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# Quantifying non-linear dynamics of mass-springs in series oscillators via asymptotic approach

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ABSTRACT

Dynamical regular response of an oscillator with two serially connected springs with nonlinear characteristics of cubic type and governed by a set of differential-algebraic equations (DAEs) is studied. The classical approach of the multiple scales method (MSM) in time domain has been employed and appropriately modified to solve the governing DAEs of two systems, i.e. with one- and two degrees-of-freedom. The approximate analytical solutions have been verified by numerical simulations.



## 1. Introduction

The massless springs connected in parallel and in series are widely applied in theory and applications of mechanisms and machines, in classical mechanics from regarding both static and dynamical cases, in civil engineering, various mechatronical devices, and more recently in micromechanical systems.

From a point of view of Classical Mechanics application of this massless springs in series element seems to be simple and well understood. However, the situation may change surprisingly, if this spring in series element occurs either as a structural or machine/mechanism member while a studied physical object is either composed of many such elements or it works in nonlinear regimes due to geometry, design or natural types non-linearity like those introduced by friction and/or impacts.

Our research presented in this paper is two-folded. We consider two relatively simple mass-spring in series systems, where springs possess a cubic type stiffness. The first studied system has one degree-of-freedom (1-DoF) and exhibits a horizontal movement (translation), whereas the second one, though having only one particle, it has 2-DoF, due to longitudinal (radial) movement of the springs and rotation around a pivot. We show not only novel non-linear phenomena exhibited by both studied systems, but we also present the modified multiple scale method to fit dynamics appropriately, which is further also validated numerically.

Another important goal of our paper is oriented on applications of the studied mass-spring in series systems in modeling and simulation of the physical systems beyond the classical applications associated with Mechanics.

Weggel et al. [1] employed the basic physics of a spring and elementary structural analyses principle to produce a powerful procedure to study structural behavior. Namely, structural members are modeled as springs assembled in parallel

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and/or series configurations to describe structural equivalent stiffness.

On the other hand the modeling of numerous produced machines and mechanisms governed by ordinary differential equations require introduction of rigid bodies linked by massless spring-damper elements. For instance, vehicle suspensions require lumped mass modeling of oil damper mounted either in parallel or in series with helical springs as well as Kelvin–Voigt models of dashpots and elastic elements (see [2,3]). It is evident that a proper construction of the vehicle suspensions including spring series elements has an important impact on the vibration transmissibility from the rough road to the vehicle, and hence to passengers.

The studied by us models can be directly included to extend the classical analysis of dynamics of two spring-connected masses in a central gravitational field [4]. The latter system serves as a simplified model for the tethered satellite systems, where the tether can be modeled by a linear/nonlinear one or two connected springs.

On the other hand mass-spring systems have been widely used in computer graphics, as it has been pointed out in reference [5]. The author claims that it is often possible to simulate the motion of a real physical body by lumping the mass into a set of points. In spite of that the exact coupling between the motions of different points on a body can be very complex, it can be approximated by a collection of springs.

Another challenging direction of application of connected in series springs is associated with modeling and simulation dynamics of particles undergoing Brownian motions, which are usually based on direct application of the Newton Second Law for a particle.

Ma and Hu [6] have used molecular dynamics approach to study the aggregation processes of systems composed of polymer chains and molecules. The neighborhood located monomers are linked by springs of five different strengths for five different systems. The non-equilibrium aggregation processes have been investigated with respect to the monomer-monomer connecting springs. It has been shown that several stages of clustering may exist while the spring's stiffness is increased.

A series of beads connected by springs is employed in Brownian dynamics modeling for a polymer [7]. In addition, in recent years an attempt to model single polymer molecule dynamics in flow using double stranded DNA, where DNA stands for a unique polymer with force-extension relation understood as an entropic spring-like force has been reported by Chu [8].

In soil dynamics and earthquake engineering the soil nonlinearity is modeled by a hybrid spring configuration consisting of nonlinear springs connected in series to an elastic spring-damper model [9]. Remarkably, the applied nonlinear springs captured the near spring-damper systems represented by Kelvin–Voigt elements, which model the for-field viscoelastic character of the soil.

Springs in series are also applied in micromachines design, as it is presented in reference [10]. A nonlinear suspended energy harvesting device composed of a permanent magnet (mass), springs in series, and two coil sets have been employed for a five pressure monitoring system. Kinematic system behavior is governed by the stiffness in two directions of the two suspended springs, whereas electromagnetic damping allows for estimation of the power output.

Therefore, nonlinear models and among them also nonlinear oscillators have been widely investigated in physics and engineering, and many interesting issues arise in relation to the oscillators having springs connected in series. In addition to the already discussed importance of systems composed with springs in series, they have potentially wide spectrum of application in tuning or suppressing vibration. However, nonlinear oscillators with serially connected springs were investigated by many authors mostly numerically. The systems consisting of two serially connected springs, where one of them is linear and the second one is nonlinear, is discussed among other, in papers [11–13].

The method of averaging is used in paper [14] to obtain an asymptotic solution of the equation of motion of a nonlinear oscillator along a rough straight line considered the motion of a system of two equal mass points along a straight line under the action of a small non symmetric Coulomb dry friction force. The mass points are connected by a non linear spring with cubical non linearity.

Telli and Kopmaz [11] showed that the motion of a mass mounted via linear and nonlinear springs connected in series, is described by a set of differential-algebraic equations. Similar situation occurs in our investigation. The one dimensional oscillator with two nonlinear springs connected in series and the spring pendulum with two nonlinear springs arranged in the same manner are investigated in this paper. The specific form of the governing equations that include apart from the differential equations also the algebraic ones causes that the standard approach in MMS should be modified in order to deal with the mathematical model of such systems. The appropriately adopted MMS has been briefly described in reference [15]. In this paper the procedure applied to solve the considered problems has been described more precisely. Unlike the approach presented in [15], three times scales are used to solve both issues considered here. Moreover, the discussion concerning the obtained results has been extended. The paper is organized in the following way. In Section 2 one-dimensional oscillator is studied analytically and numerically. Section 3 deals with the spring in series pendulum, where the asymptotical solution has been found and compared with the direct numerical simulation. Concluding remarks regarding the obtained results are outlined in Section 4.

## 2. One-dimensional oscillator

Let us consider one-dimensional motion of a body of mass $m$ attached by two massless nonlinear springs to an immovable wall shown in Fig. 1.

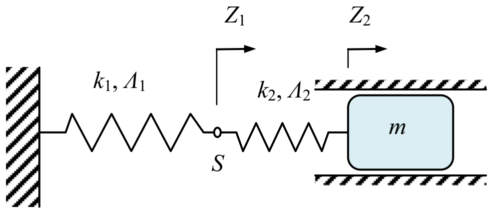


**Fig. 1.** Oscillator with series connection of two nonlinear springs.

The restoring force exerted by the spring with cubic nonlinearity has the following form

$$F_i = k_i\left(Z_i + \Lambda_i Z_i^3\right), \quad i = 1, 2 \tag{1}$$

where $Z_i$ is the elongation of the $i$th spring, $k_i$ is its stiffness coefficient, and $\Lambda_i$ stands for the nonlinearity parameter. Lengths of not tensioned springs are $L_{10}$ and $L_{20}$, respectively.

Such type of nonlinear elasticity is widely discussed in the papers concerning nonlinear dynamics [11,15]. When $\Lambda_i > 0$ the characteristics of the spring is called "hard", while for $\Lambda_i < 0$ the characteristics is called "soft". Hereafter we consider only the case $\Lambda_i > 0$.

### 2.1. Mathematical model

Two equations describe behavior of the system. One of them is the differential equation of the body motion

$$m\left(\ddot{Z}_1 + \ddot{Z}_2\right) + k_2 Z_2\left(1 + \Lambda_2 Z_2{}^2\right) = 0, \tag{2}$$

whereas the second one is the algebraic equation describing equilibrium at the massless connection point $S$ (see Fig. 1):

$$k_1 Z_1\left(1 + \Lambda_1 Z_1^2\right) - k_2 Z_2\left(1 + \Lambda_2 Z_2{}^2\right) = 0. \tag{3}$$

The above equations are supplemented by the initial conditions

$$Z_1(0) + Z_2(0) = Z_0,\ \dot{Z}_1(0) + \dot{Z}_2(0) = V_0, \tag{4}$$

where $Z_0$ and $V_0$ are initial displacement and velocity of the body, respectively.

Transformation to the more convenient dimensionless form of the governing equations yields

$$\ddot{z}_1 + \ddot{z}_2 + (1 + \lambda)z_2\left(1 + \alpha_2 z_2^2\right) = 0, \tag{5}$$

$$\lambda z_2\left(1 + \alpha_2 z_2^2\right) - z_1\left(1 + \alpha_1 z_1^2\right) = 0, \tag{6}$$

$$z_2(0) + z_1(0) = z_0,\ \dot{z}_2(0) + \dot{z}_1(0) = v_0, \tag{7}$$

where $z_0 = Z_0/L$, $v_0 = V_0/L$, $z_i = Z_i/L$, $\alpha_i = \Lambda_i L^2$ for $i = 1, 2$, $L = L_{10} + L_{20}$, $\lambda = k_2/k_1$.

Dimensionless time $\tau = t\omega_1$, where $\omega_1 = k_e/m$ and $k_e = k_1 k_2/(k_2 + k_2)$, plays a role of the characteristic quantity.

Observe that the additional relations between derivatives of the functions $z_1(\tau)$ and $z_2(\tau)$ are obtained by differentiating once and twice Eq. (6), i.e. we have

$$\left(1 + 3\alpha_1 z_1^2\right)\dot{z}_1 - \lambda\left(1 + 3\alpha_2 z_2^2\right)\dot{z}_2 = 0, \tag{8}$$

$$6\alpha_1 z_1 \dot{z}_1^2 + \left(1 + 3\alpha_1 z_1^2\right)\ddot{z}_1 - 6\lambda\alpha_2 z_2 \dot{z}_2^2 - \lambda\left(1 + 3\alpha_2 z_2^2\right)\ddot{z}_2 = 0. \tag{9}$$

Formulas (8) and (9) allow to eliminate the function $z_1(\tau)$ from Eq. (5).

### 2.2. Approximate analytical solution

The problem defined by Eqs. (5)–(7) can be solved directly using MSM [4], though the approach requires some significant modifications. The assumptions concerning smallness of the nonlinearity parameters are proposed in the following form

$$\alpha_1 = \tilde{\alpha}_1 \varepsilon^2,\ \alpha_2 = \tilde{\alpha}_2 \varepsilon^2, \tag{10}$$

where $\varepsilon$ is a small perturbation parameter.

The solution is searched in the form of series with respect to the small parameter. Namely, we have

$$z_2(\tau;\,\varepsilon) = \sum_{k=0}^{k=2} \varepsilon^k z_{2k}(\tau_0,\,\tau_1,\,\tau_2), \quad z_1(\tau;\,\varepsilon) = \sum_{k=0}^{k=2} \varepsilon^k z_{1k}(\tau_0,\,\tau_1,\,\tau_2) \tag{11}$$

where the time scales are defined in the following manner: $\tau_0 = \tau$ is the "fast" time, whereas $\tau_1 = \varepsilon\tau$ and $\tau_2 = \varepsilon^2\tau$ serve as the "slow" time. The differential operators follow:

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \varepsilon^2\frac{\partial}{\partial\tau_2},$$
$$\frac{d^2}{d\tau^2} = \frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1} + \varepsilon^2\left(\frac{\partial^2}{\partial\tau_1^2} + 2\frac{\partial^2}{\partial\tau_0\partial\tau_2}\right) + o\left(\varepsilon^3\right). \tag{12}$$

Introducing (10) and (11) into (5) and (6) and replacing the ordinary derivatives by the differential operators (12), we obtain the differential equations with small parameter $\varepsilon$. These equations should be satisfied for any value of the small parameter, so after rearranging them with respect to the powers of $\varepsilon$, we get:

– equations of order of $\varepsilon^0$

$$(1+\lambda)z_{20} + \frac{\partial^2 z_{10}}{\partial\tau_0^2} + \frac{\partial^2 z_{20}}{\partial\tau_0^2} = 0, \tag{13}$$
$$\lambda z_{20} - z_{10} = 0, \tag{14}$$

– equations of order of $\varepsilon^1$

$$(1+\lambda)z_{21} + 2\frac{\partial^2 z_{10}}{\partial\tau_0\partial\tau_1} + 2\frac{\partial^2 z_{20}}{\partial\tau_0\partial\tau_1} + \frac{\partial^2 z_{11}}{\partial\tau_0^2} + \frac{\partial^2 z_{21}}{\partial\tau_0^2} = 0, \tag{15}$$
$$\lambda z_{21} - z_{11} = 0, \tag{16}$$

– equations of order of $\varepsilon^2$

$$(1+\lambda)\left(1+\tilde{\alpha}_2 z_{20}^2\right)z_{20} + 2\frac{\partial^2 z_{10}}{\partial\tau_0\partial\tau_2} + 2\frac{\partial^2 z_{20}}{\partial\tau_0\partial\tau_2} + 2\frac{\partial^2 z_{11}}{\partial\tau_0\partial\tau_1} +$$
$$2\frac{\partial^2 z_{21}}{\partial\tau_0\partial\tau_1} + \frac{\partial^2 z_{10}}{\partial\tau_1^2} + \frac{\partial^2 z_{20}}{\partial\tau_1^2} + \frac{\partial^2 z_{12}}{\partial\tau_0^2} + \frac{\partial^2 z_{22}}{\partial\tau_0^2} = 0, \tag{17}$$
$$\lambda\tilde{\alpha}_2 z_{20}^3 + \lambda z_{22} - \tilde{\alpha}_1 z_{10}^3 - z_{12} = 0. \tag{18}$$

In this way the original equations of motion have been approximated by the system of the partial differential and algebraic equations. This system of equations is solved recursively. Then we apply the classical way of finding solution. Namely, solutions of the equations of successive approximations, starting from the first ones, are inserted into the equations of higher order approximations.

The solution of Eqs. (13)–(14) follows

$$z_{10} = \lambda\left(e^{i\tau_0}B(\tau_1,\,\tau_2) + e^{-i\tau_0}\bar{B}(\tau_1,\,\tau_2)\right), \tag{19}$$

$$z_{20} = e^{i\tau_0}B(\tau_1,\,\tau_2) + e^{-i\tau_0}\bar{B}(\tau_1,\,\tau_2), \tag{20}$$

where $B(\tau_1,\,\tau_2)$ is unknown complex-valued function. Introducing (19) and (20) into equations of the second order (15) and (16) and zeroing secular terms yields

$$\frac{\partial B(\tau_1,\,\tau_2)}{\partial\tau_1} = 0, \quad \frac{\partial\bar{B}(\tau_1,\,\tau_2)}{\partial\tau_1} = 0, \tag{21}$$

which means that the function $B$ depends only on the slowest time scale $\tau_2$.

The solutions of the Eqs. (13)–(14) of order of $\varepsilon^2$ are $z_{11} = 0$ and $z_{21} = 0$.

Elimination of secular terms from the equations of order of $\varepsilon^2$ yields

$$\frac{dB(\tau_2)}{d\tau_2} = \frac{3iB^2(\tau_2)\bar{B}(\tau_2)\left(\lambda^3\tilde{\alpha}_1 + \tilde{\alpha}_2\right)}{2(1+\lambda)}, \quad \frac{d\bar{B}(\tau_2)}{d\tau_2} = -\frac{3i\bar{B}^2(\tau_2)B(\tau_2)\left(\lambda^3\tilde{\alpha}_1 + \tilde{\alpha}_2\right)}{2(1+\lambda)}. \tag{22}$$

Using solutions of the equations of order of $\varepsilon^0$ and $\varepsilon^1$, the solutions of Eqs. (17)–(18) are as follows

$$z_{22} = e^{3i\tau_0}B^3(\tau_2)\frac{9\lambda^3\tilde{\alpha}_1 + \tilde{\alpha}_2 - 8\lambda\tilde{\alpha}_2}{8(1+\lambda)} + CC, \tag{23}$$

$$z_{12} = e^{3i\tau_0}\lambda B^3(\tau_2)\frac{9\tilde{\alpha}_2 + \tilde{\alpha}_1\lambda^2(\lambda - 8)}{8(1+\lambda)} - 3e^{i\tau_0}\lambda B^2(\tau_2)\bar{B}(\tau_2)\left(\lambda^2\tilde{\alpha}_1 - \tilde{\alpha}_2\right) + CC, \tag{24}$$

where $CC$ means complex conjugates.

It is convenient to express the function $B(\tau_2)$ in the polar form

$$B(\tau_2) = \frac{b(\tau_2)}{2}e^{i\psi(\tau_2)},\ \bar{B}(\tau_2) = \frac{b(\tau_2)}{2}e^{-i\psi(\tau_2)}, \tag{25}$$

where $b(\tau_2)$ and $\psi(\tau_2)$ are real-valued functions.

Introducing definitions (25) into the solvability condition (22) leads to the following modulation equations

$$\frac{db(\tau_2)}{d\tau_2} = 0,\quad \frac{d\psi(\tau_2)}{d\tau_2} = \frac{3b^2(\tau_2)\left(\lambda^3\tilde{\alpha}_1 + \tilde{\alpha}_2\right)}{8(1+\lambda)}. \tag{26}$$

The solution of the system (26), assuming initial conditions in the form

$$b(0) = b_0 \text{ and } \psi(0) = \psi_0, \tag{27}$$

is as follows

$$b(\tau_2) = b_0,\quad \psi(\tau_2) = \psi_0 + \frac{3b_0^2\left(\lambda^3\tilde{\alpha}_1 + \tilde{\alpha}_2\right)}{8(1+\lambda)}\tau_2. \tag{28}$$

On the other hand, the solutions of the Eqs. (13)–(18) are introduced into the power series (11). Then using the solution (28) of the modulation problem (26) and (27) the following approximate analytical solution of the problem (5)–(7) are obtained

$$z_1(\tau) = b_0\lambda\ \cos(\Gamma\tau + \psi_0) + \frac{b_0^3\lambda\left(3\alpha_2(5+8\lambda) - \alpha_1\lambda^2(16+25\lambda)\right)}{32(1+\lambda)}\cos(\Gamma\tau + \psi_0) + \frac{2b_0^3\lambda\left(9\alpha_2 + \alpha_1(\lambda - 8)\lambda^2\right)}{32(1+\lambda)}\cos(\Gamma\tau + \psi_0)\cos(2\Gamma\tau + 2\psi_0), \tag{29}$$

$$z_2(\tau) = b_0\ \cos(\Gamma\tau + \psi_0) + \frac{b_0^3\left(\alpha_2 - 8\alpha_2\lambda + 9\alpha_1\lambda^3\right)}{32(1+\lambda)}\cos(3\Gamma\tau + 3\psi_0), \tag{30}$$

where $\Gamma = 1 + 3b_0^2\left(\alpha_2 + \alpha_1\lambda^3\right)/8(1+\lambda)$ is the shortening denotation, while $b_0$ and $\psi_0$ are the initial values of the amplitude and the phase, respectively.

The relationships between the initial values $b_0$, $\psi_0$ in (27) and the initial values of the position $z_0$ and the velocity $v_0$ appearing in (7) are as follows

$$z_0 = \frac{b_0}{32}\cos(\psi_0)\left(32(1+\lambda) - b_{10}^2\left(\alpha_2 - 24\alpha_2\lambda + 25\alpha_1\lambda^3\right) + 2b_{10}^2\left(\alpha_2 + \alpha_1\lambda^3\right)\cos(2\psi_{10})\right), \tag{31}$$

$$v_0 = -b_0\left(32(1+\lambda) + 3b_0^2\left(\alpha_2 + 8\alpha_2\lambda - 7\alpha_1\lambda^3\right) + 6b_0^2\left(\alpha_2 + \alpha_1\lambda^3\right)\cos(2\psi_0)\right) \times \left(8(1+\lambda) + 3b_0^2\left(\alpha_2 + \alpha_1\lambda^3\right)\right)\sin(\psi_0)/(256(1+\lambda)). \tag{32}$$

Since the solutions of the equations of the order $\varepsilon^1$ are equal to zero, only two time scales are required. It should be mentioned that the approximate analytical solution (31) and (32) obtained using three time scales has the same form, when only two time scales are adapted while solving this problem [15].

### 2.3. Results

The time history of the generalized coordinates and their sum (describing position of the body) are presented in Fig. 2.

The comparison between the numerical and the analytical solutions confirms correctness of the employed asymptotic approach. Needless to say that the obtained the explicit form of the approximate solution allows for a deeper analysis of the motion of the studied mechanical system.

The derived analytic form of the solution yields a period of vibration

$$T = \frac{16\pi(\lambda + 1)}{3\alpha_1 b_0^2\lambda^3 + 3\alpha_2 b_0^2 + 8(\lambda + 1)}. \tag{33}$$

Expression (31) describes dependence of the period versus initial value of the amplitude of oscillations. Both nonlinearity parameters $\alpha_1$, $\alpha_2$ as well as the parameter $\lambda$ remarkably effect the character of this dependence. Since the springs are

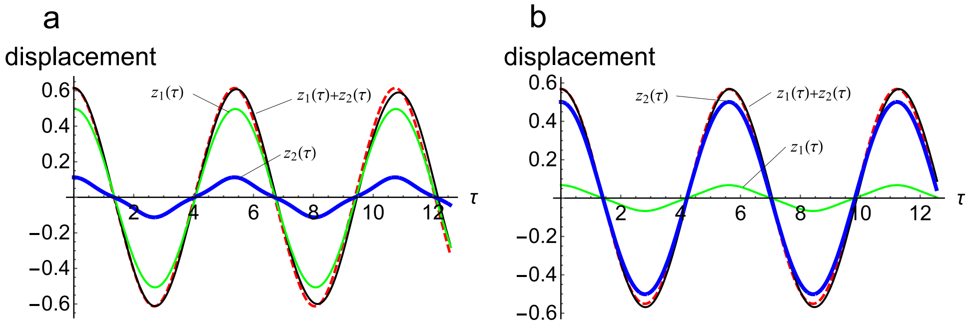


**Fig. 2.** Time history of the system motion: (a) $\alpha_1 = 0.8$ $\alpha_2 = 1.4\lambda = 8$, $b_0 = 0.1$, $\psi_0 = 0$; (b) $\alpha_1 = 1.9$ $\alpha_2 = 1.4$ $\lambda = 0.1$, $b_0 = 0.5$, $\psi_0 = 0$ (dashed line corresponds to numerical solution).

nonlinear with hard characteristics ($\alpha_i > 0$), the period of oscillations is smaller than $2\pi$. In Fig. 3 the value of the oscillations period versus nonlinearity parameters $\alpha_1$ and $\alpha_2$ is shown.

The dependences of the period $T$ and amplitude $b_0$ of the body oscillations versus $\lambda$ obtained from Eqs. (29) and (30) are presented in Fig. 4.

The position and value of the both extrema values observed in Fig. 4 strongly depend on the nonlinearity $\alpha_1$, $\alpha_2$ of the applied springs.

The analytical form of the asymptotic solution allows to perform a detailed analysis of the relationships between the kinematic aspects of the vibration and the physical parameters.

## 3. Spring pendulum

The dynamics of the nonlinear spring pendulum presented in Fig. 5 is investigated in this section. Though the system is relatively simple, it can serve as a good example for studying non-linear phenomena of physical and geometrical nature which occur in two degrees-of-freedom mechanical systems.

The investigated pendulum-type system consists of the small body of mass $m$ suspended at a fixed point via two nonlinear springs of the nominal length $L_{01}$, and $L_{02}$, whose elastic constants are denoted by $k_1$, $\Lambda_1$ and $k_2$, $\Lambda_2$, respectively. The lumped mass body can move only in the fixed vertical plane due to the introduced constraints. Moreover, we assume that the springs are collinear. We are interested in free motion of the system, thus neither external force nor damping are admitted. The total spring elongations $Z_1$, $Z_2$ and the angle $\varphi$ describe unambiguously the position of the studied system.

### 3.1. Mathematical model

The equations of motion are obtained using the Lagrange equations of the second kind. Similarly as in previous section, the differential equations are supplemented by algebraic one, which describes the equilibrium state of the massless point $S$ connecting the massless springs.

The restoring forces in the springs are of the same nonlinear type as previously, and they are governed by Eq. (1). The dimensionless form of the mathematical model follows

$$\ddot{z}_1 + \ddot{z}_2 + \left(1 + 3z_{2r}^2\alpha_2\right)(1+\lambda)z_2 + 3z_{2r}\alpha_2(1+\lambda)z_2^2 + \alpha_2(1+\lambda)z_2^3 - w^2(\cos\varphi - 1) - (1 + z_1 + z_2)\dot{\varphi}^2 = 0, \tag{34}$$

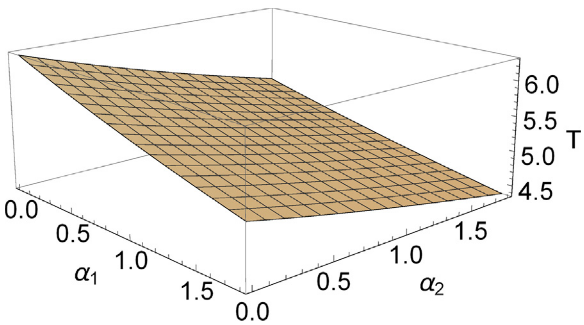


**Fig. 3.** Period $T$ versus nonlinearity parameters $\alpha_1$, $\alpha_2$ for $\lambda = 1.0$, $b_0 = 0.8$.

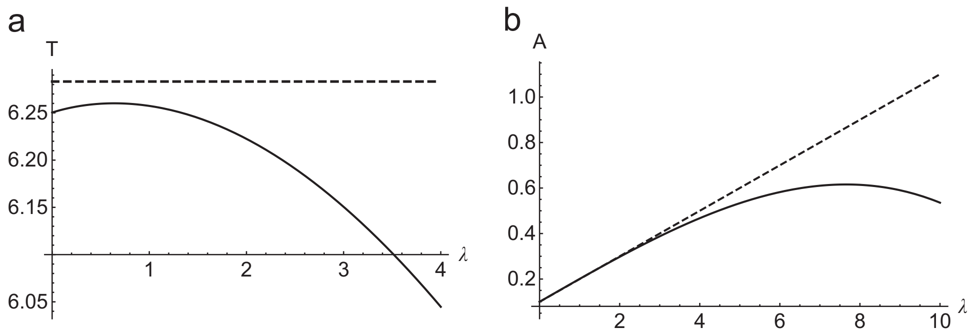


**Fig. 4.** Period $T$ and amplitude $A$ versus $\lambda$: $\alpha_1 = 0.8$, $\alpha_2 = 1.4$,$b_0 = 0.1$, $\psi_0 = 0$ (dashed line corresponds to the linear case $\alpha_1 = 0$, $\alpha_2 = 0$).

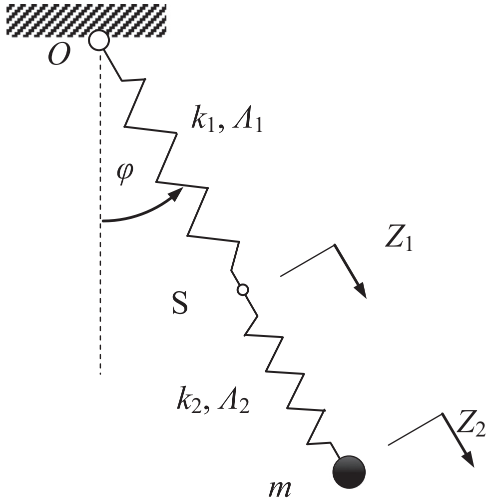


**Fig. 5.** The pendulum with two nonlinear springs connected in series.

$$(1 + z_1 + z_2)\left((1 + z_1 + z_2)\ddot{\varphi} + 2(\dot{z}_1 + \dot{z}_2)\dot{\varphi} + w^2 \sin\varphi\right) = 0, \tag{35}$$

$$\lambda z_2\left(\alpha_2 z_2(z_2 + 3z_{2r}) + 3\alpha_2 z_{2r}^2 + 1\right) - z_1\left(\alpha_1 z_1(z_1 + 3z_{1r}) + 3\alpha_1 z_{1r}^2 + 1\right) = 0. \tag{36}$$

The initial conditions are introduced in the following way

$$z_2(0) + z_1(0) = z_0,\ \dot{z}_2(0) + \dot{z}_1(0) = v_0,\ \varphi(0) = \varphi_0,\ \dot{\varphi}(0) = \omega_0. \tag{37}$$

The dimensionless parameters are defined in the same way as in the previous Section 2. The elongations of the springs at the static equilibrium position $z_{1r}$ and $z_{2r}$ satisfy the following additional conditions:

$$z_{2r}\left(1 + z_{2r}^2\alpha_2\right) = w^2/(1 + \lambda) \text{ and } z_{1r}\left(1 + z_{1r}^2\alpha_2\right) = \lambda w^2/(1 + \lambda). \tag{38}$$

The trivial solution of Eq. (34), satisfying the algebraic equations $z_1 + z_2 + 1 = 0$, will be omitted in our further investigations.

### 3.2. Approximate analytical solution

The problem (34)–(37) can be solved analytically using the multiple scale method [16], but with an introduction of the significant modification. The assumptions of smallness of the nonlinearity parameters are proposed in the form

$$\alpha_1 = \tilde{\alpha}_1\varepsilon^2,\ \alpha_2 = \tilde{\alpha}_2\varepsilon^2. \tag{39}$$

The employed solving procedure is similar to that one presented in the previous section. However, in this problem also three time scales are employed, and the solutions are searched in the form of the power expansions with respect to small parameter as follows

$$
\begin{aligned}
z_2(\tau;\,\varepsilon) &= \sum_{k=0}^{k=2} \varepsilon^k z_{2k}(\tau_0,\,\tau_1,\,\tau_2),\\
z_1(\tau;\,\varepsilon) &= \sum_{k=0}^{k=2} \varepsilon^k z_{1k}(\tau_0,\,\tau_1,\,\tau_2),\\
\varphi(\tau;\,\varepsilon) &= \sum_{k=0}^{k=2} \varepsilon^k z_{1k}(\tau_0,\,\tau_1,\,\tau_2).
\end{aligned} \tag{40}
$$

Then the expansions (40) and the assumptions (39) are introduced into the equations of motion (34)–(36) and the definitions of differential operators (12) are applied. Afterwards the differential equations of the subsequent orders of the approximation are solved and then, after excluding secular terms, they are introduced into the equations of the higher order approximation, recursively. In this way the following asymptotic analytical solution is obtained:

$$
\begin{aligned}
z_1 =\ & a_{10}\lambda\left(1-3\left(z_{1r}^2\alpha_1 - z_{2r}^2\alpha_2\right)\right)\cos\left(\left(\Gamma_1+\Gamma_2\right)\tau+\psi_{10}\right) + a_{20}^2 w^2\lambda\left(3\ \cos\left(2\left(\Gamma_5+\Gamma_6\right)\tau+2\psi_{20}\right)\right.\\
& \left.+4w^2-1\right)/4\left(4w^2-1\right)(1+\lambda) - 3a_{10}a_{20}^2 w\lambda\left(\left(1+w-2w^2\right)\cos\left(\left(\Gamma_2+\Gamma_3+2\Gamma_4\right)\tau+\psi_{10}-2\psi_{20}\right)\right.\\
& \left.+\left(-1+w+2w^2\right)\cos\left(\left(\Gamma_2+\Gamma_3-2\Gamma_4\right)\tau+\psi_{10}+2\psi_{20}\right)\right)/\left(16\left(4w^2-1\right)\right),
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
z_2 =\ & a_{10}\ \cos\left(\left(\Gamma_1+\Gamma_2\right)\tau+\psi_{10}\right) + \frac{a_{20}^2 w^2\left(4w^2-1+3\ \cos\left(2\left(\Gamma_5+\Gamma_6\right)\tau+2\psi_{20}\right)\right)}{4\left(4w^2-1\right)(1+\lambda)}\\
& - 3a_{10}a_{20}^2 w\left(\left(1+w-2w^2\right)\cos\left(\left(\Gamma_2+\Gamma_3+2\Gamma_4\right)\tau+\psi_{10}-2\psi_{20}\right)\right.\\
& \left.+\left(-1+w+2w^2\right)\cos\left(\left(\Gamma_2+\Gamma_3-2\Gamma_4\right)\tau+\psi_{10}+2\psi_{20}\right)\right)/\left(16\left(4w^2-1\right)\right),
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\varphi =\ & a_{20}\ \cos\left(\left(\Gamma_5+\Gamma_6\right)\tau+\psi_{20}\right) + a_{10}a_{20}w(1+\lambda)\left(\left(-2-3w+2w^2\right)\cos\left(\left(-\Gamma_1-\Gamma_2+\Gamma_5+\Gamma_6\right)\tau-\psi_{10}+\psi_{20}\right)\right.\\
& \left.+\left(2-3w-2w^2\right)\cos\left(\left(\Gamma_1+\Gamma_2+\Gamma_5+\Gamma_6\right)\tau+\psi_{10}+\psi_{20}\right)\right)/\left(2\left(4w^2-1\right)\right)\\
& + a_{20}a_{10}^2 w\left(6-5w+w^2\right)(1+\lambda)^2\ \cos\left(\left(-2\left(\Gamma_2+\Gamma_3\right)-\Gamma_4/2\right)\tau-2\psi_{10}+\psi_{20}\right)/16(2w-1)\\
& + a_{20}a_{10}^2 w\left(-6+7w+9w^2+2w^3\right)(1+\lambda)^2\ \cos\left(\left(2\left(\Gamma_2+\Gamma_3\right)-\Gamma_4/2\right)\tau+2\psi_{10}+\psi_{20}\right)/192\left(4w^2-1\right)\\
& + a_{20}^3\left(1-49w^2\right)\cos\left(3\left(\Gamma_5+\Gamma_6\right)\tau+3\psi_{20}\right)/192\left(4w^2-1\right),
\end{aligned} \tag{43}
$$

$$
\Gamma_1 = 1+\frac{3\left(z_{2r}^2\alpha_2+z_{1r}^2\alpha_1\lambda\right)}{2(1+\lambda)},\ \Gamma_2=\frac{3a_{20}^2w^2\left(w^2-1\right)}{4\left(4w^2-1\right)},\ \Gamma_3=\frac{2+3z_{2r}^2\alpha_2+2\lambda+3z_{1r}^2\alpha_1\lambda}{2(1+\lambda)},
$$

$$
\Gamma_4=\frac{12a_{10}^2(1+\lambda)^2\left(w-w^3\right)+w\left(8a_{20}^2w^4+\left(16-a_{20}^2\right)-w^2\left(64+7a_{20}^2\right)\right)}{8\left(4w^2-1\right)},
$$

$$
\Gamma_5=\frac{12wa_{10}^2(1+\lambda)^2\left(w^2-1\right)}{64w^2-16},\ \Gamma_6=\frac{-8a_{20}^2w^5-w\left(16-a_{20}^2\right)+w^3\left(64+7a_{20}^2\right)}{64w^2-16}.
$$

where $a_{10}$, $a_{20}$, $\psi_{10}$, $\psi_{20}$ are the initial values of the amplitudes and the phases of $z_2$ and $\varphi$, respectively. They are related to the initial values $z_0$, $v_0$, $\phi_0$ and $\omega_0$ by the conditions (37) and solutions (41)–(43) at instant $\tau = 0$.

### 3.3. Results

Time histories of the coordinates describing position of the analyzed particle are presented in Fig. 6. In both graphs, the solid curve represents the asymptotic solution according to (41)–(43), while the dashed curve describes the solution obtained numerically.

The period of oscillations depends, among other, on the parameters $\lambda$ and $w$. These relations can be derived from the asymptotic solutions. The period of the longitudinal $T_z$ and the swing $T_\varphi$ oscillations versus parameter $\lambda$ is presented in Fig. 7, whereas both periods are presented in Fig. 8 as functions of the parameter $w$.

## 4. Conclusions

The mechanical lumped mass systems containing two nonlinear springs connected in series have been investigated. The

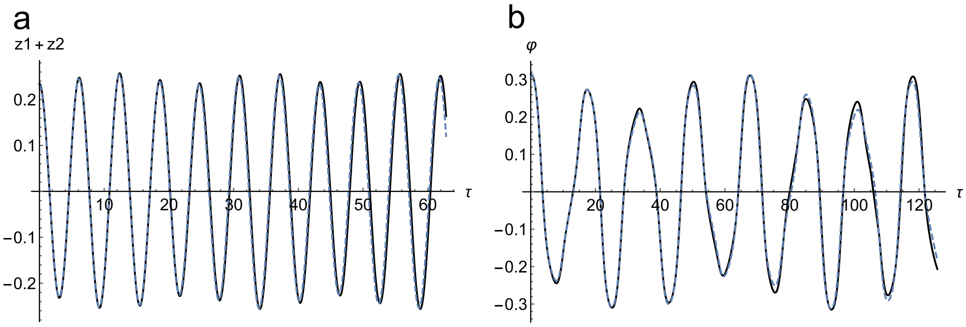


**Fig. 6.** Time history of the particle position for $\alpha_1 = 0.35$, $\alpha_2 = 0.25$, $\lambda = 2.5$, $a_{10} = 0.07$, $a_{20} = 0.27$, $\psi_{10} = 0$, $\psi_{20} = 0$ (dashed curve corresponds to numerical solution).

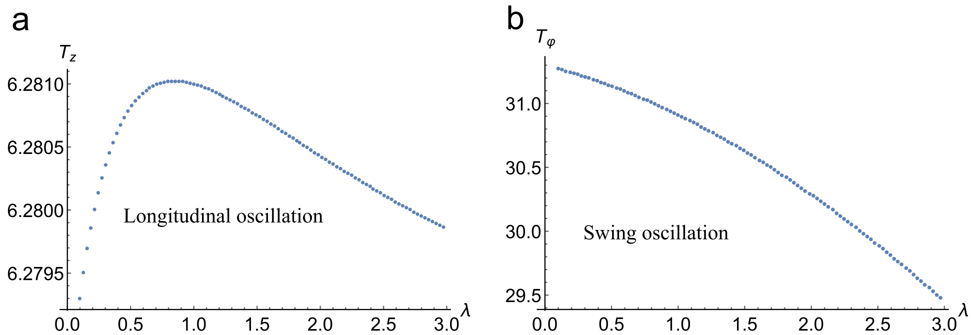


**Fig. 7.** Period $T$ versus $\lambda$ of longitudinal and swing oscillations for $\alpha_1 = 0.35$, $\alpha_2 = 0.25$, $a_{10} = 0.07$, $a_{20} = 0.27$, $\psi_{10} = 0$, $\psi_{20} = 0$.

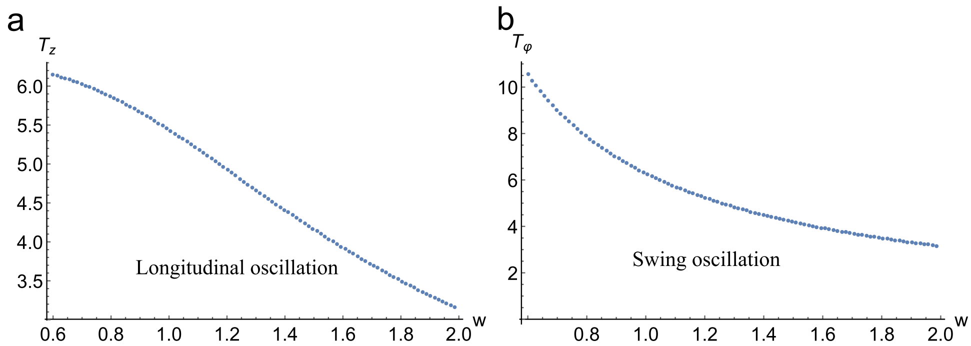


**Fig. 8.** Period $T$ versus $w$ of longitudinal and swing oscillations for $\alpha_1 = 0.35$, $\alpha_2 = 0.25$, $a_{10} = 0.07$, $a_{20} = 0.27$, $\psi_{10} = 0$, $\psi_{20} = 0$.

mathematical model of that kind of objects consists of the differential and algebraic equations. Properly modified multiple scales method in time domain have been applied to solve this problem effectively, and to obtain the approximate analytical solutions. The range of the parameters responsible for the physical type nonlinearity is limited according to the assumptions of the MSM. The correctness of the results has been confirmed by direct numerical simulations.

The approximate analytical solution obtained using appropriately modified MSM allows to analyse the influence of the system parameters on the motion. The influence of some parameters on the period and amplitude has been discussed in the case of regular oscillations.

It turned out that in the case of one dimensional oscillator the approximate analytical solutions obtained using three time scales has the same form when only two time scales are adapted while solving this problem. Hence, introducing the intermediate time scale does no influence the quality of the received solution and application of only “slow” and “fast” time scales are sufficient, in fact. In more sophisticated systems, in which geometrical nonlinearity appears, each of the three time scales used in MSM may play a significant role.

All presented results, both analytical and numerical, has been obtained using the Wolfram Mathematica software.

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