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Mathematical model of a three-layer micro- and nano-beams based on the hypotheses of the Grigolyuk–Chulkov and the modified couple stress theory

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A B S T R A C T

The mathematical model of three-layered beams developed based on the hypothesis of the Grigolyuk–Chulkov and the modified couple stress theory and the size depended equations governing the layers motions on the micro- and nano-scales is constructed. The Hamilton's principle yields the novel equations of motion as well as the boundary/initial conditions regarding beams displacement. The latter ones clearly exhibit the size dependent dynamics of the studied micro- and nano-beams, and the introduced theory overlaps with the classical beam equations for large enough layer thickness. In particular, a three-layer beam with the micro-layer thickness has been investigated with respect to the classical theory of Grigolyuk–Chulkov. The derived boundary problem is of sixth order and can be solved analytically in the case of statics. The carried out numerical experiments allowed to detect and explain size dependent effects exhibited by the micro-beams. The beam deflections and stress yielded by the employed couple stress model are less than those predicted by the classical Grigolyuk–Chulkov theory, while the estimated eigen frequencies are higher, respectively. It has been shown that the proposed model can be reduced to the classical three-layer Grigolyuk–Chulkov beam through increase of the layers thickness, which validates our approach.

1. Introduction

It is well known that the three-layer structures have found applications in the mechanical and civil engineering, aviation, ship design, as well as in the airplanes and cosmic industries. In recent years they are employed in the design of numerous micro- and nano-devices, including the micro- and nano-sensors as well as the electromagnetic sensors.

A general theory of static/dynamic behavior of the three-layer structural construction with respect to their size dependent behavior subjected to various external loading has been proposed by Grigolyuk and Chulkov (1973).

Remarkably, three-layer constructions do not only imply the structural non-homogeneity regarding the beam thickness, but they require inclusions of the middle layer due to occurrence of the transversal shear and compression, as well as coupling effects of the layers should be taken into account. Grigolyuk and Chulkov (1973) introduced a hypothesis of the linear distribution of the tangent displacements with respect to both weight of the beam package and condition of the lack of compression in the structural package while constructing their theory of three-layered structures. In contrary to the Bernoulli-Euler hypothesis, in the Grigolyuk–Chulkov model a perpendicular line to the initial surface becomes not perpendicular to the deformed surface, since it rotates on amount of a certain angle due to transversal shear generated by the middle layer. On the other hand, the external layers fit with the Bernoulli–Euler hypotheses, whereas the internal layer follows the Timoshenko hypothesis. It should be emphasized that the mentioned theory generalizes the classical beam theory. Grigolyuk–Chulkov, using the hypothesis of straight cross-sections for the internal layer, constructed the equilibrium equations and studied the stability and vibrations of the carrying load by the

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three layer beam. Namely, the external layers have been made from materials of infinite stiffness against shear and transversal compression, whereas the internal layer had the infinite stiffness against the transversal compression. The static and dynamic behavior of multi-layer beams based on the Bernoulli–Euler and Timoshenko hypotheses have been employed for the whole structural package in references (Zenkour, 1999; Miller and Shenoy, 2000; Chong et al., 2001; Yang et al., 2002; Lam et al., 2003; Park and Gao, 2006; Sun et al., 2007; Awrejcewicz et al., 2008; Krysko et al., 2012). For multilayer anisotropic plates and shells, Andreev and Nemirovsky developed a general theory based on a broken line (Andreev and Nemirovskii, 2001). A regular and chaotic contact/no-contact nonlinear dynamics of the multi-layer structure composed of one plate and three Euler–Bernoulli beams coupled only by boundary condition have been studied in the work of Awrejcewicz et al. (2016).

The functional nano-materials used as plies and layers put on the surfaces of the rigid bodies essentially improve the exploitation characteristics of the industrial products. If the multi-layer beams with the plane thin external layers do not have the required loading ability, the latter can be increased/improved via application of the reinforced external layering, i.e. employing the layers/plies/films having large Young moduli in the form of nano-layers and the micro-layers made from carbon.

On the basis of the standard computations regarding the material resistance, the thickness of the layers of a micro-beam should achieve tenth of microns in order to satisfy the industrial requirements. However, as the experimental investigations show, the mechanical properties of the micro- and nano-size elements depend on their sizes. This is why a novel name has been introduced emphasizing the size dependent effect and characterizing the change of the properties of the structures composed of elements of the size of microns and nano-meters. Different features of the size-depended effects exhibited by the micro- and nano-elements are widely described in the existing literature, and among them the gradient effects play a significant role.

The size dependent behaviour of elastic elements have been observed experimentally while bending and turning of the micro-beams (Fleck et al., 1994; Stolken and Evans, 1998; Chong et al., 2001; Lam et al., 2003;). In reference Fleck et al. (1994), it has been reported that the torsion stiffness of a copper made wires increases simultaneously decreasing their diameter from 170 down to 12 µm (observe that the size decrease should imply zero stiffness of a wire). In reference Stolken and Evans (1998) the increase in the bending stiffness of a nickel foil corresponding to its height yielded the decrease in its thickness from 50 to 12.5 µm. The micro- and nano-oriented investigations (Miller and Shenoy, 2000; Yang et al., 2002) show that the stiffness of the pure and poly-crystal metallic materials can be doubled while decreasing the thickness from 10 to 1 µm. In order to estimate the material resistance and keep the optimal design of the mentioned products, the key role is played by the reliable and highly accurate analysis of the stress-strain states of the investigated micro- and nano-mechanical objects.

Importance of the size dependent behaviour being an inherent property of materials has been presented experimentally by McFarland and Colton (2005) and Kong et al. (2008). It has been demonstrated that in micron/sub-micron scale regions, i.e. when characteristic diameter of thickness is close to the internal material length scale parameter, the classical continuum mechanical theories cannot be employed to study the beam static/dynamic behaviour.

In references Chong et al. (2001) and Sun et al. (2007) investigations regarding explanation of the so far mentioned effects on a basis of the strain gradient theory for the problems of bending and torsions have been carried out.

More recently Srinivasa and Reddy (2013) proposed a systematic treatment of higher gradient theories in a nonlinear context.

The size dependent models of the Bernoulli–Euler beam have been studied in reference Park and Gao (2006) and Awrejcewicz et al. (2012), whereas the Timoshenko beam has been analyzed in Park and Gao (2008), and the Reddy–Levinson beam has been studied in references Ma et al. (2010) and Reddy (2011) using the modified couple stress theory proposed in reference Yang et al. (2002) (the latter mainly employs only size depended material length).

The origin and development of the couple stress elasticity theory can be found in the seminal works of Koiter (1964), Toupin (1962), Mindlin and Tiersten (1962) and Mindlin (1963).

As it has been already mentioned, a modified couple stress theory has been developed by Yang et al. (2002), where only one scale parameter of material length as well as symmetric couple stress tensor have been employed. This concept has been expanded in reference Park and Gao (2008), where the variational formalism associated with this theory has been introduced.

Park and Gao (2006) utilized the concepts of the modified couple stress theory (MCST) to develop a new model for the bending of a Bernoulli–Euler beam. It contains an internal material length scale parameter and in contrary to the classical Bernoulli–Euler beam model, it captures the size scale effects. Considering a cantilever beam, it has been shown, that the bending rigidity estimated through their model is larger than that yielded by the classical model.

The seminal work of Eringen (1983) serves as a source to develop microstructure-dependent non-local theories of beams being based on the Hamilton's principle and the non-local constitutive relations including the Bernoulli–Euler, Timoshenko, Reddy/Levinson models (Yang et al., 2002; Peddieson et al., 2003; Polizzotto, 2003; Wang, 2005; Wang et al., 2006). Ma et al. (2008) constructed a microstructure-dependent Timoshenko beam model based on a modified couple stress bending and axial principle, where both bending and axial deformations, as well as the Poisson effect, have been taken into account.

In reference Arbind and Reddy (2013) solving equations for the models of functionally graded Bernoulli–Euler, Timoshenko and Reddy–Levinson beams with respect to their thickness on the basis of the modified couple stress theory and taking into account the size dependent state equations have been derived and analyzed.

Asghari et al. (2010) utilized the MCST to study nonlinear behavior of Timoshenko hinged-hinged beam including the mid-plane stretching. In the case of static bending the non-linear size depended phenomena has been studied numerically, whereas a solution to the free-vibrations problem has been solved analytically. The latter work has been extended by Ke and Wang (2011) and Ke et al. (2011) to study non-linear beam vibrations with an emphasis put to the stability estimation while including the axial displacement in their study.

Reddy (2011) extended theories related to nonlinear Euler–Bernoulli and Timoshenko beam taking into account through-thickness power-law vibration of a two-constituent material and moderate rotation of transverse normal through the von Kármán nonlinear strain. The proposed model, based on a modified couple stress theory, power-law variation of the material, and the von Kármán geometric nonlinearity uses only one material length scale parameter and captures the size effect in a functionally graded material.

Santos and Reddy (2012) presented comparison among classical elasticity, nonlocal elasticity, and modified couple stress theories for free vibration analysis of the Timoshenko beams, where the rotary inertia and nonlocal parameter have been taken into account. Convergence of the theories has been demonstrated taking into account increase of the beam global dimension.

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Reddy and Arbind (2012) reformulated the classical beam theories, i.e. the Bernoulli–Euler and Timoshenko, using a modified couple stress theory and employing thickness power-law variation of a functionally graded material. The algebraic relationships have been derived for the beam deflections, slopes and stress resultants. They have been validated through examples of straight beams with simply supported and clamped boundary conditions.

The nonlinear resonant dynamics of a microscale beam based on the modified couple stress theory has been analyzed numerically in reference Ghayesh et al. (2013). First Hamilton's principle has been employed to derive a PDE governing motion using the modified couple stress theory, and then the Galerkin technique has been utilized to obtain a set of coupled nonlinear ODEs. The effect of different system parameters on the resonant dynamics system response has been studied.

Kahrobaiyan et al. (2014) proposed a new comprehensive Timoshenko beam element based on the modified couple stress theory. Then the mass and stiffness matrices have been computed using energy approach and Hamilton's principle. In particular, the static deflection of a short microbeam and pull-in voltage of an electrostatically actuated micro cantilever made of silicon are estimated using this new beam element.

A microstructure-dependent nonlinear third-order beam theory which accounts for through-thickness power-law variation of a two-constituent material has been developed by Arbind et al. (2014) based on modified couple stress theory the influence of the material length has been investigated.

Thai and Vo (2013) studied static bending, buckling and free vibration behaviors of size dependent functionally graded sandwich microbeams based on the modified couple stress theory and Timoshenko beam theory. Two kinds of the sandwich beams have been analyzed: functionally graded skins and homogeneous core and functionally graded core and homogeneous skins. It has been shown that inclusion of the size effect resulted in an increase in the beam stiffness.

In reference Alashti and Abolghasemi (2014) a size-dependent formulation for the Bernoulli–Euler beam based on the couple stress theory has been given. It has been shown that the natural frequencies obtained using the utilized couple stress model are higher than these predicted by the classical theory.

A new modified couple stress theory containing three material length scale parameters has been developed by Chen and Li (2014) for anisotropic elasticity and microscale laminated Kirchhoff plate model. The principle of minimum total potential energy has been employed, and the curvature tensor has been taken as asymmetric, whereas the couple stress moment tensor has been used as symmetric. The carried out numerical simulation have validated the proposed Kirchhoff plate model, which captures the scale effect of the microstructures.

Mohammad-Abadi and Daneshmehr (2015) carried out the vibration analysis of the composite laminated beams in order of micron based on the modified couple stress theory. In particular, the Euler–Bernoulli, Timoshenko and Reddy beam model have been studied with respect to the differences in estimation of shear deformation. The governing equations have been solved using three boundary conditions and four types of lamination.

The transverse vibration of rotary tapered microbeam has been analyzed by Shafiei et al. (2015) using a modified couple stress theory and Euler–Bernoulli beam model. In particular, the effect of the small-scale parameter, beam length, rate of cross-section change, hub radius and non-dimensional angular velocity on the microbeam vibration process have been illustrated and discussed.

In reference Dehrouyeh-Semnani et al. (2015) the size dependent model of a three-layer beam employing the modified version of the couple stress theory and length parameter influence associated with all layers and its impact on the damping characteristics of the micro-beam vibrations have been studied.

Rajneesh (2016) solved the problem of thermoelastic beam using the modified couple stress theory. Both governing equation for the modified couple stress theory and heat conduction equation for coupled thermoelasticity have been employed to study the vibrations in homogenous isotropic thin beam by applying the Euler–Bernoulli theory. The lateral deflection, thermal moment, axial stress average due to normal heat flux have been derived and studied numerically.

In reference Khorshidi et al. (2016) shear deformable functionally graded nano-beams in post-buckling based on modified couple stress theory have been studied. The governing equations and boundary conditions are yielded by the principle of minimum potential energy. Exact and generalized differential quadrature solutions for the static postbuckling response of the functionally graded nanobeams under different boundary conditions have been derived. Effects of length-scale parameter, material gradient, length-to-thickness ratio and Poisson's ratios have been illustrated and analyzed, among other.

In all mentioned works, in contrary to the Grigolyuk–Chulkov hypothesis, only one hypothesis, i.e. either the Bernoulli–Euler or Timoshenko assumptions (for all layers) have been employed. It does not allow to take into account a large difference between thickness of the layers (for example when the external layers are made from the thin emulsion membranes).

We have employed the modified couple stress theory in order to take into account the size depended effects in the three-layer beam with both stiffening and softening external layers. This choice is motivated by an observation that in all works using the mentioned theory, the stiffening effect is exhibited to explain the size depended effects (see, for instance the references Fleck et al., 1994; Stolken and Evans, 1998; Miller and Shenoy, 2000; Yang et al., 2002). Owing to the experimental results published in the work Abazari et al. (2015), where in Table 1 the size dependency of micro/nano structures is reported for different materials, the majority of the analysed materials show stiffening effects.

In our work, we study the beam having the external layer made from copper (Cu) and the internal layer made from epoxide tar. Owing to the mentioned Table 1, those materials exhibit stiffening effects. On the other hand a comparison of the couple stress theory and non-local Eringen's theory has been carried out by Tsiatas and Yiotis (2015), where the effects of the beam softening based on non-local Eringen's theory and the beam stiffening due to the modified couple stress theory have been reported. In reference Santos and Reddy (2012) it is pointed out while investigating the Timoshenko beams dynamics that the use of non-local Eringen's theory implies a decrease in the frequencies of free beam vibrations, which stands in contrast to the results obtained using the modified couple stress theory.

Relevance of the gradient and non-local (integral) elastic models to include the size dependent effects is widely illustrated and discussed for instance in reference Challamell and Wang (2008). The gradient models are considered as weak non-local models. Furthermore, in many cases, the non-local models may yield a paradox, since the obtained solutions based on the classical theory are the same as in the case of the non-local theory, i.e. there is a lack of the size dependent effect. The mentioned paradox does not appear while using the modified couple stress theory.

In this work we propose the theory of three-layer beams based on the hypotheses of Grigolyuk and Chulkov (1973) as well as the modified couple stress theory and the derived size-dependent equations of motion for the micro- and nano-order thickness of the layers. The Hamilton principle yields new equations of motion as well as the boundary and initial conditions regarding displacements for the micro-beams. The obtained equations allow to

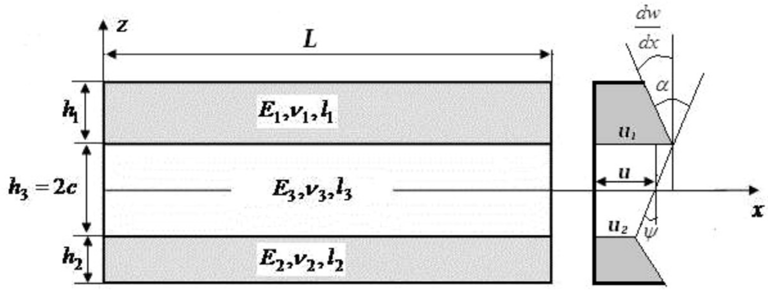

Fig. 1. The longitudinal cross-section of the three layers beams.

explain the size dependent behavior of a micro-beam and they coincide with the classical equations if the layer thickness becomes large enough. A numerical example of computation of the three-layer beam with micro-level thickness of the layers has been given, and a comparison of its behavior versus the classical Grigolyuk–Chulkov theory has been conducted.

2. Theory of bending including shear effects and the modified couple stress theory

In the process of the beam deformation the transversal cross sections of the middle layer being perpendicular to the beam axis, in contrary to Bernoulli–Euler hypothesis, rotate as a rigid body on amount of angle ψ. Here, in contrary to the hypothesis of the plane cross-sections, we do not require cross-sections to be perpendicular to the bended beam axis while deformation process, but in general, this behavior is not excluded. This stands for a general hypothesis, which is not contradicting the Bernoulli–Euler one. Namely, it tends to the latter one if the middle layer stiffness against shear is of infinite magnitude. The external layers material is assumed to be linearly elastic, and they obey Hooke's law. The homogenous internal layer also obeys Hooke's law.

The beam is analyzed using the rectangular coordinates Oxz (see Fig. 1). The axis x overlaps with the middle beam layer line, whereas the axis z goes in direction opposite to the Earth gravity. The carrying out load layer located in the positive z axis direction is called the first layer and the middle layer is called the third layer.

Let h_k ($k=1, 2, 3$) denote the layer thickness ($h_3=2c$); $h=h_1+h_2+h_3$ stands for the thickness and b is the width of a beam wall, respectively; E_k is the elasticity material modulus, whereas ν_k denotes the Poisson's coefficient of a layer with number k; G_3 is the modulus regarding the transversal shear of the middle beam, l_k stands for the internal material length scale parameter. In order to keep a compact formulation of the problem, we introduce the following averaged elasticity modulus

$$E = (E_1 h_1 + E_2 h_2 + E_3 h_3)/h, \tag{1}$$

as well as the non-dimensional stiffness characteristics γ_k and the non-dimensional thickness of the layer t_k as follows

$$\gamma_k = E_k h_k / Eh, \quad t_k = h_k/h. \tag{2}$$

Owing to the concept of continuum mechanics, the deformations ε_{ij}, displacements u_i and the curvatures χ_{ij} satisfy the following relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad \chi_{ij} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right), \tag{3}$$

where $\boldsymbol{\theta}$ is the infinite small vector of rotations with components θ_i. Observe that $\boldsymbol{\theta} = (rot(\mathbf{u}))/2$.

In order to simplify our analysis and in order to get explicit results, we employ the couple stress theory (Yang et al., 2002; Park and Gao, 2006, 2008), in which the deformation energy contains only one parameter of the material length and is governed by a symmetric function of deformation and is associated with the symmetric curvatures. Relations between deformations and symmetric components of the curvatures and coupled with them moments of higher orders follow

$$\sigma_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij}, \tag{4}$$

$$m_{ij} = 2l^2 G\chi_{ij}, \tag{5}$$

where: E, G and ν is the Young modulus, the shear modulus and the Poisson's coefficient of a rigid body, respectively, and l stands for the internal material length scale parameter.

In the modified couple stress theory the summed energy of deformation of an elastic body of space Ω is governed by the following formula (it differs from the classic formula only with respect to the second term)

$$U = (1/2)\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right)d\Omega \tag{6}$$

Owing to the assumption that the material of all three layers is non-compressed in the transversal direction, the deflection w does not depend on the transversal coordinate z, i.e. we have

$$w = w(x). \tag{7}$$

In the case of the middle layer, we take into account the Timoshenko hypothesis and the longitudinal displacements of the points read

$$u_3 = u + z\psi, \qquad -c \le z \le c. \tag{8}$$

Material of the carrying load layers is assumed to be absolutely stiff with respect to shear, and hence the shear angles in the first and second layers are equal to zero, i.e.

$$\alpha_1 = \frac{\partial u_1}{\partial z} + \frac{\partial w}{\partial x} = 0, \ \ (c \le z \le c + h_1),$$
$$\alpha_2 = \frac{\partial u_2}{\partial z} + \frac{\partial w}{\partial x} = 0, \ \ (-c - h_2 \le z \le -c). \tag{9}$$

The last formula, taking into account Eqs. (7) and (8) and assuming lack of the relative sliding of the layers, yields the following formulas governing the longitudinal displacements of the points of the transversal beam cross-section

$$u(x,z) = \begin{cases} u + c\psi - (z-c)\dfrac{\partial w}{\partial x}, & (c \le z \le c+h_1) \\ u + z\psi, & (-c \le z \le c) \\ u - c\psi - (z+c)\dfrac{\partial w}{\partial x}, & (-c-h_2 \le z \le -c). \end{cases} \tag{10}$$

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Introducing α instead of the shear angle ψ, due to relations $\alpha = \psi + \partial w/\partial x$, Eq. (10) yields

$$u(x,z)=\begin{cases} u+c\alpha-z\dfrac{\partial w}{\partial x}, & (c\le z\le c+h_1)\\ u+z\alpha-z\dfrac{\partial w}{\partial x}, & (-c\le z\le c)\\ u-c\alpha-z\dfrac{\partial w}{\partial x}, & (-c-h_2\le z\le -c).\end{cases} \tag{11}$$

For the given displacements, the deformations of each layer lying in a distance z from the averaged line of the middle layer, have the following form

$$\varepsilon_{xx}(x,z)=\begin{cases} \dfrac{\partial u}{\partial x}+c\dfrac{\partial \alpha}{\partial x}-z\dfrac{\partial^2 w}{\partial x^2}, & (c\le z\le c+h_1)\\ \dfrac{\partial u}{\partial x}+z\dfrac{\partial \alpha}{\partial x}-z\dfrac{\partial^2 w}{\partial x^2}, & (-c\le z\le c)\\ \dfrac{\partial u}{\partial x}-c\dfrac{\partial \alpha}{\partial x}-z\dfrac{\partial^2 w}{\partial x^2}, & (-c-h_2\le z\le -c).\end{cases} \tag{12}$$

$$\varepsilon_{xz}=\begin{cases}\alpha_1=0, & (c\le z\le c+h_1),\\ \alpha_3(x)=\alpha(x), & (-c\le z\le c),\\ \alpha_2=0, & (-c-h_2\le z\le -c),\end{cases}$$
$$\varepsilon_{xy}=\varepsilon_{yz}=0. \tag{13}$$

Symmetric components of the curvature possess one non-zero component

$$\chi_{xy}=\chi_{yx}=\begin{cases} -\dfrac{1}{2}\dfrac{\partial^2 w}{\partial x^2}, & (c\le z\le c+h_1)\\ \dfrac{1}{4}\left(\dfrac{\partial \alpha}{\partial x}-2\dfrac{\partial^2 w}{\partial x^2}\right), & (-c\le z\le c)\\ -\dfrac{1}{2}\dfrac{\partial^2 w}{\partial x^2}, & (-c-h_2\le z\le -c).\end{cases} \tag{14}$$

Now, having in hand the deformations and employing the Hook's law, we find the normal stresses in the layers:

$$\sigma_{xx}(x,z)=\begin{cases} E_1\left(\dfrac{\partial u}{\partial x}+c\dfrac{\partial \alpha}{\partial x}-z\dfrac{\partial^2 w}{\partial x^2}\right), & (c\le z\le c+h_1)\\ E_3\left(\dfrac{\partial u}{\partial x}+z\dfrac{\partial \alpha}{\partial x}-z\dfrac{\partial^2 w}{\partial x^2}\right), & (-c\le z\le c)\\ E_2\left(\dfrac{\partial u}{\partial x}-c\dfrac{\partial \alpha}{\partial x}-z\dfrac{\partial^2 w}{\partial x^2}\right), & (-c-h_2\le z\le -c),\end{cases} \tag{15}$$

the tangential stresses in the middle layer

$$\sigma_{xz}=\tau=\begin{cases} G_1\alpha_1=0, & (c\le z\le c+h_1);\\ G_3\alpha_3(x)=G_3\alpha(x), & (-c\le z\le c); \quad \sigma_{xy}=\sigma_{yz}=0,\\ G_2\alpha_2=0, & (-c-h_2\le z\le -c);\end{cases} \tag{16}$$

as well as the higher order moments

$$m_{xy}=\begin{cases} -l_1^2G_1\dfrac{\partial^2 w}{\partial x^2}, & (c\le z\le c+h_1)\\ \dfrac{l_3^2G_3}{2}\left(\dfrac{\partial \alpha}{\partial x}-2\dfrac{\partial^2 w}{\partial x^2}\right), & (-c\le z\le c)\\ -l_2^2G_2\dfrac{\partial^2 w}{\partial x^2}, & (-c-h_2\le z\le -c).\end{cases} \tag{17}$$

Applying Eqs. (15)–(17), we find the forces and moments in each beam layer. In the case of the first carrying load layer $c \le z \le c+h_1$, we have

$$N_1=b\int_c^{c+h_1}\sigma_{xx}dz=B\gamma_1\frac{\partial u}{\partial x}+K\gamma_1\left[t_3\frac{\partial \alpha}{\partial x}-(t_1+t_3)\frac{\partial^2 w}{\partial x^2}\right], \tag{18}$$

$$\begin{aligned} M_1 &= M_1^0+M_1^h=b\int_c^{c+h_1}\left[\sigma_{xx}(z-c)+m_{xy}^1\right]dz\\ &= -cN_1+K\gamma_1t_1\frac{\partial u}{\partial x}+D\gamma_1t_1\Theta^{-1}\\ &\quad\times\left[3t_3\frac{\partial \alpha}{\partial x}-\left(4t_1\left(1+\frac{3}{2(1+\nu_1)}\left(\frac{l_1}{h_1}\right)^2\right)+3t_3\right)\frac{\partial^2 w}{\partial x^2}\right]. \end{aligned} \tag{19}$$

In the case of the middle layer $-c \le z \le c$, we obtain:

$$N_3=b\int_{-c}^{c}\sigma_{xx}dz=B\gamma_3\frac{\partial u}{\partial x}, \tag{20}$$

$$\begin{aligned} M_3 &= M_3^0+M_3^h=b\int_{-c}^{c}\left[\sigma_{xx}z+m_{xy}^3\right]dz\\ &= D\gamma_3t_3^2\Theta^{-1}\left[\left(1+\frac{3}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2\right)\frac{\partial \alpha}{\partial x}\right.\\ &\quad\left.-\left(1+\frac{6}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2\right)\frac{\partial^2 w}{\partial x^2}\right], \end{aligned} \tag{21}$$

$$Q_3=b\int_{-c}^{c}G_3\alpha dz=G_3hbt_3\alpha. \tag{22}$$

Finally, for the second layer $(-c-h_2 \le z \le c-c)$, we get

$$N_2=b\int_{-c-h_2}^{-c}\sigma_{xx}dz=B\gamma_2\frac{\partial u}{\partial x}-K\gamma_2\left[t_3\frac{\partial \alpha}{\partial x}-(t_2+t_3)\frac{\partial^2 w}{\partial x^2}\right], \tag{23}$$

$$\begin{aligned} M_2 &= M_2^0+M_2^h=b\int_{-c-h_2}^{-c}\left[\sigma_{xx}(z+c)+m_{xy}^2\right]dz\\ &= cN_2-K\gamma_2t_2\frac{\partial u}{\partial x}+D\gamma_2t_2\Theta^{-1}\left[3t_3\frac{\partial \alpha}{\partial x}\right.\\ &\quad\left.-\left(4t_2\left(1+\frac{3}{2(1+\nu_2)}\left(\frac{l_2}{h_2}\right)^2\right)+3t_3\right)\frac{\partial^2 w}{\partial x^2}\right]. \end{aligned} \tag{24}$$

where in Eqs. (18)–(24) the following notation has been introduced

$$B=Ehb,\qquad K=\frac{1}{2}Eh^2b,\qquad D=\frac{Eh^3b}{12}\Theta. \tag{25}$$

Furthermore, D in Eq. (25) denotes a usual classical minimum beam bending stiffness, where the introduced parameter Θ is not defined yet. The full longitudinal force follows: $N=N_1+N_2+N_3$, and taking into account Eqs. (18), (23), and (20) we finally get

$$N=B\frac{\partial u}{\partial x}+K\left[c_{12}\frac{\partial \alpha}{\partial x}-c_{13}\frac{\partial^2 w}{\partial x^2}\right]. \tag{26}$$

If, instead of the displacement u, we introduce a new generalized displacement V due to the formula

$$V=u+\frac{1}{2}h\left(c_{12}\alpha-c_{13}\frac{\partial w}{\partial x}\right) \tag{27}$$

then the full longitudinal force Eq. (26) takes the form

$$N=B\frac{\partial V}{\partial x} \tag{28}$$

The full moment computed in relations to the averaged line of the third layer follows

$$\hat{M} = D\left(\gamma \frac{\partial \alpha}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + \frac{h}{2}c_{13}N + M^h, \tag{29}$$

where M^h stands for an additional moment due to the size dependent additives yielded by the occurred size effects. The latter is governed by the formula

$$M^h = D\Theta^{-1}\left\{\frac{3\gamma_3 t_3^2}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2 \frac{\partial \alpha}{\partial x} - \left[\frac{6\gamma_1 t_1^2}{1+\nu_1}\left(\frac{l_1}{h_1}\right)^2 + \frac{6\gamma_3 t_3^2}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2 + \frac{6\gamma_2 t_2^2}{1+\nu_2}\left(\frac{l_2}{h_2}\right)^2\right]\frac{\partial^2 w}{\partial x^2}\right\}. \tag{30}$$

On this step we introduce a new moment $\hat{H}$, which does not have analogy in theory of homogeneous beams, and which defines the transversal shear in the middle layer. It is called the shear moment and it is defined in the following way

$$\hat{H} = M_3 + cN_1 - cN_2 \tag{31}$$

or equivalently

$$\hat{H} = \frac{h}{2}c_{12}N + D\gamma\left(\frac{\gamma}{1-\vartheta}\frac{\partial \alpha}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + H^h \tag{32}$$

where

$$H^h = \frac{D}{2\Theta}\left\{\frac{3\gamma_3 t_3^2}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2\frac{\partial \alpha}{\partial x} - \frac{6\gamma_3 t_3^2}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2\frac{\partial^2 w}{\partial x^2}\right\} \tag{33}$$

stands for the size dependent additive term to the shear moment.

In the previous formulas the following notation has been introduced

$$\Theta = c_{33} - 3c_{13}^2;\ \gamma = (c_{23} - 3c_{12}c_{13})\Theta^{-1};$$
$$\vartheta = 1 - \gamma\frac{c_{23} - 3c_{12}c_{13}}{c_{22} - 3c_{12}^2}, \tag{34}$$

where the explicit form of the coefficients c_{ij} read

$$c_{12} = t_3(\gamma_1 - \gamma_2);\ c_{13} = \gamma_1(t_1 + t_3) - \gamma_2(t_2 + t_3);$$
$$c_{22} = t_3^2(3\gamma_1 + 3\gamma_2 + \gamma_3);$$
$$c_{23} = 3\gamma_1 t_3(t_1 + t_3) + 3\gamma_2 t_3(t_2 + t_3) + \gamma_3 t_3^2;$$
$$c_{33} = \gamma_1\left(4t_1^2 + 6t_1t_3 + 3t_3^2\right) + \gamma_2\left(4t_2^2 + 6t_2t_3 + 3t_3^2\right) + \gamma_3 t_3^2. \tag{35}$$

3. Equations of motion and boundary conditions for the three-layer beam

The state/motion equations as well as the associated boundary conditions corresponding to the assumed kinematic hypotheses are yielded directly from the Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T - U + W)\,dt = 0, \tag{36}$$

where U is the energy of deformation of the bended isotropic three layer linearly elastic beam, T is its kinetic energy, whereas W presents the work of external forces. Therefore, we study the three layer beam of length L subjected to both external transversal load $q(x)$ and external stresses: normal $b\sigma$ and tangential $b\tau$ with intensities on boundaries

$$\sigma_0,\ \tau_0|_{x=0},\ \ \sigma_L,\ \tau_L|_{x=L}. \tag{37}$$

We introduce virtual displacements, i.e. normal

$$\delta w,\ \ (-c - h_2 \le z \le c + h_1) \tag{38}$$

and tangential

$$\delta u(x,z) = \begin{cases} \delta u + c\delta\alpha - z\dfrac{\partial \delta w}{\partial x}, & (c \le z \le c + h_1) \\ \delta u + z\delta\alpha - z\dfrac{\partial \delta w}{\partial x}, & (-c \le z \le c) \\ \delta u - c\delta\alpha - z\dfrac{\partial \delta w}{\partial x}, & (-c - h_2 \le z \le -c). \end{cases} \tag{39}$$

The virtual displacements generate the following virtual deformations in the layers

$$\delta\varepsilon_{xx}(x,z) = \begin{cases} \dfrac{\partial \delta u}{\partial x} + c\dfrac{\partial \delta\alpha}{\partial x} - z\dfrac{\partial^2 \delta w}{\partial x^2}, & (c \le z \le c + h_1) \\ \dfrac{\partial \delta u}{\partial x} + z\dfrac{\partial \delta\alpha}{\partial x} - z\dfrac{\partial^2 \delta w}{\partial x^2}, & (-c \le z \le c) \\ \dfrac{\partial \delta u}{\partial x} - c\dfrac{\partial \delta\alpha}{\partial x} - z\dfrac{\partial^2 \delta w}{\partial x^2}, & (-c - h_2 \le z \le -c), \end{cases} \tag{40}$$

$$\delta\alpha = \begin{cases} 0, & (c \le z \le c + h_1) \\ \delta\alpha, & (-c \le z \le c) \\ 0, & (-c - h_2 \le z \le -c), \end{cases} \tag{41}$$

$$\delta\chi_{xy} = \begin{cases} -\dfrac{1}{2}\dfrac{\partial^2 \delta w}{\partial x^2}, & (c \le z \le c + h_1) \\ \dfrac{1}{4}\left(\dfrac{\partial \delta\alpha}{\partial x} - 2\dfrac{\partial^2 \delta w}{\partial x^2}\right), & (-c \le z \le c) \\ -\dfrac{1}{2}\dfrac{\partial^2 \delta w}{\partial x^2}, & (-c - h_2 \le z \le -c). \end{cases} \tag{42}$$

Since $\sigma_{yy} = \sigma_{zz} = 0$, then using Eqs. (3)–(5) and (12)–(17), the internal energy U of the deformed isotropic three layer beams takes the following form

$$\begin{aligned} U &= (1/2)b\int_0^L\int_{-c-h_2}^{c+h_1}(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})dxdz \\ &= (1/2)b\int_0^L\left[\int_c^{c+h_1}\left(E_1\left(\frac{\partial u}{\partial x} + c\frac{\partial \alpha}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)^2 + l_1^2G_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right)dz\right. \\ &\quad + \int_{-c}^{c}\left(E_3\left(\frac{\partial u}{\partial x} + z\frac{\partial \alpha}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)^2 + G_3\alpha^2 + \frac{l_3^2G_3}{4}\left(\frac{\partial \alpha}{\partial x} - 2\frac{\partial^2 w}{\partial x^2}\right)^2\right)dz \\ &\quad \left. + \int_{-c-h_2}^{-c}\left(E_2\left(\frac{\partial u}{\partial x} - c\frac{\partial \alpha}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)^2 + l_2^2G_2\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right)dz\right]dx. \end{aligned} \tag{43}$$

Let us introduce to the layers the virtual displacements δu, δw, $\delta\alpha$ owing to Eqs. (38)–(42). Therefore, the variation of the internal energy takes the form

$$\begin{aligned} \delta U = b\int_0^L&\left[\int_c^{c+h_1}\left(\sigma_{xx}\left(\frac{\partial \delta u}{\partial x} + c\frac{\partial \delta\alpha}{\partial x} - z\frac{\partial^2 \delta w}{\partial x^2}\right)\right.\right. \\ &\left. - m_{xy}\left(\frac{\partial^2 \delta w}{\partial x^2}\right)\right)dz + \int_{-c}^{c}\left(\sigma_{xx}\left(\frac{\partial \delta u}{\partial x} + z\frac{\partial \delta\alpha}{\partial x} - z\frac{\partial^2 \delta w}{\partial x^2}\right)\right. \\ &\left. + \tau\delta\alpha + m_{xy}\frac{1}{2}\left(\frac{\partial \delta\alpha}{\partial x} - 2\frac{\partial^2 \delta w}{\partial x^2}\right)\right)dz \\ &+ \int_{-c-h_2}^{-c}\left(\sigma_{xx}\left(\frac{\partial \delta u}{\partial x} - c\frac{\partial \delta\alpha}{\partial x} - z\frac{\partial^2 \delta w}{\partial x^2}\right)\right. \\ &\left.\left. - m_{xy}\left(\frac{\partial^2 \delta w}{\partial x^2}\right)\right)dz\right]dx. \end{aligned} \tag{44}$$

Now, introducing the general displacement v instead if u, and taking into account Eq. (27) we get

$$\delta u = \delta V - \frac{1}{2}h\left(c_{12}\delta\alpha - c_{13}\frac{\partial \delta w}{\partial x}\right)$$

and hence Eq. (44) yields

$$\delta U = \int_0^L \left[\frac{\partial N}{\partial x}\delta V + \left(\frac{\partial H}{\partial x} - Q_3\right)\delta\alpha + \frac{\partial^2 M}{\partial x^2}\delta w\right]dx - \left[N\delta V + M\left(\gamma\delta\alpha - \frac{\partial \delta w}{\partial x}\right) + \left(\frac{H}{\gamma} - M\right)\gamma\delta\alpha + \frac{\partial M}{\partial x}\delta w\right]_{x=0}^{x=L}, \quad (45)$$

In the above the following notations have been introduced

$$M = \hat{M} - \frac{h}{2}c_{13}N = D\left(\gamma\frac{\partial\alpha}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + M^h, \quad (46)$$

$$H = \hat{H} - \frac{h}{2}c_{12}N = D\gamma\left(\frac{\gamma}{1-\vartheta}\frac{\partial\alpha}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + H^h. \quad (47)$$

Notice that M^h, H^h are defined by Eqs. (30) and (33). Since the kinetic energy T has the following form

$$T = (1/2)b\int_0^L \int_{-c-h_2}^{c+h_1} \times\rho\left\{\left[V_{,t}^2 + \left(z - \frac{1}{2}hc_{12}\right)\alpha_{,t} - \left(z - \frac{1}{2}hc_{13}\right)w_{,xt}\right]^2 + w_{,t}^2\right\}dxdz \quad (48)$$

then

$$\begin{aligned}\delta T = &-\int_0^L \Big\{\left[B^* V_{,tt} + K^*\left(c_{12}^* - c_{12}\right)\alpha_{,tt} - K^*\left(c_{13}^* - c_{13}\right)w_{,xtt}\right]\delta v \\ &+ \left[K^*\left(c_{12}^* - c_{12}\right)V_{,tt} + D^*\gamma^*\left(\tfrac{\gamma^*}{1-\vartheta^*}\alpha_{,tt} - w_{,xtt}\right)\right]\delta\vartheta \\ &+ \left[K^*\left(c_{13}^* - c_{13}\right)V_{,xtt} + B^* w_{,tt} + D^*\left(\gamma^*\alpha_{,xtt} - w_{,xxtt}\right)\right]\Big\}dx \\ &+ \left[K^*\left(c_{13}^* - c_{13}\right)V_{,tt} + D^*\left(\gamma^*\alpha_{,tt} - w_{,xtt}\right)\right]\Big|_{x=0}^{x=L}.\end{aligned} \quad (49)$$

where $\gamma_k^* = \rho_k h_k / \rho h$. Parameters c_{ik}^* are computed through (35), as it was with the parameters c_{ik} (we should use γ_k^* instead of γ_k). The remaining parameters introduced while computing variation of the inertia forces are as follows:

$$\begin{aligned}B^* &= \rho hb, \ K^* = \frac{1}{2}\rho h^2 b, \ D^* = \frac{\rho h^3 b}{12}\Theta^*, \\ \Theta^* &= c_{33}^* - 6c_{13}^* c_{13} + 3c_{13}^2, \\ \gamma^* &= \left(c_{23}^* - 3c_{12}^* c_{13} - 3c_{13}^* c_{12} + 3c_{12}c_{13}\right)/\Theta, \\ \vartheta^* &= 1 - \gamma^*\frac{c_{23}^* - 3c_{12}^* c_{13} - 3c_{13}^* c_{12} + 3c_{12}c_{13}}{c_{22}^* - 6c_{12}c_{12}^* + 3c_{12}^2}.\end{aligned} \quad (50)$$

The work of the external forces on the virtual displacement follows

$$\delta W = \int_0^L q\delta w\, dx + \left[N_p\delta V + M_p\left(\gamma\delta\alpha - \frac{\partial\delta w}{\partial x}\right) + Q_p\delta w\right]_{x=0}^{x=L}. \quad (51)$$

Here N_p, Q_p, M_p denote external forces and moments acting on the beam

$$N_p = b\int_{-c-h_2}^{c+h_1}\sigma_p dz, \ Q_p = b\int_{-c-h_2}^{c+h_1}\tau_p dz, \quad M_p = b\int_{-c-h_2}^{c+h_1} z\sigma_p dz - \frac{1}{2}hc_{13}N_p, \quad (52)$$

and for $x=0$ we have $\sigma_p = \sigma_0$, whereas for $x=L$ we have $\sigma_p = \sigma_L$.

Employing the Hamilton principle, and comparing to zero multipliers standing by variation of the independent displacements, one gets

$$\begin{aligned}\frac{\partial N}{\partial x} &= B^*\frac{\partial^2 V}{\partial t^2} + K^*\frac{\partial^2}{\partial t^2}\left[\left(c_{12}^* - c_{12}\right)\alpha - \left(c_{13}^* - c_{13}\right)\frac{\partial w}{\partial x}\right], \\ \frac{\partial H}{\partial x} - Q_3 &= K^*\left(c_{12}^* - c_{12}\right)\frac{\partial^2 V}{\partial t^2} + D^*\gamma^*\frac{\partial^2}{\partial t^2}\left[\frac{\gamma^*}{1-\vartheta^*}\alpha - \frac{\partial w}{\partial x}\right], \\ \frac{\partial^2 M}{\partial x^2} + q &= K^*\left(c_{13}^* - c_{13}\right)\frac{\partial^3 V}{\partial x\partial t^2} + B^*\frac{\partial^2 w}{\partial t^2} + D^*\frac{\partial^2}{\partial t^2}\left[\gamma^*\frac{\partial\alpha}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right].\end{aligned} \quad (53)$$

The boundary conditions are

$$\begin{aligned}&[N - N_p]\,|_{x=0,\,l} = 0 \text{ or } \delta u|_{x=0,\,L} = 0, \\ &[M - M_p]\,|_{x=0,\,L} = 0 \text{ or } \delta\left(\gamma\alpha - \frac{\partial w}{\partial x}\right)\Big|_{x=0,\,L} = 0, \\ &\left[\frac{H}{\gamma} - M\right]\Big|_{x=0,\,L} = 0 \text{ or } \delta\alpha\,|_{x=0,\,L} = 0, \\ &\left[\frac{\partial M}{\partial x} - Q_p - K^*\left(c_{13}^* - c_{13}\right)\frac{\partial^2 V}{\partial t^2} - D^*\Theta^*\frac{\partial^2}{\partial t^2}\left(\gamma^*\alpha - \frac{\partial w}{\partial x}\right)\right]\Big|_{x=0,\,L} \\ &= 0 \text{ or } \delta w\,|_{x=0,\,L} = 0.\end{aligned} \quad (54)$$

4. Static transversal bending of the three layer beam

In order to get directly the counterpart static problem we omit dynamic terms in Eq. (53), and the following system of equilibrium equations is obtained

$$\frac{dN}{dx} = 0, \ \frac{dH}{dx} - Q_3 = 0, \ \frac{d^2 M}{dx^2} + q = 0. \quad (55)$$

Now, proceeding to displacements, we obtain

$$\frac{d^2 V}{dx^2} = 0, \quad (56)$$

$$D\gamma^2\left[\left(1 + \frac{3}{2(1+\nu_3)}\frac{(1-\vartheta)\gamma_3 t_3^2}{\gamma^2\Theta}\left(\frac{l_3}{h_3}\right)^2\right)\gamma\frac{d^2\alpha}{dx^2} - (1-\vartheta)\left(1 + \frac{3\gamma_3 t_3^2}{\gamma\Theta}\left(\frac{l_3}{h_3}\right)^2\right)\frac{d^3 w}{dx^3}\right] - (1-\vartheta)G_3 bht_3\gamma\alpha = 0, \quad (57)$$

$$D\left\{\left(1 + \frac{3}{2(1+\nu_3)}\frac{\gamma_3 t_3^2}{\gamma\Theta}\left(\frac{l_3}{h_3}\right)^2\right)\gamma\frac{d^3\alpha}{dx^3} - \left(1 + \frac{6}{\Theta}\left[\frac{\gamma_1 t_1^2}{1+\nu_1}\left(\frac{l_1}{h_1}\right)^2 + \frac{\gamma_3 t_3^2}{1+\nu_3}\left(\frac{l_3}{h_3}\right)^2 + \frac{\gamma_2 t_2^2}{1+\nu_2}\left(\frac{l_2}{h_2}\right)^2\right]\right)\frac{d^4 w}{dx^4}\right\} + qb = 0. \quad (58)$$

After employment of the following simplifications

$$a_h^2 = \frac{3}{2(1+\nu_3)}\frac{(1-\vartheta)\gamma_3 l_3^2}{\gamma^2\Theta}, \ b_h^2 = \frac{3\gamma_3 l_3^2}{\gamma\Theta}, \quad c_h^2 = \frac{3}{2(1+\nu_3)}\frac{\gamma_3 l_3^2}{\gamma\Theta}, \ d_h^2 = \frac{6}{\Theta}\left[\frac{\gamma_1 l_1^2}{1+\nu_1} + \frac{\gamma_3 l_3^2}{1+\nu_3} + \frac{\gamma_2 l_2^2}{1+\nu_2}\right], \quad (59)$$

Eqs. (57) and (58) take the following form

$$D\gamma^2\left[\left(1+\left(\frac{a_h}{h}\right)^2\right)\gamma\frac{d^2\alpha}{dx^2}-(1-\vartheta)\left(1+\left(\frac{b_h}{h}\right)^2\right)\frac{d^3w}{dx^3}\right]-(1-\vartheta)G_3bht_3\gamma\alpha=0, \tag{60}$$

$$D\left\{\left(1+\left(\frac{c_h}{h}\right)^2\right)\gamma\frac{d^3\alpha}{dx^3}-\left(1+\left(\frac{d_h}{h}\right)^2\right)\frac{d^4w}{dx^4}\right\}+qb=0. \tag{61}$$

Therefore, the system of equilibrium Eqs. (56)–(58) split into Eq. (56) and the system of Eqs. (60) and (61) regarding the functions $\gamma\alpha$ and w. It is suitable for a further analysis to reduce the whole problem to only one equation by introducing w and $\gamma\alpha$ as a function χ being differentiable the required times:

$$w=\left\{1-\frac{h^2}{\beta}\left[1+\left(\frac{a_h}{h}\right)^2\right]\frac{d^2}{dx^2}\right\}\chi,$$
$$\gamma\alpha=-(1-\vartheta)\frac{h^2}{\beta}\left[1+\left(\frac{b_h}{h}\right)^2\right]\frac{d^3\chi}{dx^3}. \tag{62}$$

Substituting the expression Eq. (62) into Eq. (60), we find

$$\beta=\frac{12G_3t_3(1-\vartheta)}{E\gamma^2\Theta}. \tag{63}$$

Introducing Eq. (62) into Eq. (61), the following equation regarding the displacement function χ is obtained

$$D^h\left[1-\frac{h^2}{\beta}\left(\left(1+\left(\frac{a_h}{h}\right)^2\right)-(1-\vartheta)p_h\right)\frac{d^2}{dx^2}\right]\frac{d^4\chi}{dx^4}=qb, \tag{64}$$

where

$$D^h=D\left(1+\left(\frac{d_h}{h}\right)^2\right),$$
$$p_h=\left(1+\left(\frac{c_h}{h}\right)^2\right)\left(1+\left(\frac{b_h}{h}\right)^2\right)\Big/\left(1+\left(\frac{d_h}{h}\right)^2\right). \tag{65}$$

Eq. (64) describes the transversal bending of the three layer beam. Since the function χ keeps the displacements, and consequently the moments and transverse forces, we call Eq. (64) the solving equation.

In order to choose the formulation of the problem regarding deformation of the three layer beam, it is necessary to attach boundary conditions to the equilibrium Eqs. (56) and (64). The latter express the influences of both the boundaries action on the beam edges and the boundary loads action. In the case of Eq. (56), where the stiffness is equal to zero (the edge is free), we have

$$\left.\frac{dV}{dx}\right|_{x=0,\,L}=0,\ \text{or}\ N|_{x=0,\,L}=0. \tag{66}$$

In contrary, if the coupling possesses an infinite stiffness, then the boundary condition takes the form $V|_{x=0,\ L}=0$.

In what follows we proceed to formulation of the boundary conditions for the moments. If there is no coupling, then the edge is free, and boundary conditions for χ have the form:

$$\chi=\frac{d^2\chi}{dx^2}=\left.\frac{d^4\chi}{dx^4}\right|_{x=0,\,L}=0, \tag{67}$$

which follow from the boundary conditions $w|_{x=0,L}=\frac{\partial^2 w}{\partial x^2}|_{x=0,L}=\frac{\partial\gamma}{\partial x}|_{x=0,L}=0$.

The boundary conditions for clamped ends have the form:

$$\left\{1-\frac{h^2}{\beta}\left[1+\left(\frac{a_h}{h}\right)^2\right]\frac{d^2}{dx^2}\right\}\chi=\frac{d\chi}{dx}=\left.\frac{d^3\chi}{dx^3}\right|_{x=0,\,L}=0. \tag{68}$$

Introducing the following non-dimensional quantity

$$n^2=\frac{\beta l^2}{h^2\left(\left(1+\left(\frac{a_h}{h}\right)^2\right)-(1-\vartheta)p_h\right)}, \tag{69}$$

we reduce the solving Eq. (64) to the form

$$\left(\frac{d^2}{dx^2}-\frac{n^2}{l^2}\right)\frac{d^4\chi}{dx^4}=-\frac{qn^2}{D^hl^2}. \tag{70}$$

A general solution to the counterpart homogenous equation of Eq. (70) follows:

$$\chi_0(x)=A_0+A_1\frac{x}{l}+A_2\left(\frac{x}{l}\right)^2+A_3\left(\frac{x}{l}\right)^3+A_4sh\frac{nx}{l}+A_5ch\frac{nx}{l}, \tag{71}$$

whereas the particular equation, when $q(x)=q_0=const$, takes the following form

$$\bar{\chi}(x)=\frac{q_0}{24D^h}x^4. \tag{72}$$

Let us find a solution to Eq. (70). For this purpose we define coefficients A_i in general solution Eq. (71) employing the boundary conditions Eq. (67):

$$A_0=-\frac{24q_0l^4}{D^h},\ A_1=\frac{q_0l^4(12-n^2)}{D^h},\ A_2=-\frac{12q_0l^4}{D^h},$$
$$A_3=\frac{2q_0l^4n^2}{D^h},A_4=-\frac{24q_0l^4[1-ch(n)]}{D^hn^2sh(n)},\ A_5=\frac{24q_0l^4}{D^hn^2}. \tag{73}$$

Therefore $\chi(x)=\chi_0(x)+\bar{\chi}(x)$ and both functions $\gamma\alpha$ and w are defined now through formulas Eq. (62).

5. Vibrations of a three layer beam

Let us turn back to dynamic problem. The governing equations are based on the introduced hypotheses and since they take into account the full inertial force. The whole problem is governed by a relatively complex system of *parabolic equations*. Observe that assuming the layers carrying the load (layers 1, 2) are membranes ($t_1=t_2=0$), then the system becomes *hyperbolic*. In the general case, however, the studied system of PDEs can be reduced to only one PDE of eight order with even derivatives. Its coefficients allow to carry out a deep analysis regarding the system behavior. However, in the current study our investigations will be limited to only the main inertial term, assuming that the influence of the remaining inertial terms can be neglected. Namely, we take $K^*=0,\ \ D^*=0$ in Eq. (53), and we obtain

$$\frac{\partial N}{\partial x}=B^*\frac{\partial^2V}{\partial t^2}, \tag{74}$$

$$\frac{\partial H}{\partial x}-Q_3=0, \tag{75}$$

$$\frac{\partial^2M}{\partial x^2}+q=B^*\frac{\partial^2w}{\partial t^2}, \tag{76}$$

or the equivalent form with respect to displacements as follows

$$B\frac{d^2V}{dx^2}=B^*\frac{\partial^2V}{\partial t^2}, \tag{77}$$

$$D\gamma^2\left[\left(1+\left(\frac{a_h}{h}\right)^2\right)\gamma\frac{d^2\alpha}{dx^2}-(1-\vartheta)\left(1+\left(\frac{b_h}{h}\right)^2\right)\frac{d^3w}{dx^3}\right]-(1-\vartheta)G_3bht_3\gamma\alpha=0, \tag{78}$$

$$D\left\{\left(1+\left(\frac{c_h}{h}\right)^2\right)\gamma\frac{d^3\alpha}{dx^3}-\left(1+\left(\frac{d_h}{h}\right)^2\right)\frac{d^4w}{dx^4}\right\}+qb=B^*\frac{\partial^2 w}{\partial t^2}. \tag{79}$$

We take $a^2=E/\rho$, and hence Eq. (77) yields

$$\frac{d^2V}{dx^2}=\frac{1}{a^2}\frac{\partial^2 V}{\partial t^2}. \tag{80}$$

i.e. we have got the PDE governing longitudinal vibrations of the beam.

Introducing, owing to Eq. (62), the function $\chi(x)$, two independent dynamic equations are obtained (the second Eq. (78) is satisfied identically). The third PDE Eq. (79) takes the form

$$D\left(1+\left(\frac{d_h}{h}\right)^2\right)\left[1-\frac{h^2}{\beta}\left(\left(1+\left(\frac{a_h}{h}\right)^2\right)-(1-\vartheta)p_h\right)\frac{\partial^2}{\partial x^2}\right]\frac{\partial^4\chi}{\partial x^4}+\frac{Ehb}{a^2}\frac{\partial^2}{\partial t^2}\left[1-\frac{h^2}{\beta}\left(1+\left(\frac{a_h}{h}\right)^2\right)\frac{\partial^2}{\partial x^2}\right]\chi=qb. \tag{81}$$

We begin with a study of free vibrations of the beam $q\equiv 0$. For this purpose we modify Eq. (81) using the non-dimensional coordinate $\xi=\pi x/L$ and presenting the function $\chi(\xi,\ t)$ in its counterpart form $\chi(\xi,t)=\frac{L}{\pi}X(\xi)e^{i\omega t}$. Dividing the equation for $X(\xi)$ by $e^{i\omega t}$ and by $D^h\frac{h^2}{\beta}\frac{\pi^6}{l^6}\tilde{\vartheta}$, where

$$\tilde{\vartheta}=\left(1+\left(\frac{a_h}{h}\right)^2\right)-(1-\vartheta)p_h, \tag{82}$$

we get

$$X^{VI}-\frac{1}{k\tilde{\vartheta}}X^{IV}-\frac{\omega_*^2\left(1+\left(\frac{a_h}{h}\right)^2\right)}{\tilde{\vartheta}\left(1+\left(\frac{d_h}{h}\right)^2\right)}X^{II}+\frac{\omega_*^2}{k\tilde{\vartheta}\left(1+\left(\frac{d_h}{h}\right)^2\right)}X=0, \tag{83}$$

where the following non-dimensional parameters have been introduced

$$k=\frac{h^2\pi^2}{l^2\beta},\quad \omega_*^2=\frac{12\,l^4\omega^2}{a^2h^2\pi^4\Theta}. \tag{84}$$

It has been shown in Grigolyuk and Chulkov (1973) that the characteristic equation associated with Eq. (83) reads:

$$s^3-\frac{1}{k\tilde{\vartheta}}s^2-\frac{\omega_*^2\left(1+\left(\frac{a_h}{h}\right)^2\right)}{\tilde{\vartheta}\left(1+\left(\frac{d_h}{h}\right)^2\right)}s+\frac{\omega_*^2}{k\tilde{\vartheta}\left(1+\left(\frac{d_h}{h}\right)^2\right)}=0, \tag{85}$$

and consequently, it may have one real and negative root. Introducing the following notations $s_1=-\lambda_1^2,\ s_2=\lambda_2^2,\ s_3=\lambda_3^2$, the general solution to Eq. (83) can be presented in the following form

$$X(\xi)=C_1\sin(\lambda_1\xi)+C_2\cos(\lambda_1\xi)+C_3sh(\lambda_2\xi)+C_4ch(\lambda_1\xi)+C_5sh(\lambda_3\xi)+C_6ch(\lambda_3\xi). \tag{86}$$

On this step we need only to construct the characteristic equation to find the roots $\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2$ by satisfying the homogeneous boundary conditions. Note that the boundary conditions are the same as in the case if the static bending, and hence we have to find the roots of rather a complex transcendental equation. For the case of $a_h=0,\ \ d_h=0$ (classical case) and for a $a_h\neq 0,\ \ d_h\neq 0$ (couple stress theory), the values of $\lambda_1,\ \lambda_2,\ \lambda_3$ are the same and they depend only on the boundary conditions. Since $-\lambda_1^2$ is a root, then Eq. (85) implies

$$\omega_*^2=\lambda_1^4\frac{\left(\lambda_1^2k\tilde{\vartheta}+1\right)\left(1+\left(\frac{d_h}{h}\right)^2\right)}{k\lambda_1^2\left(1+\left(\frac{a_h}{h}\right)^2\right)+1}. \tag{87}$$

In the classical case we have (Grigolyuk and Chulkov, 1973)

$$\omega_{clas}^2=\lambda_1^4\frac{\lambda_1^2k\vartheta+1}{k\lambda_1^2+1}. \tag{88}$$

Let us investigate the ratio $r(\lambda_1^2)=\omega_*^2/\omega_{clas}^2$ expressed explicitly in the form

$$r(\lambda_1^2)=\frac{\left(\lambda_1^2k\tilde{\vartheta}+1\right)\left(1+\left(\frac{d_h}{h}\right)^2\right)}{k\lambda_1^2\left(1+\left(\frac{a_h}{h}\right)^2\right)+1}\frac{k\lambda_1^2+1}{\lambda_1^2k\vartheta+1}, \tag{89}$$

where λ_1^2 plays a role of the control parameter.

4. Numerical results and their validation

Owing to the literature reports (Sun et al., 2007; Awrejcewicz et al., 2008; Krysko et al., 2014), the values of the size dependent parameter l are quite different and still awaiting the estimation for many materials. Since the three layer beams can be metallic, polymer and made from the biological tissues, therefore the scalar length l parameter can change within a rather large interval. For instance, in the case of Al and Si materials, the scalar parameter regarding length l equals to 10^{-10}m. The results of the molecular modeling reported in the reference Fleck et al. (1994) show that the gradient effects occur the thickness of amount $l\approx 10^{-9}$m. In reference Yang et al. (2002), in the case of investigation of the copper wire the value of the length parameter achieves $l=3\cdot 10^{-3}$m. On the other hand, in the case of the rubber epoxides materials, in reference Miller and Shenoy (2000) has been shown that the scalar parameter has been found experimentally as $l=12\cdot 10^{-3}$m for the Young moduli $E=1.44\cdot$ GPa and the Poisson's coefficient $\nu=0.38$.

As an example we consider a three layer beam with the micro-structural effect by taking into account the following length parameters $l_1=l_2$, l_3 for the given values $t_1=t_2=0.125$, $t_3=0.75$, the thickness $h=32\cdot 10^{-6}$ m and length $L=240\cdot 10^{-6}$ m, being under action of the constant and uniformly distributed load $q_0=1\,$N/m.

The numerical examples have been carried out for the three layer micro-beam composed of two copper made external layers and the middle layer made from the rubber epoxide material:

$$E_1=E_2=120\,\text{GPa},\ \nu_1=\nu_2=0.38,\ \ l_1=l_2=3\cdot 10^{-3}\text{m},$$
$$E_3=1.44\text{GPa},\ \nu_3=0,38,\ \ l_3=12\cdot 10^{-3}\text{m}$$

4.1. Static bending

In the beginning the deflections w_0^c in the three layer beam center are computed where the simple boundary conditions are taken and the classical theory of the Grigolyuk–Chulkov is employed ($l_1=l_2=l_3=0,\ \ D^h=D$). In the next step we find the values of the deflection w_0^h in the beam center, which have been obtained based on our introduced theoretical background for the different fixed values of $l_1/h_1=l_2/h_2$ for $l_3=0$ and $l_3=12\cdot 10^{-3}$m.

In Fig. 2 the dependencies characterizing the relative values of the deflection of the beam center w_0^h/w_0^c versus the ratio $l/h=l_1/h_1$ are obtained (solid/dashed curve corresponds to $l_3=0$ / $l_3=12\cdot 10^{-3}$m).

The reported results show that for $l_3=0$ and $l_3=12\cdot 10^{-3}$m the deflection decreases while l_1/h_1 increases. The similar like results has been obtained in reference Fleck et al. (1994) for the plates modelled by the Kirchhoff hypotheses and taking into account the gradient effects. Also in reference McFarland and Colton (2005), where the microstructural effects for a simply supported Timoshenko beam with the employment of the modified couple stress theory of elasticity (Kong et al., 2008) the similar effect has been reported. Observe that for $l_3=12\cdot 10^{-3}$m the beam center deflections become less, i.e. the beam becomes more stiff while taking into account the micro-structural effect in the middle layer.

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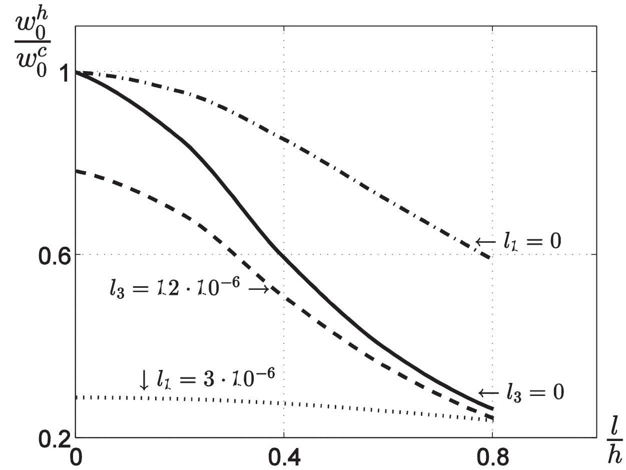

Fig. 2. Relative values of the beam center deformation vs. l/h.

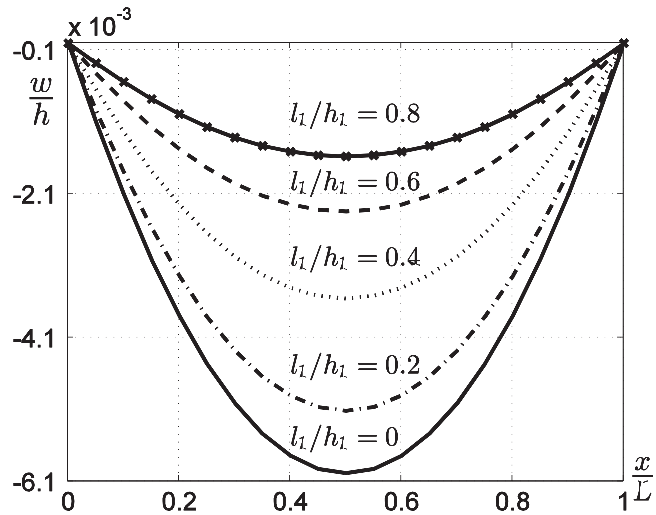

Fig. 3. Profiles of the relative beam deflection value for different values l_1/h_1 for $l_3/h_3=0$.

The figure presents also the dependence (dotted-dashed curve for $l_1=0$ and dotted curve for $l_1=3\cdot10^{-3}$m) w_0^h/w_0^c versus the parameter $l/h=l_3/h_3$. In the latter case the deflection decreases vs. increase of l_3/h_3.

Furthermore, in Fig. 3 the profiles regarding the deflection value $w(x)/h$ of the simply supported three layer beam for $l_3/h_3=0$ and for the different values of the non- dimensional length parameters l_1/h_1 are reported ($l_1/h_1=0$ corresponds to the classical Grigolyuk–Chulkov solution).

Results reported in Fig. 3 imply that the obtained deflections of our model are less than those yielded by the classical theory of Grigolyuk–Chulkov on amount of six times for $l_1/h_1=0.8$. A decrease of the difference between two models while increasing the thickness (when l_1/h_1 is decreased) means that the size effect is visible only on the micro-scale. The same conclusions have been carried out in references Stolken and Evans (1998) and McFarland and Colton (2005), where the difference of the classical Bernoulli–Euler beams as well as the simply supported Timoshenko beams have been studied.

In Fig. 4 the profiles regarding the deflection values $w(x)/h$ of the simply supported three layer beam for $l_1/h_1=l_2/h_2=0$ and the different values of the scalar non-dimensional length parameter l_3/h_3 ($l_3/h_3=0$ corresponds to the classical Grigolyuk–Chulkov solutions) are shown.

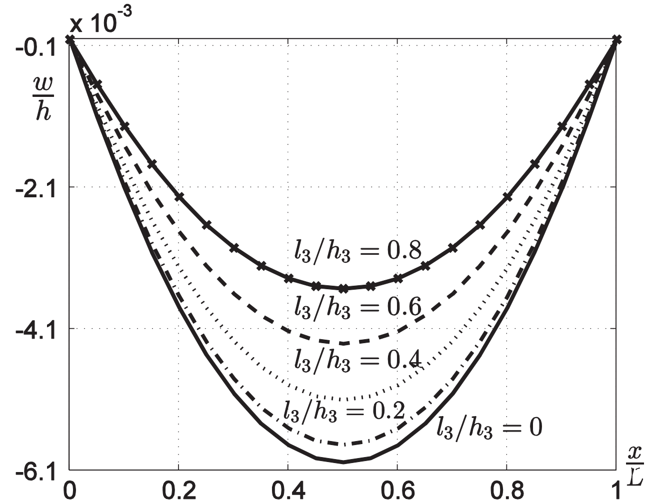

Fig. 4. Profiles regarding the beam deflection for different values of l_3/h_3 for $l_1/h_1=0$.

Though the obtained deflections of the given model are less than those yielded by the classical theory of Grigolyuk–Chulkov, but the influence of the parameter l_3/h_3 is essentially less. For all types of the boundary conditions, deflections obtained using our model are always less than the corresponding values obtained based on the classical theory. Besides, the decrease of a difference between the results obtained using two models (our and classical) while increasing the size effect is essential only on the nano-scale domain.

4.2. Free vibrations

In what follows we investigate qualitatively the influence of the size effects in the beam on the frequencies of its free vibrations taking into account the value of the negative root $(-\lambda_1^2)$ yielded by the characteristic Eq. (85). For this purpose, based on the formula Eq. (89), we find the ratio $r(\lambda_1^2)$.

In Fig. 5 graphs of the values $r(\lambda_1^2)$of the three layer beam versus the ratio $l/h=l_1/h_1$ are reported (solid curve corresponds to $l_3=0$, whereas the dashed curve to $l_3=12\cdot10^{-3}$m; note that $l_1/h_1=0$ corresponds to the classical Grigolyuk–Chulkov solution).

The mentioned figure includes also dependencies $r(\lambda_1^2)$ against $l/h=l_3/h_3$ (dashed-dotted curve for $l_1=0$ and dotted curve for $l_1=3\cdot10^{-3}$m for the parameter $\lambda_1^2=1,\ 100(l_3/h_3=0$ corresponds to the classical Grigolyuk–Chulkov solution).

For all values of the boundary conditions the ratio of the non-dimensional frequencies $r(\lambda_1^2)$obtained through the classical Grigolyuk–Chulkov model always increases while increasing the parameter l/h in an arbitrary beam layer. However, increase of the ratio l/hin the middle layer has less essential consequence in comparison to the increase of the l/hin the remaining layers.

5. Conclusions

Based on both Grigolyuk–Chulkov and modified couple stress theories, the new model validated by both static and dynamic analyses of the three layer micro-beams including only one scalar/length parameter has been constructed, which takes into account the size effect. The employed Hamilton principle yielded the governing equation of the motion as well as general boundary and initial conditions regarding displacements formulated for the micro-beams.

The proposed model of the micro-beam deformation is one of the most simple models and it includes the only one scalar length

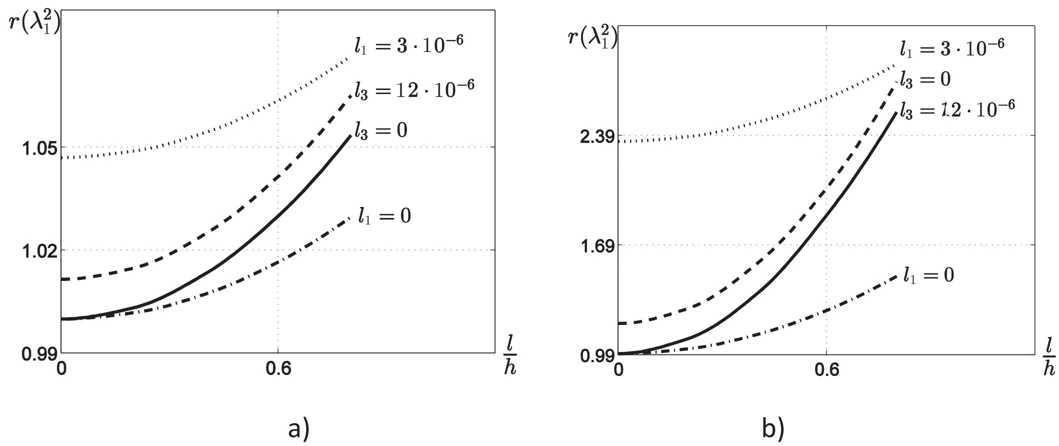

Fig. 5. Relative values $r(\lambda_1^2) = \omega_*^2/\omega_{clas}^2$ versus l/h: a) - $\lambda_1^2 = 1$, b) $\lambda_1^2 = 100$.

parameter. However, it allows to take into account the microstructural effects in both external as well as internal beam layers for any boundary conditions. The finally formulated boundary value problem is of sixth order, and in the case of the static problem it is solved analytically.

The carried out numerical results show that the studied beam model can explain the scale effect exhibited by the micro-beams. The obtained deflections and stresses based on the introduced modified couple stress model are less in comparison to the classical three layer beam model of the Grigolyuk–Chulkov while increasing beam thickness.

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