfaces on phase boundaries are analysed in Section 3. Physical equivalence of composite structures is considered in Section 4. Composites with hexagonal lattices of circular inclusions are investigated in Section 5. Finally, Section 6 presents concluding remarks.

The formulas given in Part I are referenced here by adopting dual indices, where the first number (1). indicates Part I of the paper.

2. Models of composites with curvilinear rhombic inclusions

The structure of a composite cell with curvilinear rhombic inclusions is shown in Fig. 1.

* Corresponding author at: Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz, Poland.

E-mail addresses: igor_andrianov@hotmail.com (I.V. Andrianov), awrejcew@p.lodz.pl (J. Awrejcewicz), gs_gala-star@mail.ru (G.A. Starushenko).

http://dx.doi.org/10.1016/j.compstruct.2017.07.068
0263-8223/

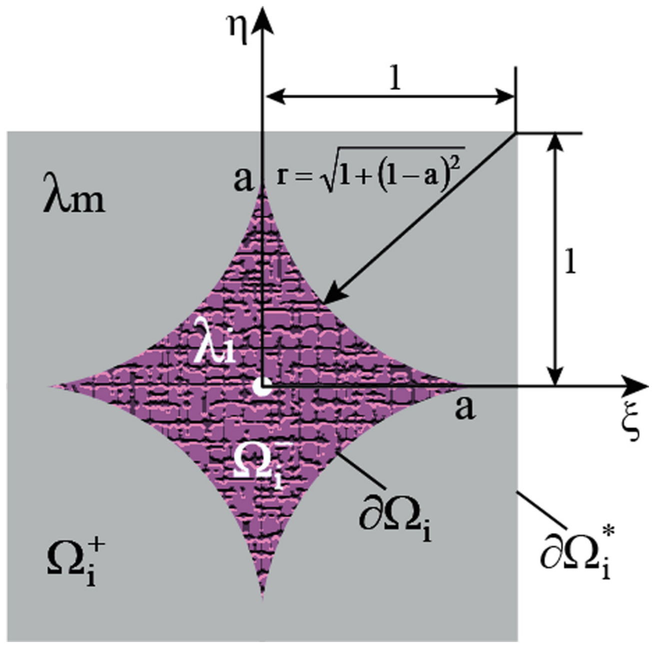


Fig. 1. Characteristic cell of a composite with curvilinear rhombic inclusions.

In order to carry out asymptotic investigation of the composites with inclusions of large sizes ($0 \ll a < 1$) and large conductivity ($\lambda \gg 1$), the lubrication theory is employed [9]. The inclusion size a depends on the coordinate ξ in the following way (see Fig. 2):

$$a(\xi) = 1 - \sqrt{1 + (1-a)^2 - (1-\xi)^2}. \tag{1}$$

Solution of the following local problems

$$\frac{\partial^2 u_{11}^+}{\partial \eta^2} = 0 \text{ in } \Omega_{i1}^+, \frac{\partial^2 u_{11}^-}{\partial \xi^2} + \frac{\partial^2 u_{11}^-}{\partial \eta^2} = 0 \text{ in } \Omega_{i1}^-,$$

$$u_{11}^+ = u_{11}^-, \ \frac{\partial u_1^+}{\partial \eta} - \lambda \frac{\partial u_1^-}{\partial \eta} = (\lambda - 1)\frac{\partial u_0}{\partial \eta} \text{ for } \eta = a,$$

$$u_{11}^- = 0 \text{ for } \eta = 0,$$

$$u_{11}^+ = 0 \text{ for } \eta = 1$$

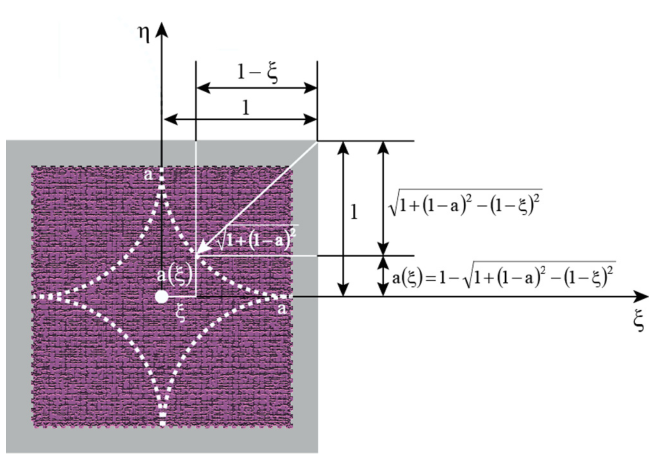


Fig. 2. Approximation of the curvilinear rhombic inclusion in the model based on the lubrication theory.

yields

$$u_{11}^+ = (-1+\eta)B_0^* \frac{\partial u_0}{\partial y}, \tag{2}$$

$$u_{11}^- = D_0^* \eta \frac{\partial u_0}{\partial y}, \tag{3}$$

where

$$B_0^* = -(1-\lambda\Delta), \ D_0^* = -(1-\Delta), \ \Delta = (a + \lambda(1-a))^{-1}. \tag{4}$$

Employing solutions (2)–(4) and relations

$$u_1^\pm = u_{11}^\pm + u_{12}^\pm, \ u_{12}^\pm = u_{11}^\pm \left(\frac{\partial u_0}{\partial y} \to \frac{\partial u_0}{\partial x}\right),$$

the homogenised equation

$$\frac{1}{|\Omega_i|}\left[\iint_{\Omega_i^+}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1^+}{\partial x \partial \xi} + \frac{\partial^2 u_1^+}{\partial y \partial \eta}\right) d\xi d\eta \right.$$
$$\left. + \lambda \iint_{\Omega_i^-}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1^-}{\partial x \partial \xi} + \frac{\partial^2 u_1^-}{\partial y \partial \eta}\right) d\xi d\eta\right] = F$$

can be recast to the following one

$$q\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) = F,$$

where q stands for the homogenised coefficient

$$q = \frac{1}{|\Omega_i|}\left(\iint_{\Omega_i^+}(1+B_0^*)d\xi d\eta + \lambda \iint_{\Omega_i^-}(1+D_0^*)d\xi d\eta\right). \tag{5}$$

In (5), integration is carried over the domain $\Omega_i = \Omega_i^+ \cup \Omega_i^-$. Note that the inclusion size $a = a(\xi)$ is governed by formula (1).

As a result of integration in (5), the following effective conductivity coefficient for $\lambda \gg 1, \ 0 \ll a < 1$, $\left(\lambda > 1 + \frac{1}{\sqrt{1+(1-a)^2}}\right)$ is obtained:

$$q = 1 - a + \frac{\lambda}{\lambda - 1}\left(\frac{\pi}{2} - \frac{1}{\sqrt{\Delta_1 - 1}} \ln \frac{\sqrt{\Delta_1 - 1} + \sqrt{\Delta_1} - 1}{\sqrt{\Delta_1 - 1} - \sqrt{\Delta_1} + 1}\right), \tag{6}$$

where $\Delta_1 = (\lambda - 1)^2(1 + (1-a)^2)$.

In particular cases, formula (6) yields the following asymptotic relations:

a) in the case of absolutely conducting inclusions ($\lambda \to \infty$), the size of which is close to the maximal possible value ($0 \ll a < 1$), we have

$$q = 1 - a + \frac{\pi}{2} - \frac{\ln\left(2\lambda\sqrt{1+(1-a)^2}\right)}{\lambda\sqrt{1+(1-a)^2}}, \tag{7}$$

for $\lambda \to \infty$, $a \to 1$, we get

$$q = \frac{\pi}{2} - \frac{\ln 2\lambda}{\lambda}. \tag{8}$$

b) In the case of inclusions of the maximal possible size $a = 1$ and large conductivity ($\lambda \gg 1$), we have

$$q = \frac{\lambda}{\lambda - 1}\left(\frac{\pi}{2} - \frac{1}{\sqrt{\lambda(\lambda-2)}} \ln \frac{\sqrt{\lambda} + \sqrt{\lambda - 2}}{\sqrt{\lambda} - \sqrt{\lambda - 2}}\right). \tag{9}$$

For the maximal possible size ($a \to 1$) of absolutely conducting inclusions ($\lambda \to \infty$), we obtain

$$q_{rd}^{(\infty)} = \frac{\pi}{2} - \frac{\ln 2\lambda}{\lambda}. \tag{10}$$

For composites with curvilinear large ($0 \ll a < 1$) rhombic inclusions and small conductivity $\lambda \ll 1$ $\left(\text{note that } \lambda < 1 - \frac{1}{1+\sqrt{1+(1-a)^2}}\right)$, the asymptotic representation of the effective conductivity coefficient can be derived with the help of the formula (6) and Keller's theorem [13]:

$$q = \frac{(1-\lambda)\sqrt{\Delta_2 - 1}}{\left((1-a)(1-\lambda) + \frac{\pi}{2}\right)\sqrt{\Delta_2} - \ln\frac{\sqrt{\Delta_2}+\sqrt{\Delta_2-1}}{\sqrt{\Delta_2}-\sqrt{\Delta_2+1}}} \quad \text{for } \lambda \ll 1,\ 0 \ll a < 1, \tag{11}$$

where $\Delta_2 = \left(\frac{1-\lambda}{\lambda}\right)^2(1 + (1-a)^2)$.

Formulas (7)–(10) can be recast to the following ones:

a) For non-conducting inclusions of a large, close to the maximal possible, size

$$q = \frac{2}{2(1-a) + \pi - \frac{2\lambda}{\sqrt{1+(1-a)^2}}\ln\left(\frac{2\sqrt{1+(1-a)^2}}{\lambda}\right)} \quad \text{for } \lambda \to 0, 0 \ll a < 1, \tag{12}$$

and in particular,

$$q = \frac{2}{\pi - 2\lambda\ln\left(\frac{2}{\lambda}\right)} \quad \text{for } \lambda \to 0,\ a \to 1; \tag{13}$$

b) Inclusions of the maximal possible size and low conductivity:

$$q = \frac{2(1-\lambda)}{\pi - \frac{2\lambda}{\sqrt{1-2\lambda}}\ln\frac{1+\sqrt{1-2\lambda}}{1-\sqrt{1-2\lambda}}} \quad \text{for } a = 1,\ \lambda \ll 1; \tag{14}$$

or for almost absolutely non-conducting curvilinear rhombic inclusions:

$$q_{rd}^{(0)} = \frac{2}{\pi - 2\lambda la\left(\frac{2}{\lambda}\right)} \quad \text{for } a = 1,\ \lambda \to 0. \tag{15}$$

3. Composite structures with curvilinear rhombic inclusions and an interface at the phase boundary

We consider a composite with highly conducting curvilinear large rhombic inclusions contacting with each other through a thin interface of the conductivity $\lambda_{\text{int}} = \frac{\lambda_i+\lambda_m}{2}$ (see Fig. 3).

Note that asymptotic formula (9) holds for $\lambda > 2$, and hence it cannot be directly employed for $\tilde{\lambda} \sim 1$. For this purpose, we rewrite (6) to satisfy the condition $0 < \lambda < 1 + \frac{1}{\sqrt{1+(1-a)^2}}$:

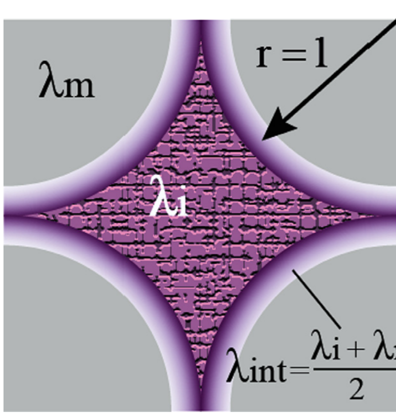


Fig. 3. Composite with highly conducting curvilinear rhombic inclusions contacting through a thin interface.

$$q = 1 - a + \frac{\lambda}{\lambda - 1}\left(\frac{\pi}{2} - \frac{2}{\sqrt{1-\Delta_1}}\arctan\sqrt{\frac{1-\sqrt{\Delta_1}}{1+\sqrt{\Delta_1}}}\right). \tag{16}$$

Due to Keller's theorem, formula (16) yields [13] $\left(\text{for } 1 - \frac{1}{1+\sqrt{1+(1-a)^2}} < \lambda < \infty\right)$:

$$q = \frac{1}{1 - a + \frac{1}{1-\lambda}\left(\frac{\pi}{2} - \frac{2}{\sqrt{1-\Delta_2}}\arctan\sqrt{\frac{1-\sqrt{\Delta_2}}{1+\sqrt{\Delta_2}}}\right)}. \tag{17}$$

For $a = 1$, formula (17) takes the following form

$$q = \frac{\lambda - 1}{\frac{2\lambda}{\sqrt{2\lambda-1}}\arctan\sqrt{2\lambda-1} - \frac{\pi}{2}},$$

or for $\lambda \gg 1$:

$$q = \frac{2(\lambda-1)}{\pi\sqrt{2\lambda}}. \tag{18}$$

Eq. (18), presented in the form of the sawtooth transformation of the argument with an account for conductivities of inclusions and the matrix (1.36), (1.37), is coupled with formula (1.43) representing the effective parameter yielded by the lubrication theory for the absolutely conducting circular inclusion ($\lambda \to \infty$) in the following manner:

$$q_{rd} = \frac{\tilde{\lambda}}{q_r}, \tag{19}$$

where q_{rd} and q_r denote the effective parameters of composites with curvilinear rhombic and circular inclusions, respectively.

Consequently, asymptotic estimations (1.44), (1.48) are also valid for the studied case. Therefore, for a composite with curvilinear rhombic inclusions of the absolute conductivity ($\lambda \to \infty$), contacting each other ($a = 1$) through a thin interface at the boundary between phases, the effective parameter of conductivity takes the below form

$$q_{rd\,\text{int}}^{(\infty)} = \frac{2(\ln\lambda - \ln 2)}{\pi}. \tag{20}$$

A scheme of the asymptotic investigation of the effective coefficient of large $0 \ll a < 1$ inclusions of small conductivity $\lambda \ll 1$, contacting each other through a thin interface is identical as described earlier. Thus, only the final results are reported here:

(i) for $1 - \frac{1}{1+\sqrt{1+(1-a)^2}} < \lambda < \infty$, formula (11) is substituted by (17);
(ii) Keller's theorem [13] yields the generalisation of (17) for $0 < \lambda < 1 + \frac{1}{\sqrt{1+(1-a)^2}}$, which has the form of (16);
(iii) for contacting inclusions ($a = 1$), formula (16) takes the following form (for $\lambda < 2$)

$$q = \frac{\lambda}{\lambda - 1}\left(\frac{\pi}{2} - \frac{2}{\sqrt{\lambda(2-\lambda)}}\arctan\sqrt{2-\lambda}\right); \tag{21}$$

(iv) for $\lambda \ll 1$, Eq. (21) yields

$$q = \frac{\pi}{1-\lambda}\sqrt{\frac{\lambda}{2}}; \tag{22}$$

(v) formula (22), employed in the non-smooth argument substitution, is coupled with formula (1.50) for the effective parameter of heat transfer of a composite with non-conducting circular inclusions ($\lambda \to 0$) by means of relation (19).

In what follows, the asymptotic formulas (1.51), (1.52) are applied to estimate (22) adopting the generalised relations for the conductivity of inclusions and the matrix governed by formulas (1.36), (1.37).

In the case of the composite structure with curvilinear rhombic non-conducting inclusions ($\lambda \to 0$) contacting each other ($a = 1$) through a thin interface, the following effective heat transfer coefficient holds

$$q^{(0)}_{rd\ \mathrm{int}} = \frac{\pi}{2(\ln \lambda^{-1} - \ln 2)}. \quad (23)$$

4. On the physical equivalence of the composite structures

Theoretical investigations of composite materials are based on quantifying the effective conductivity by means of employing relative quantities describing the conductivity of the matrix instead of the absolute ones. However, in a large class of composites of a regular structure, the notion of a matrix and inclusions is rather relative and depends on the decision of the researcher. For instance, the composite shown in Fig. 4 can be treated either as the structure with rhombic inclusions with sharp corners or as a structure composed of rhombic inclusions with rounded corners.

Another important issue regarding this investigation concerns the description of the physical characteristics (i.e. conductivity) of the matrix and inclusions. It is clear that the term "absolutely conducting/nonconducting inclusions", widely employed in mechanics of composites, refer rather to the mathematical abstraction, while real properties of phases can be more rigorously described by the following relations:

$$\lambda_i \gg \lambda_m \quad \Rightarrow \quad \lambda = \frac{\lambda_i}{\lambda_m} \gg 1, \quad (24)$$

and

$$\lambda_i \ll \lambda_m \quad \Rightarrow \quad \lambda = \frac{\lambda_i}{\lambda_m} \ll 1. \quad (25)$$

However, if the normalisation is adopted in relations (24), (25), i.e. if we define λ as relative to the conductivity of inclusions instead of to the one of the matrix (or, equivalently, we change the role of the phases "inclusion-matrix"), then the structure with absolutely conducting inclusions can be described as a structure with non-conducting inclusions, and vice versa.

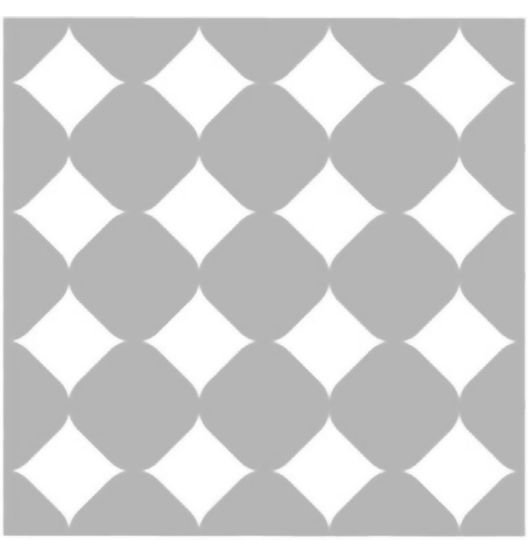


Fig. 4. Composite material of a regular structure.

The so far given observations are valid not only in the limiting cases of the conductivity ($\lambda \to 0$ or $\lambda \to \infty$), but for its arbitrary values.

For example, for the composite shown in Fig. 4, we take:

(i) Phase I: rhombs with rounded corners;
(ii) Phase II: rhombs with sharp corners;
(iii) if $\lambda_I \ll \lambda_{II}$, the composite structure can be equivalently described;
(iv) A composite with absolutely conducting rhombic inclusions with sharp corners (Fig. 5a);
(v) A composite with non-conducting rhombic inclusions with rounded corners (Fig. 5b).

It is evident that the models reported in Fig. 4a,b present the same composite, i.e. they present composites of physically equivalent structures.

Observe that the effective parameters λ_{eff} of physically equivalent composite structures satisfy the following formula

$$\lambda^{(1)}_{eff.}(\lambda) = \lambda \cdot \lambda^{(2)}_{eff.}(\lambda^{-1}). \quad (26)$$

Formula (26) holds for the earlier described effective parameters for the following cases:

a) Contact of absolutely conducting circular (1.49) and non-conducting curvilinear rhombic (23) inclusions through a thin interface;
b) Non-conducting circular (1.53) and absolutely conducting curvilinear rhombic (20) inclusions in the case of presence of a thin interface.

Besides, employing the relation (26) and the earlier established asymptotic representation of the effective parameters of the composites with rounded corners and with curvilinear rhombic inclusions, for the case of their almost absolutely large/small conductivity (see (1.28), (1.33), (10), (15)), one may prove the equivalence of the considered structures by means of adopting generalised asymptotics.

Asymptotic representations of the effective parameters of the composites with different structure with almost absolutely large/small conductivities of the inclusions are reported in Table 1.

Analysis of the asymptotic formulas shown in Table 1 implies that all of them, if properly chosen in pairs, satisfy the so-called "complementary system" condition reported in Ref. [10], which can be treated as a generalisation of Dykhne's formula holding for composites with equally distributed phases. Namely, consider a two-phase composite. If the conductivities of phases I and II are denoted by λ_I and λ_{II}, and their concentrations are denoted by c and $1 - c$, respectively, where $c \neq 1/2$, then either a change $\lambda_I \rightleftarrows \lambda_{II}$ or $c \rightleftarrows 1 - c$ does not violate the effective parameters of the structures $\lambda_{eff.}(c)$ and $\lambda_{eff.}(1 - c)$, which satisfy the following relation

$$\lambda_{eff.}(c) \cdot \lambda_{eff.}(1 - c) = \lambda_I \cdot \lambda_{II}. \quad (27)$$

In what follows, it is shown that Dykhne's relation (31) is satisfied for the asymptotic formulas given in Table 1 ($\lambda_r = \lambda_I$, $\lambda_{rd} = \lambda_{II}$):

(i) for relations (1.28) and (29)

$$\lambda_{eff.}(c) \cdot \lambda_{eff.}(1 - c) = \left(\pi\sqrt{\frac{\lambda_I}{2\lambda_{II}}}\lambda_{II}\right) \cdot \left(\frac{1}{\pi}\sqrt{\frac{2\lambda_I}{\lambda_{II}}}\lambda_{II}\right) = \lambda_I \cdot \lambda_{II};$$

(ii) for relations (27) and (10)

$$\lambda_{eff.}(c) \cdot \lambda_{eff.}(1 - c) = \left(\frac{2\lambda_I}{\pi\lambda_{II}}\lambda_{II}\right) \cdot \left(\frac{\pi}{2}\lambda_{II}\right) = \lambda_I \cdot \lambda_{II};$$

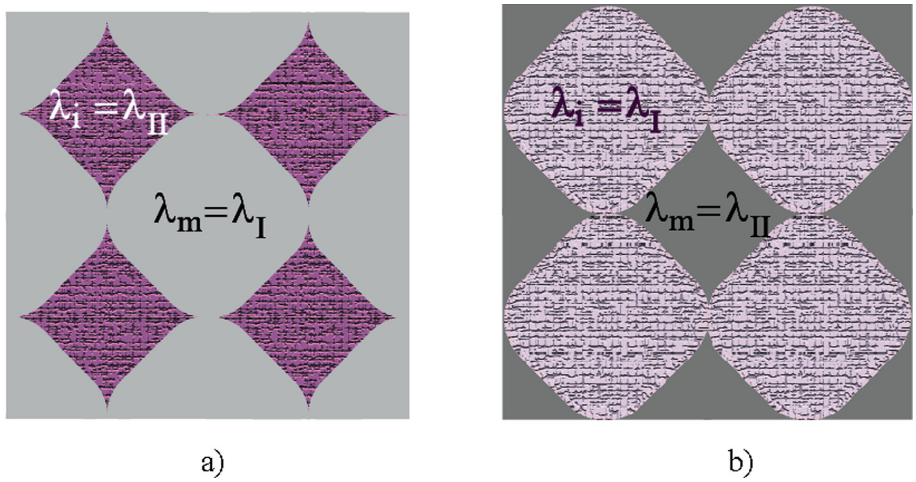


Fig. 5. Physically equivalent composite structures: a) composite with highly conducting rhombic inclusions with sharp corners: $\lambda_m = \lambda_I,\ \lambda_i = \lambda_{II},\ \lambda^{(1)} = \frac{\lambda_i}{\lambda_m} = \frac{\lambda_{II}}{\lambda_I} \gg 1$; b) composite with low conducting rhombic inclusions with rounded corners: $\lambda_m = \lambda_{II},\ \lambda_i = \lambda_I,\ \lambda^{(2)} = \frac{\lambda_i}{\lambda_m} = \frac{\lambda_I}{\lambda_{II}} \ll 1$.

(iii) for relations (1.49) and (20)

$$\lambda_{eff.}(c)\cdot\lambda_{eff.}(1-c) = \left(\frac{\pi\lambda_I}{2\lambda_{II}\left(\ln\frac{\lambda_1}{\lambda_{II}} - \ln 2\right)}\lambda_{II}\right)\cdot\left(\frac{2}{\pi}\left(\ln\frac{\lambda_I}{\lambda_{II}} - \ln 2\right)\lambda_{II}\right) = \lambda_I\cdot\lambda_{II};$$

(iv) for relations (1.33) and (30)

$$\lambda_{eff.}(c)\cdot\lambda_{eff.}(1-c) = \left(\frac{1}{\pi}\sqrt{\frac{2\lambda_I}{\lambda_{II}}}\lambda_{II}\right)\cdot\left(\pi\sqrt{\frac{\lambda_I}{2\lambda_{II}}}\lambda_{II}\right) = \lambda_I\cdot\lambda_{II};$$

(v) for relations (28) and (15)

$$\lambda_{eff.}(c)\cdot\lambda_{eff.}(1-c) = \left(\frac{\pi\lambda_I}{2\lambda_{II}}\lambda_{II}\right)\cdot\left(\frac{2}{\pi}\lambda_{II}\right) = \lambda_I\cdot\lambda_{II};$$

(vi) for relations (1.53) and (23)

$$\lambda_{eff.}(c)\cdot\lambda_{eff.}(1-c) = \left(\frac{2}{\pi}\frac{\lambda_I}{\lambda_{II}}\left(\ln\frac{\lambda_{II}}{\lambda_I} - \ln 2\right)\lambda_{II}\right)\cdot\left(\frac{\pi}{2}\frac{1}{\left(\ln\frac{\lambda_{II}}{\lambda_I} - \ln 2\right)}\lambda_{II}\right) = \lambda_I\cdot\lambda_{II}.$$

5. Hexagonal lattice of circular inclusions

In Ref. [12], the effective conductivity of the composite of a hexagonal structure with large ($a \to 1$) circular inclusions of large conductivity ($\lambda \to \infty$) are derived with the help of the lubrication theory.

In the limiting case of the inclusions size $a = 1$ (Fig. 6), the main term of the obtained asymptotic follows

$$q = \frac{\pi\sqrt{3}}{2} f_1\left(\tilde{\lambda}\right), \tag{28}$$

where $f_1(\tilde{\lambda}) = 2\frac{\tilde{\lambda}-1}{\tilde{\lambda}+1}\Big/\sqrt{1-\frac{\tilde{\lambda}-1}{\tilde{\lambda}+1}}$.

In further investigations, a composite of a hexagonal structure with circular inclusions of high conductivity $\lambda \gg 1$ and large size $a \to 1$, having a thin interface λ_{int} at the phase boundary (Fig. 7), is considered.

The non-smooth argument substitution (Appendix A [4]) is applied with the assumption that conductivities of inclusions and the matrix are presented in the form of analytical relations (1.36), (1.37).

A magnitude of $f_1\left(\tilde{\lambda}\right)$ can be estimated using (31) for $\tilde{\lambda} \to 1$, i.e.

$$f_1\left(\tilde{\lambda}\right) \sim f_0\left(\tilde{\lambda}\right) \text{ for } \tilde{\lambda} \to 1, \tag{29}$$

where

$$f_0(\tilde{\lambda}) = \left(2\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}-1\right)^{\frac{3}{2}}\left(\frac{\tilde{\lambda}}{2}\right)^{\frac{1}{4}}\right)\Bigg/\left(\ln\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}\right)\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}\right)^{\frac{1}{2}}\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}+1\right)^{\frac{1}{4}}\right).$$

As in the case of a square lattice, for the hexagonal lattice, the approximating function $f_0\left(\tilde{\lambda}\right)$ (33) has the following properties:

(i) It correctly describes the function $f_1(\tilde{\lambda})$ in the vicinity of the point $\tilde{\lambda} = 1$; series of functions $f_1(\tilde{\lambda})$, $f_0(\tilde{\lambda})$ coincide with accuracy up to the terms of order $(\tilde{\lambda}-1)^4$ (see the Appendix);
(ii) Its shape is analogous to the shape of the symmetric saw-tooth function (Fig. 8);
(iii) It allows for a correct transition from the sawtooth argument substitution to the original smooth parameter. Taking into account relations (1.46), (1.47) for $\lambda \to \infty$, we have

$$f_0\left(\tilde{\lambda}\right) \sim \frac{\lambda}{\ln\frac{\lambda}{2}}. \tag{30}$$

Therefore, the formula for the effective parameter of the composite of the hexagonal structure with large ($a = 1$), absolutely conducting inclusions ($\lambda \to \infty$), and with a thin interface at the boundary between phases, after taking into account formulas (32)–(34), has the form

$$q_{hex}^{(\infty)} = \frac{\pi\lambda\sqrt{3}}{2(\ln\lambda - \ln 2)}. \tag{31}$$

The main part of the asymptotic formula (35) coincides with the known result [16,17]

$$q_{hex}^{(\infty)} = \frac{\pi\lambda\sqrt{3}}{2\ln\lambda}.$$

Using the formula for the effective parameter, reported in Ref. [12], for $\lambda \ll 1$, the main term of the asymptotic series is

Table 1
Asymptotic representations of the effective parameters of composites of different structures.

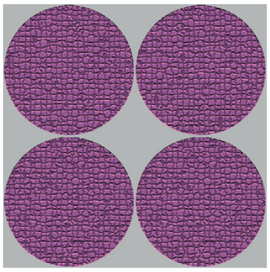
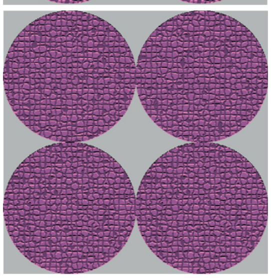
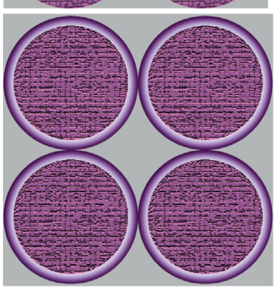
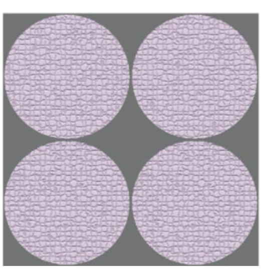
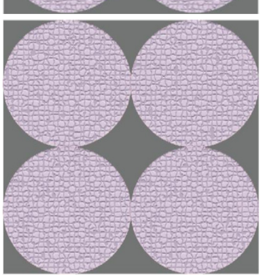
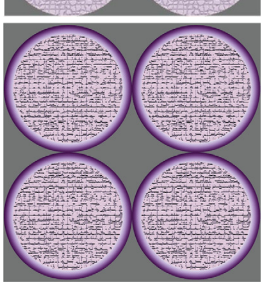
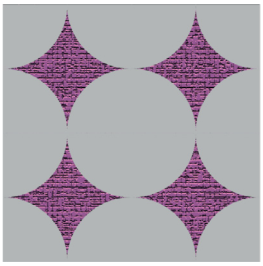
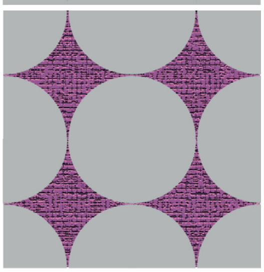
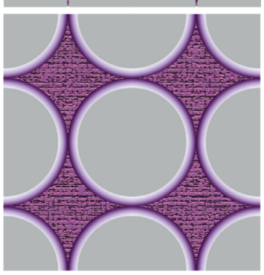
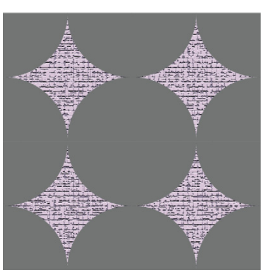
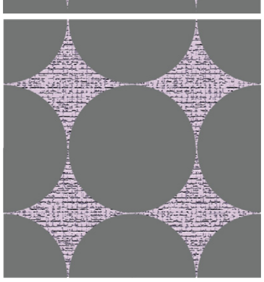
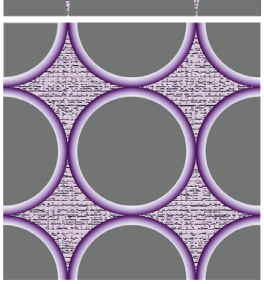
Composite structure form	Asymptotic representations of the effective parameters
1. Absolutely conducting circular inclusions $\lambda \to \infty$	
	1.1. Contact through the matrix material (curvilinear rhombs) $\lambda = \frac{\lambda_r}{\lambda_{rd}}$ $q = \pi\sqrt{\frac{\lambda}{2}}$ Formula (1.28)
	1.2. Contact through the inclusion material $\lambda = \frac{\lambda_r}{\lambda_{rd}}$ $q = \frac{2\lambda}{\pi}$ Formula (27) (yielded by (26) and (15))
	1.3. Contact with the interface $\lambda = \frac{\lambda_r}{\lambda_{rd}}$ $q = \frac{\pi\lambda}{2(\ln\lambda - \ln 2)}$ Formula (1.49)
2. Non-conducting circular inclusions $\lambda \to 0$	
	2.1. Contact through the matrix material (curvilinear rhombs) $\lambda = \frac{\lambda_r}{\lambda_{rd}}$ $q = \frac{\sqrt{2\lambda}}{\pi}$ Formula (1.33)
	2.2. Contact through the inclusion material $\lambda = \frac{\lambda_r}{\lambda_{rd}}$ $q = \frac{\pi\lambda}{2}$ Formula (28) (yielded by (26) and (10))
	2.3. Contact with the interface $\lambda = \frac{\lambda_r}{\lambda_{rd}}$ $q = \frac{2\lambda(\ln\lambda^{-1} - \ln 2)}{\pi}$ Formula (1.53)

Table 1 (*continued*)

Composite structure form	Asymptotic representations of the effective parameters
3. Curvilinear absolutely conducting rhombic inclusions $\lambda \to \infty$	
	3.1. Contact through the matrix material $\lambda = \frac{\lambda_{rd}}{\lambda_r}$ $q = \frac{\pi}{2}$ Formula (10)
	3.2. Contact through the material of inclusions (curvilinear rhombs) $\lambda = \frac{\lambda_{rd}}{\lambda_r}$ $q = \frac{\sqrt{2\lambda}}{\pi}$ Formula (29) (yielded by (26) and (1.33))
	3.3. Contact with the interface $\lambda = \frac{\lambda_{rd}}{\lambda_r}$ $q = \frac{2(\ln\lambda - \ln 2)}{\pi}$ Formula (20)
4. Curvilinear non-conducting rhombic inclusions $\lambda \to 0$	
	4.1. Contact through the matrix material $\lambda = \frac{\lambda_{rd}}{\lambda_r}$ $q = \frac{2}{\pi}$ Formula (15)
	4.2. Contact through the material of inclusions (curvilinear rhombs) $\lambda = \frac{\lambda_{rd}}{\lambda_r}$ $q = \pi\sqrt{\frac{\lambda}{2}}$ Formula (30) (yielded by (26) and (1.28))
	4.3. Contact with the interface $\lambda = \frac{\lambda_{rd}}{\lambda_r}$ $q = \frac{\pi}{2(\ln\lambda^{-1} - \ln 2)}$ Formula (23)

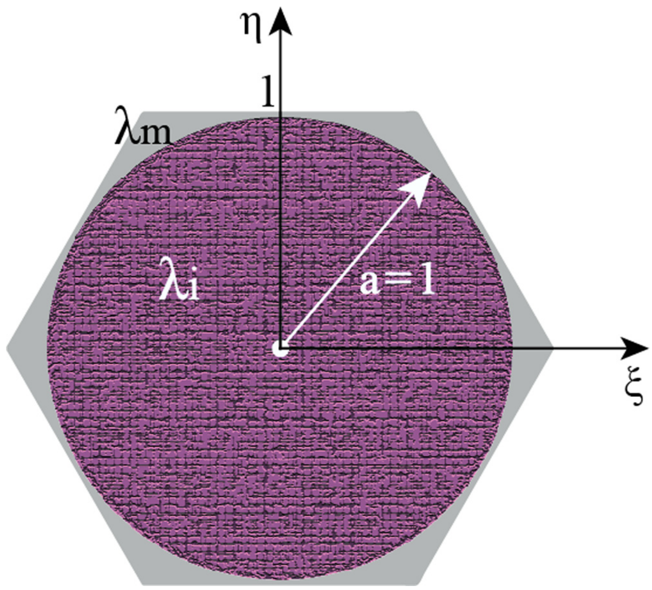


Fig. 6. Characteristic cell of a composite of a hexagonal structure.

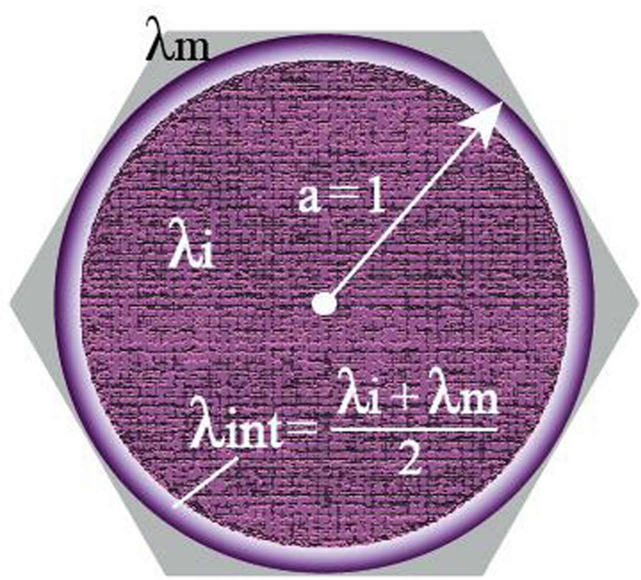


Fig. 7. Composite of a hexagonal structure with a thin interface an the boundary between phases.

$$q = \frac{f_2\left(\tilde{\lambda}\right)}{2\pi\sqrt{3}}, \tag{32}$$

where: $f_2(\tilde{\lambda}) = f_1(\tilde{\lambda}^{-1})$.

Presenting formula (36) in terms of the non-smooth argument and estimating relation $f_2(\tilde{\lambda})$ of the asymptotically equivalent function, one gets

$$f_2\left(\tilde{\lambda}\right) = f_1\left(\tilde{\lambda}^{-1}\right) \sim f_0\left(\tilde{\lambda}^{-1}\right) \text{ for } \tilde{\lambda} \to 1. \tag{33}$$

Expansions of the functions of both parts of the equations (37) into series coincide with accuracy up to the terms $(1-\tilde{\lambda})^4$ (see the Appendix).

Owing to relations (1.46), (1.47) for $\lambda \to 0$, one obtains

$$f_0\left(\tilde{\lambda}^{-1}\right) \sim \frac{1}{\lambda \ln \frac{1}{2\lambda}}, \tag{34}$$

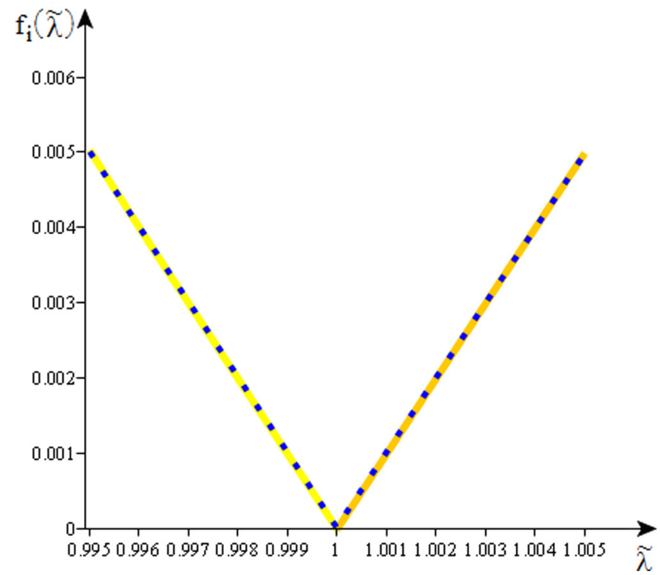


Fig. 8a. Graphs of the asymptotically equivalent (for $\tilde{\lambda} \to 1$) function. $f_1(\tilde{\lambda})$ $f_2(\tilde{\lambda}) = f_1(\tilde{\lambda}^1)$ $f_0(\tilde{\lambda})$.

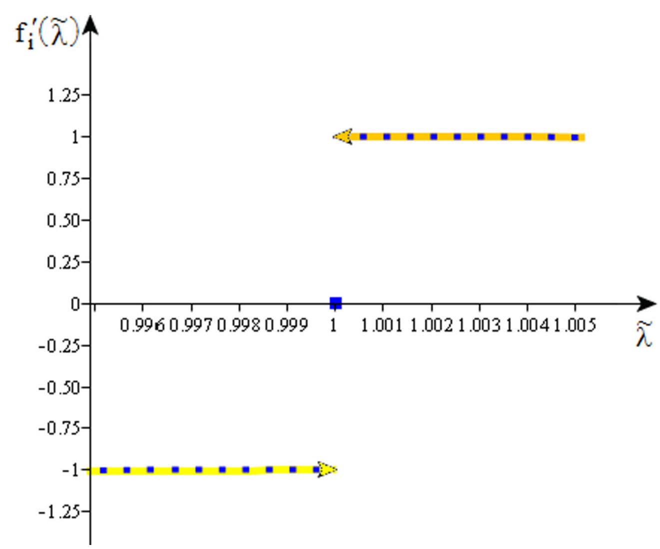


Fig. 8b. Graphs of the asymptotically equivalent derivatives of the functions. $f_1'(\tilde{\lambda})$ $f_2'(\tilde{\lambda})$ $f_0'(\tilde{\lambda})$.

and therefore the relation for the effective parameter (36) can be recast to the following form

$$q_{hex}^{(0)} = \frac{2\lambda(\ln\lambda^{-1} - \ln 2)}{\pi\sqrt{3}}. \tag{35}$$

Relations (35), (39), obtained for absolutely conducting/non-conducting inclusions of a maximally large size $a = 1$ with a thin interface at the boundary between phases, satisfy Keller's theorem [13]:

$$q_{hex}^{(\infty)}(\lambda) = \frac{1}{q_{hex}^{(0)}(\lambda^{-1})}.$$

6. Concluding remarks

Based on the lubrication theory, the asymptotic representation of the effective conductivity of fibre composites with curvilinear rhombic inclusions have been derived for both absolutely conducting or non-conducting inclusions.

Employing the non-smooth argument substitution for description of local and discretely varying properties of the inhomogeneous structures and using the asymptotically equivalent functions, the model of the composite with curvilinear rhombic inclusions and a thin interface on the phase boundary has been constructed.

The conditions of the matrix-inclusions contact for absolutely conducting/non-conducting circular and curvilinear rhombic inclusions with an account for the thin interface effects have been obtained.

The physical equivalence of composite structures has been discussed with respect to composites effective properties and possibility of generalisation of Dykhne's formula. In addition, asymptotic solutions to the effective conductivity of the hexagonal lattice of circular inclusions of a large size and absolutely conducting/non-conducting properties are obtained.

Appendix A.

Series of the asymptotically equivalent functions

$$f_1(\tilde{\lambda}) = \frac{2\,\frac{\tilde{\lambda}-1}{\tilde{\lambda}+1}}{\sqrt{1-\frac{\tilde{\lambda}-1}{\tilde{\lambda}+1}}} = \tilde{\lambda} - 1 - \frac{1}{4}(\tilde{\lambda}-1)^2 + \frac{3}{32}(\tilde{\lambda}-1)^3 - \frac{5}{128}(\tilde{\lambda}-1)^4 + \frac{35}{2048}(\tilde{\lambda}-1)^5 + O(\tilde{\lambda}-1)^6; \tag{A1}$$

$$f_0(\tilde{\lambda}) = \frac{2\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}-1\right)^{\frac{3}{2}}\left(\frac{\tilde{\lambda}}{2}\right)^{\frac{1}{4}}}{\ln\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}\right)\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}\right)^{\frac{1}{2}}\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}+1\right)^{\frac{1}{4}}} = \tilde{\lambda} - 1 - \frac{1}{4}(\tilde{\lambda}-1)^2 + \frac{3}{32}(\tilde{\lambda}-1)^3 - \frac{5}{128}(\tilde{\lambda}-1)^4 + \frac{169}{6144}(\tilde{\lambda}-1)^5 + O(\tilde{\lambda}-1)^6; \tag{A2}$$

$$f_1(\tilde{\lambda}^{-1}) = \frac{2\left(\frac{1-\tilde{\lambda}}{1+\tilde{\lambda}}\right)}{\sqrt{1-\frac{1-\tilde{\lambda}}{1+\tilde{\lambda}}}} = 1 - \tilde{\lambda} + \frac{3}{4}(1-\tilde{\lambda})^2 + \frac{19}{32}(1-\tilde{\lambda})^3 + \frac{63}{128}(1-\tilde{\lambda})^4 + \frac{867}{2048}(1-\tilde{\lambda})^5 + O(1-\tilde{\lambda})^6; \tag{A3}$$

$$f_0(\tilde{\lambda}^{-1}) = \frac{2\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}-1\right)^{\frac{3}{2}}}{\ln\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}\right)\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}\right)^{\frac{1}{2}}\left(\frac{\tilde{\lambda}}{2}+\frac{1}{2\tilde{\lambda}}+1\right)^{\frac{1}{4}}(2\tilde{\lambda})^{\frac{1}{4}}} = 1 - \tilde{\lambda} + \frac{3}{4}(1-\tilde{\lambda})^2 + \frac{19}{32}(1-\tilde{\lambda})^3 + \frac{63}{128}(1-\tilde{\lambda})^4 + \frac{2665}{6144}(1-\tilde{\lambda})^5 + O(1-\tilde{\lambda})^6. \tag{A4}$$

References

[1] Andrianov IV, Awrejcewicz J, Danishevs'kyy VV, Ivankov AO. Asymptotic methods in the theory of plates with mixed boundary conditions. Chichester: Wiley; 2014.
[2] Andrianov IV, Awrejcewicz J, Starushenko GA. Application of an improved three-phase model to calculate effective characteristics for a composite with cylindrical inclusions. Lat Amer J Sol Struct 2013;10(1):197–222.
[3] Andrianov IV, Starushenko GA, Danishevskii VV. Asymptotic determination of the effective thermal conductivity of a pile field. Soil Mech Found Eng 1999;36 (1):31–6.
[4] Andrianov IV, Awrejcewicz J, Starushenko GA. Asymptotic models for transport properties of the high contrast, highly packed fibre composites. Part I: Square lattice of circular inclusions. Compos Struct (in press).
[5] Bakhvalov NS, Panasenko GP. Averaging Processes in Periodic Media. Mathematical Problems in Mechanics of Composite Materials. Dordrecht: Kluwer; 1989.
[6] Bensoussan A, Lions J-L, Papanicolaou G. Asymptotic analysis for periodic structures. Amsterdam: North-Holland Publishing Company; 1978.
[7] Buannic N, Cartraud P, Quesnel T. Homogenization of corrugated core sandwich panels. Compos Struct 2003;59(3):299–312.
[8] Challagulla KS, Georgiades AV, Kalamkarov AL. Asymptotic homogenization modeling of thin network structures. Compos Struct 2007;79(3):432–44.
[9] Christensen RM. Mechanics of Composite Materials. Mineola, New York: Dover Publications; 2005.
[10] Dykhne AM. Conductivity of a two-dimensional two-phase system. Sov Phys JETP 1971;32(1):63–5.
[11] Kalamkarov AL, Andrianov IV, Danishevs'kyy VV. Asymptotic homogenization of composite materials and structures. Appl Mech Rev 2009;62(3):030802-1–030802-20
[12] Kalamkarov AL, Andrianov IV, Pacheco PMCL, Savi MA, Starushenko GA. Asymptotic analysis of fiber-reinforced composites of hexagonal structure. J Multiscale Model 2016;7(3). 1650006 (32 pages).
[13] Keller JB. A theorem on the conductivity of a composite medium. J Math Phys 1964;5(4):548–9.
[14] Milton GW. The theory of composites. Cambridge: Cambridge University Press; 2002.
[15] Nayfeh AH. Perturbation Methods. New York: Wiley; 2000.
[16] O'Brien RW. Ph. D. thesis. University of Cambridge; 1977.
[17] Perrins WT, McKenzie DR, McPhedran RC. Transport properties of regular array of cylinders. Proc Roy Soc London A 1979;369(1737):207–25.
[18] Pilipchuk VN. Nonlinear dynamics: between linear and impact limits. Berlin: Springer; 2010.
[19] Pilipchuk VN, Starushenko GA. A version of non-smooth transformations for one-dimensional elastic systems with a periodic structure. J Appl Math Mech 1997;61(2):265–74.
[20] Starushenko G, Krulik N, Tokarzewski S. Employment of non-symmetrical saw-tooth argument transformation method in the elasticity theory for layered composites. Int J Heat Mass Trans 2002;45(14):3055–60.