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Alternating chaos versus synchronized vibrations of interacting plate with beams

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ABSTRACT

We study networks of coupled oscillators governed by ODEs and yielded by physically validated sets of a few PDEs governing dynamics of structural members (plate and beams), chaos and phase synchronization and contact/no-contact non-linear dynamics of structural members coupled via boundary conditions. We have detected, illustrated and discussed a few novel kinds of hybrid states of the studied plate-beam(s) contact/no-contact interactions as well as novel scenarios of transition into chaos exhibited by the interplay of continuous objects. Classical (time histories, phase portraits, Poincaré maps, FFT, Lyapunov exponents) and non-classical (2D Morlet wavelets) approaches are used while monitoring non-linear dynamics of the interacting spatial structural members. Our results include examples from structural mechanics and the studied objects are modelled by validated mechanical hypotheses and assumptions. Novel non-linear phenomena including switching to different vibration regimes and phase chaotic synchronization are illustrated and discussed.

1. Introduction

There are numerous applications in civil and mechanical engineering regarding structures consisting of linked systems of elastic bodies (beams and plates) with constraints. There is also a large amount of papers and books devoted to modelling of structural members, but surprisingly there is rather a small amount of published research aimed at mathematical modelling of interacting spatial members putting emphasis on their bifurcational and chaotic dynamics. In what follows we briefly review the literature associated with our study, though, our investigation reported in this paper extends our previous studies devoted to chaotic dynamics of structural slender members putting emphasis on interplay of the plate-beams systems.

Moon and Shaw [1] studied chaotic vibrations of a beam with non-linear boundary condition. Then Shaw [2], based on a single mode approximation, analyzed regular and chaotic dynamics as well as subharmonic resonances of an elastic beam with one-sided amplitude constraint subjected to periodic excitation. Non-smooth modelling method has been applied to study non-linear dynamics of a flexible impacting beam by Wagg [3]. An N-degree-of-freedom modal model has been derived using Galerkin's reduction and the numerical simulations have been compared with experimentally recorded data for a one-sided constrained beam. Applicability of simplified models of continuous mechanical systems using example of a conveyor belt has been discussed in reference [4]. Trindade et al. [5] carried out a study of the vibrations of a vertical slender beam, clamped at its upper extreme, pinned in its lower one and constrained inside an outer cylinder in its lower portion. Non-linear stress-strain relations have been taken into account while applying a non-linear finite-element model and using the Karhunen-Loève decomposition. Dumont and Paoli [6] proposed mathematical models governing dynamics of an elastic beam clamped at its left end and located between two rigid obstacles. The studied infinite dimensional contact problem of fully discretized approximations and their convergence have been proved. Seifield [7] studied the transverse impact of steel spheres on aluminium beams numerically and experimentally. Modally-reduced models have been applied in combinations with local finite element contact models and the numerical simulation results have been validated by experimental investigation using Laser-Doppler-Vibrometer. Transient response of a cantilever beam driven by a periodic force and repeated impacting against a rod-like stop have been analyzed by Yin et al. [8]. In both impact and separation phases, the transient wave propagations have been solved using the expansion of transient wave functions. Several transient phenomena have been illustrated and discussed

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including impact-induced waves, sub-impact phases, long term impact motion, chatter, sticking motion, synchronous and non-synchronous impact and impact loss. Ervin and Wickert [9] investigated the forced response dynamics of a clamped-clamped beam with an attached rigid body. Periodic responses, period-doubling bifurcations, grazing impacts, sub-harmonic regions, fractional harmonic resonances as well as chaotic responses have been reported. The finite element method has been employed to study an impacting plate-beam system by Labuschagne et al. [10], when the Reissner-Mindlin plate model combined with the Timoshenko beam model has been used. The paper has been aimed at exhibition of the interface conditions at the contact between the plate and beams and the impact of regularity on the enforcement of certain interface conditions. Ervin [11] studied the repetitive impact dynamics of two orthogonal pinned-pinned beams subjected to base excitation in an analytical way at specified frequency and acceleration. The vibrational process has been monitored in a piecewise fashion as switching between the linear in-contact and not-in-contact states and compatibility conditions have been introduced at their junctions. The studied parameters have been performed on contact stiffness, relative beam stiffness, contact location, modal damping, and stand-off gap. Wang et al. [12] analyzed the resonance characteristic of a two-span continuous beam traversed by moving high speed trains at a constant velocity where each span is modelled as a continuous Bernoulli-Euler beam and the moving trains are represented as a series of two-degrees-of-freedom mass-spring-damper systems. Ryu et al. [13] developed the beam structure models with impact or contact parts under impact forces. Both theoretical and experimental investigations for the dynamics relationships between the impacting device and impact mechanism have been utilized. Hu and Ervin [14] studied Euler-Bernoulli beam with adjustable boundary conditions and variable contact location numerically under a pulse loading. Numerous test cases have been employed with varied pulse duration, pulse amplitude and clearance, and the computational results have been compared with experimental investigations. Yang et al. [15] carried out an analytical study on chaotic characteristics of viscoelastic beams based on the evolution of non-linear stiffness. Quasi-periodic and chaotic responses associated with period doubling bifurcations have been detected and discussed. Geiyer and Kauffman [16] studied linear cantilevered, piezoelectric energy harvesters (piezomagnetic beams) using benefits of chaotic phenomena by stabilizing high energy periodic orbits located within a chaotic attractor. The authors applied a single time series to compute orbit selection, local linearization and control perturbation.

The carried out analysis of the state-of-the art of research devoted to interaction nonlinear dynamics of contacting structural members shows that there is a lack of a study of bifurcational and chaotic dynamics of a few movable continuous mechanical objects being either in contact or non-contact while vibrating, i.e. exhibiting the so called *switching type non-linear dynamics*.

The literature review shows that mainly contact dynamics of a beam/plate with rigid obstacles have been investigated. There are only a few papers dealing with the mentioned switching type non-linear dynamics of two interacting beams, and mainly the so called reduced-order models have been derived possessing their own drawbacks with respect to neglection of the multi-modes interaction.

It is also rather difficult to find papers dealing with interaction of non-homogeneous continuous members represented for instance by a plate and a series of interacting beams, which is a subject of this paper investigation.

In spite of the physical features of the paper, we offer also important hints to the Mechanics and Mechatronics. The derived PDEs are yielded by modelling of a gyroscopic device, where a contact interaction of a plate and transversal set of beams plays a crucial role in the gyroscope working regimes. Furthermore, the studied in this paper slender structures have a wide application in different navigation devices, electronic techniques and in particular in gyroscopes serving as flat, multi-layer micromechanical accelerometers.

Structural systems consisting of beams and a plate with small clearances between them are studied, and the action of a load subjected to an upper layer yield (plate) interaction between the layers and the phase chaotic synchronization is analyzed. On each time step the contact problem is solved while the structural members are coupled only through the boundary conditions. Different cases of interactions of one plate and one, two, or three beams with small clearances are analyzed deeply, among other.

The reported research extends our earlier studies devoted to chaotic and synchronized dynamics of multi-layer beams [17,18].

The paper is organized in the following way. In Section 2 the governing equations are presented including a switching function exhibiting the contact/no-contact non-linear dynamics. Sections 3 and 4 is devoted to study of chaotic and synchronized state of one plate and one (two) beam(s). Section 5 deals with chaotic and switching type dynamics of the plate and three beams. The last Section 6 reports concluding remarks.

2. The governing equations

In this work we study chaotic vibrations and contact interactions of a plate supported by one or a few beams. The upper plate is governed by the Germain-Lagrange PDE, where beam vibrations are described by the Euler-Bernoulli equations. We assume that the plate and beams are isotropic, a clearance between them is small, and that vibrations of the structural elements are uncoupled and they are governed only by the separated boundary conditions (a contact between structural members yields action of all boundary conditions). The following PDEs govern the plate-beams system dynamics:

$$\frac{1}{12(1-\mu^2)}\nabla_\lambda^4 + \frac{\partial^2 w_1}{\partial t^2} + \varepsilon\frac{\partial w_1}{\partial t} - q_1(x, y,$$
$$t) + P_{1x}\frac{\partial^2 w_1}{\partial x^2} + P_{1y}\frac{\partial^2 w_1}{\partial y^2} - \sum_{i=2}^{n} K(w_1 - w_i - h_k)\Psi_{i-1} = 0,$$
$$\frac{1}{12}\frac{\partial^4 w_i}{\partial x^4} + \frac{\partial^2 w_i}{\partial t^2} + \varepsilon\frac{\partial w_i}{\partial t} - q_i(x, t) + P_i\frac{\partial^2 w_i}{\partial x^2} + K(w_1 - w_i - h_k)\Psi_{i-1} = 0,$$
$$\nabla_\lambda^4 = \frac{1}{\lambda^2}\frac{\partial^4}{\partial x^4} + \lambda^2\frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2\partial y^2}, \tag{1}$$

where the switching function is $\psi_{i-1} = \frac{1}{2}[1 + sign(w_1 - h_k - w_i)]$. Relation $K(w_1 - w_i - h_k)\Psi_{i-1}$ models a contact pressure between the layers, whereas the contact interaction between plates and beams is governed by the Winkler model. We take $q_i(x, t) = 0$. Observe that $\Psi_{i-1} = 1$ if $w_1 > w_i + h_k$, then a contact between plate and beam occurs, otherwise $\Psi_{i-1} = 0$; w_1, w_i are the functions of deflections of the plate and beams, respectively; K is the stiffness coefficient of the transversal structural coupling in a contact zone, and h_k stands for clearances between the structural members.

Eq. (1) are recast to their counterpart non-dimensional form via the following relations: $x = a\bar{x}$, $y = b\bar{y}$; $q = \bar{q}\frac{E(2h)^4}{a^2b^2}$, $\tau = \frac{ab}{2h}\sqrt{\frac{\gamma}{Eg}}$, $\lambda = \frac{a}{b}$, where a, b are the plate dimensions regarding x, y, respectively; t – time, ε – damping coefficient, w – deflection,$2h$ – plate thickness, $\mu = 0.3$ – Poisson's coefficient, g – acceleration of gravity, E – Young modulus, $q_1(x, y, t)$ – transverse load acting on the plate, $q_i(x, t)$ – transverse loads acting on a beam, $P_{1x}(y, t)$, $P_{1y}(x, t)$– longitudinal loads acting on the plate, $P_i(t)$– longitudinal loads acting on the beams, γ/g– unit mass density of the structural material. Bars over non-dimensional parameters in Eq. (1) are omitted.

The governing equations are supplemented by the boundary and initial conditions. In addition, a condition of a lack of penetration between contacting flexible structural members should be added. The obtained set of non-linear PDEs (1) is reduced to a counterpart set of the second-order ODEs using the Bubnov-Galerkin approach. Functions w_1 and w_2 are the solutions of Eq. (1) and they are approximated by the following series

$$w_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} A^{(1)}{}_{ij}(t)\varphi^{(1)}{}_{ij}(x, y), \quad w_k = \sum_{i=1}^{N} A^{(k)}{}_i(t)\varphi^{(k)}{}_i(x), \tag{2}$$

where k denotes the beam number. It means that the earlier problem of infinite dimension has been reduced to that of finite dimension, i.e. PDEs are modelled by finite large sets of the second-order ODEs with respect to time. *In other words, our structural members under investigation are reduced to the similar problems dealing with a network of coupled oscillators*. The system of non-linear second-order ODEs is reduced to a system of the first-order ODEs, and the former one is solved using the fourth-order Runge-Kutta method with respect to time.

The so far defined problem is solved with the help of known methods of non-linear dynamics as well the qualitative theory of differential equations. Namely, time histories (signals), phase portraits, Poincaré maps, FFT (Fast Fourier Transform), wavelet transformations, as well as the Lyapunov exponents regarding all structural members are constructed and monitored while carrying out the numerical computations. Though various types of wavelets have been tested, the Morlet wavelet has been eventually applied as most suitable for our purpose. Furthermore, a direct application of the Morlet wavelet yielded a novel approach to study the chaotic phase synchronization understood here as the chaotic phase locking though the signal amplitudes still exhibited chaotic behaviour. Note that the phase locking phenomenon implies locking of the frequencies. In the case of the wavelet transform application, the wavelet surface $W(s, t_0) = \|W(s, t_0)\| \exp[j\phi_s(t_0)]$ presents the system behaviour regarding each time scale s at an arbitrary time instant t_0. The magnitude $\|W(s, t_0)\|$ characterizes the occurrence and intensity of the counterpart time scale s at time instant t_0. In addition, the integral energy distribution of the wavelet spectrum versus time scale $E(s) = \int \|W(s, t_0)\|^2 dt_0$ is introduced. The monitored phase $\phi_s(t_0) = \arg W(s, t)$ is defined for each of the time scale s, i.e. we are able to follow dynamical behaviour of each time scale s with the help of the counterpart phases. The occurrence of phase synchronization implies phase locking effects on the synchronized time scales s, i.e. $\|\phi_{s_1}(t) - \phi_{s_2}(t)\| < const$.

3. Chaotic and synchronized interactions of a plate and one beam

In this section we study chaotic vibration and contact interaction of a plate and one beam (see Fig. 1). The non-dimensional equations of the mentioned two-layer structure are as follows:

$$\frac{1}{12(1-\mu^2)}\nabla_\lambda^4 + \frac{\partial^2 w_1}{\partial t^2} + \varepsilon\frac{\partial w_1}{\partial t} - q_1(x, y,$$
$$t) + P_{1x}\frac{\partial^2 w_1}{\partial x^2} + P_{1y}\frac{\partial^2 w_1}{\partial y^2} - K(w_1 - w_2 - h_k)\Psi = 0,$$
$$\frac{1}{12}\frac{\partial^4 w_2}{\partial x^4} + \frac{\partial^2 w_2}{\partial t^2} + \varepsilon\frac{\partial w_2}{\partial t} - q_2(x, t) + P_2\frac{\partial^2 w_2}{\partial x^2} + K(w_1 - w_2 - h_k)\Psi = 0, \tag{3}$$

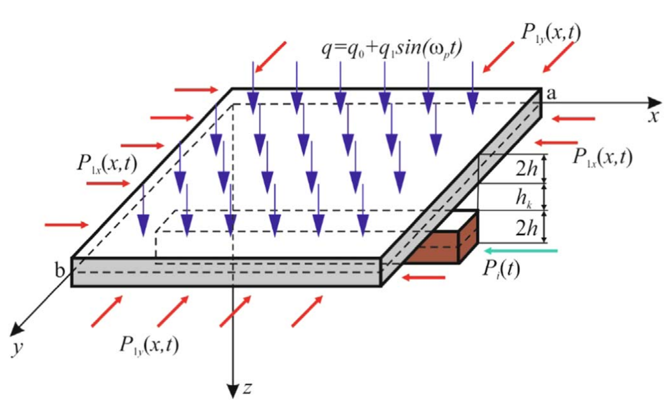

Fig. 1. System composed of a plate and one beam.

where $\psi = \frac{1}{2}[1 + sign(w_1 - h_k - w_2)]$.

We consider first the case of uniformly distributed load $q_0 = const$ for $t = 0$. The applied boundary conditions are as follows:

(i) simple support along the plate contour

$$w_1 = 0; \quad \frac{\partial^2 w_1}{\partial x^2} = 0; \quad \text{for } x = 0;\ 1; \quad w_1 = 0; \quad \frac{\partial^2 w_1}{\partial y^2} = 0; \quad \text{for } y = 0;\ 1; \tag{4}$$

(ii) simple support of the beam

$$w_2 = 0; \quad \frac{\partial^2 w_2}{\partial x^2} = 0; \quad \text{for } x = 0;\ 1. \tag{5}$$

The following initial conditions are introduced:

$$w_1(x, y)|_{t=0} = 0, \quad w_2(x)|_{t=0} = 0, \quad \left.\frac{\partial w_m}{\partial t}\right|_{t=0} = 0, \ m = 1, 2,$$

and the following forms of functions w_1 and w_2 are assumed:

$$w_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} A^{(1)}{}_{ij}(t)\sin(i\pi x)\sin(j\pi y), \quad w_k = \sum_{i=1}^{N} A^{(k)}{}_i(t)\sin(i\pi x), \tag{6}$$

where k stands for the number of the beam (here k=1).

The following parameters are fixed: $K = 5000$, $q_1 = q_0 = const = 1$, $q_2 = 0$, $h_k = 0.01$,$\varepsilon = 0.5$, $\omega_0^{(1)} = 5.9887$, $\omega_0^{(2)} = 2.84$ (the two latter quantities denote natural frequencies of the linear vibrations). In Fig. 2 the following vibration characteristics are reported: (a, d) wavelet spectra; (b, e) phase portraits; (c, f) FFT spectra for the plate and the beam, respectively; zones of the synchronized frequencies (g), as well as time histories of common beam and plate vibrations in their central point (h). Dark colour on the phase difference clearance (g) exhibits frequencies involved in the synchronization process. Vibrations take place at the frequency $\omega_0 = 5$, i.e. the natural frequency of the two-layer structure, which is illustrated by the Morlet wavelet spectrum (a, d) presenting an energy component of each of the frequencies at a given time instant. Phase synchronization is observed for the frequency interval $\omega \in [3, 7]$. Blue/red colour in graph (h) corresponds to vibration of the plate/beam. In the whole vibrational process full amplitudes locking of the plate and the beam takes place at time instants in the interval $t \in [5; 15]$ (h). Further, when time is passing the additional frequencies are involved into the synchronized regime (a, d), and then chaotic components appear. In addition, on the graph of common structural members vibrations (h) complex vibrations of the system are exhibited. Next, for $q_1 \neq const$ we study the case $\omega_p = \omega_0 = 5$ which also holds for the plate support consisting of two and three beams.

When the clearance between the plate and beam is decreased in time evolution, the amplitudes are also locked, i.e. the full (phase and amplitude) synchronization is observed for smaller values of the applied load q_0. In other words, a decrease in the clearance implies an increase in the synchronization. Furthermore, the influence of damping coefficient ε on the vibration regimes was studied. Though a decrease in ε results in the increase of higher complexity of vibrations, both phase and amplitude synchronizations are increased and the full synchronization of two structural members is observed. The so far the discussed results have been obtained for the following fixed parameters: $K = 5000$, $q = q_0 = const = 4.6$, $h_k = 0.01$, $\varepsilon = 0.25$, $\omega_0^{(1)} = 5.9887$, $\omega_0^{(2)} = 2.84$, but they are not presented here in the graphical forms.

Let us study the almost conservative case for $\varepsilon = 0.001$. Here *switch on/switch off* synchronization phenomena are detected. In other words, a competition between synchronized and chaotic states of coupled structural members appears for a long time. The full synchro-

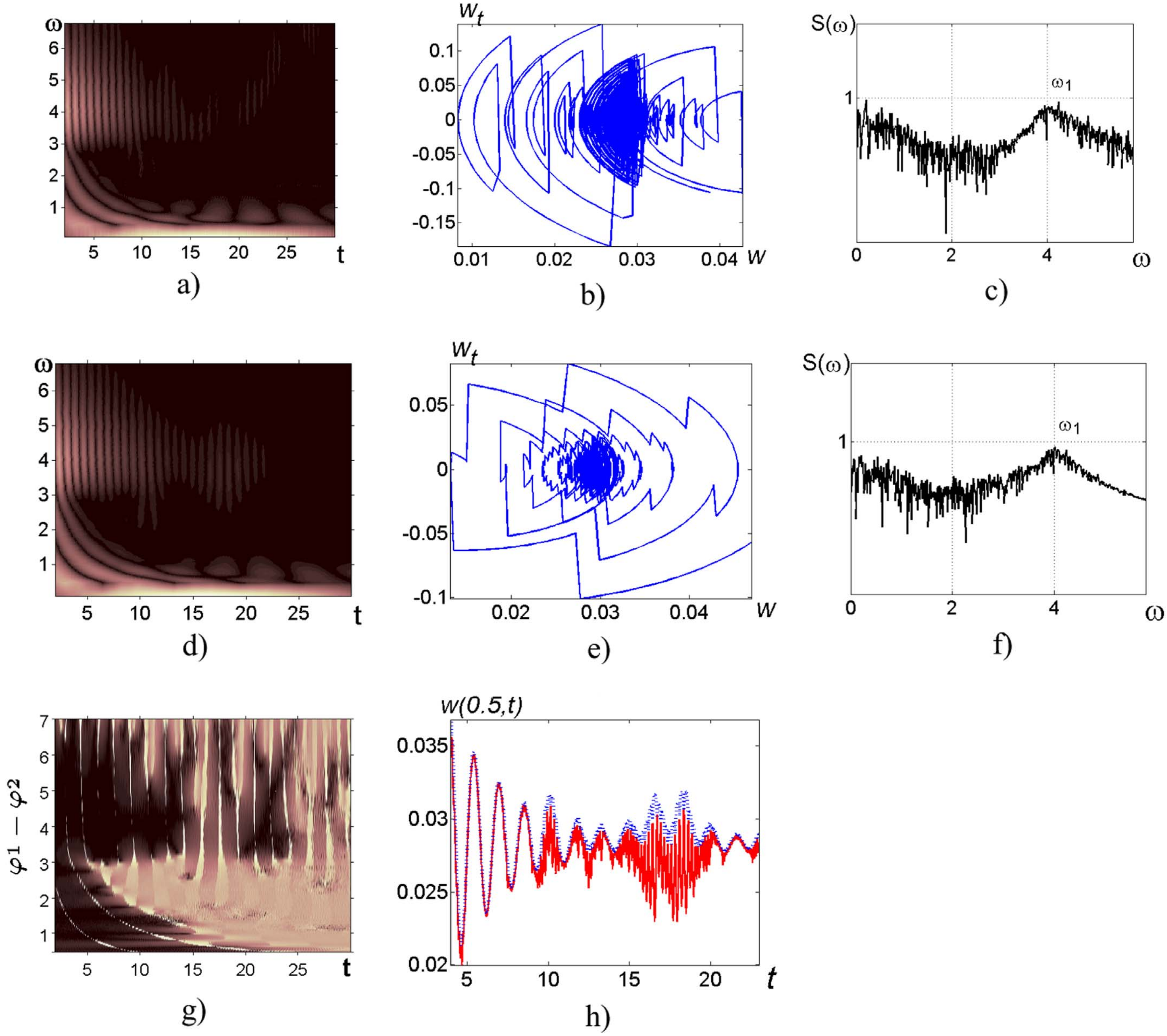

Fig. 2. Plate and beam vibrations for $q = const = 1$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

nization of both structural members is illustrated in Fig. 3, where various dynamic characteristics for the following fixed parameters are presented: $q = q_0 = const = 0.3$, $h_k = 0.01$, $\omega_0^{(1)} = 5.9887$, $\omega_0^{(2)} = 2.84$. An increase in the load implies an increase in the synchronization (Fig. 3g). The vibrations take place on the two-layer system with natural frequency, and a new frequency followed by the Hopf bifurcation appears. Simultaneous frequency locking as well the amplitude locking processes are exhibited in certain time instants. A further increase in the load implies full synchronization regarding the frequency and the amplitude (Fig. 3h).

Our research includes also investigation of the contact interaction of the plate and beam subjected to the action of the harmonic transversal uniformly disturbed load. We limit our considerations here to two cases of the beam location with respect to the plate: (1) *symmetric location* ($y = 0.5$); (2) *asymmetric location* ($y = 0.33$). Here, our aim is the analysis of dynamical phenomena of a two-layer structure composed of the beam and plate and subjected to the continuous load $q = q_0 \sin(\omega_p t)$ for the following fixed parameters: $h_k = 0.01$, $\varepsilon = 1$, $\omega_p = \omega_0 = 5$, $\lambda = 1$, $K = 5000$.

The chosen system of parameters allow us to follow the phase synchronization within regimes of chaotic vibrations. Fig. 4 shows the following characteristics: time histories of the common plate and beam vibrations, phase portraits, Fourier spectra (FFT), and the phase differences (dark colour denotes synchronized frequencies).

In both cases of the beam location regarding the plate centre and for the excitation amplitude $q_0 = 0.065$, a contact between the beams and the plate takes place. Though the plate exhibits harmonic vibrations with $\omega_p = 5$, the beam exhibits damped vibrations. This example plays an important role in our investigations. In spite of the contact between the vibrating objects, we have observed different and easy to follow switching dynamical states since both structural members behave in a qualitatively different manner. An increase in the load up to $q_0 = 0.07$ yields the period tripling bifurcation in agreement with Sharkovskiy's theorem, and the synchronization of the plate and beam occurs (Fig. 4). However, for the load $q_0 = 0.08$, after a period tripling bifurcation the system is transited into a chaotic regime. For $q_0 = 0.1$ the fully developed chaotic vibrations are dominant, the Fourier power spectrum has a broad band base, and the beam vibrations take place

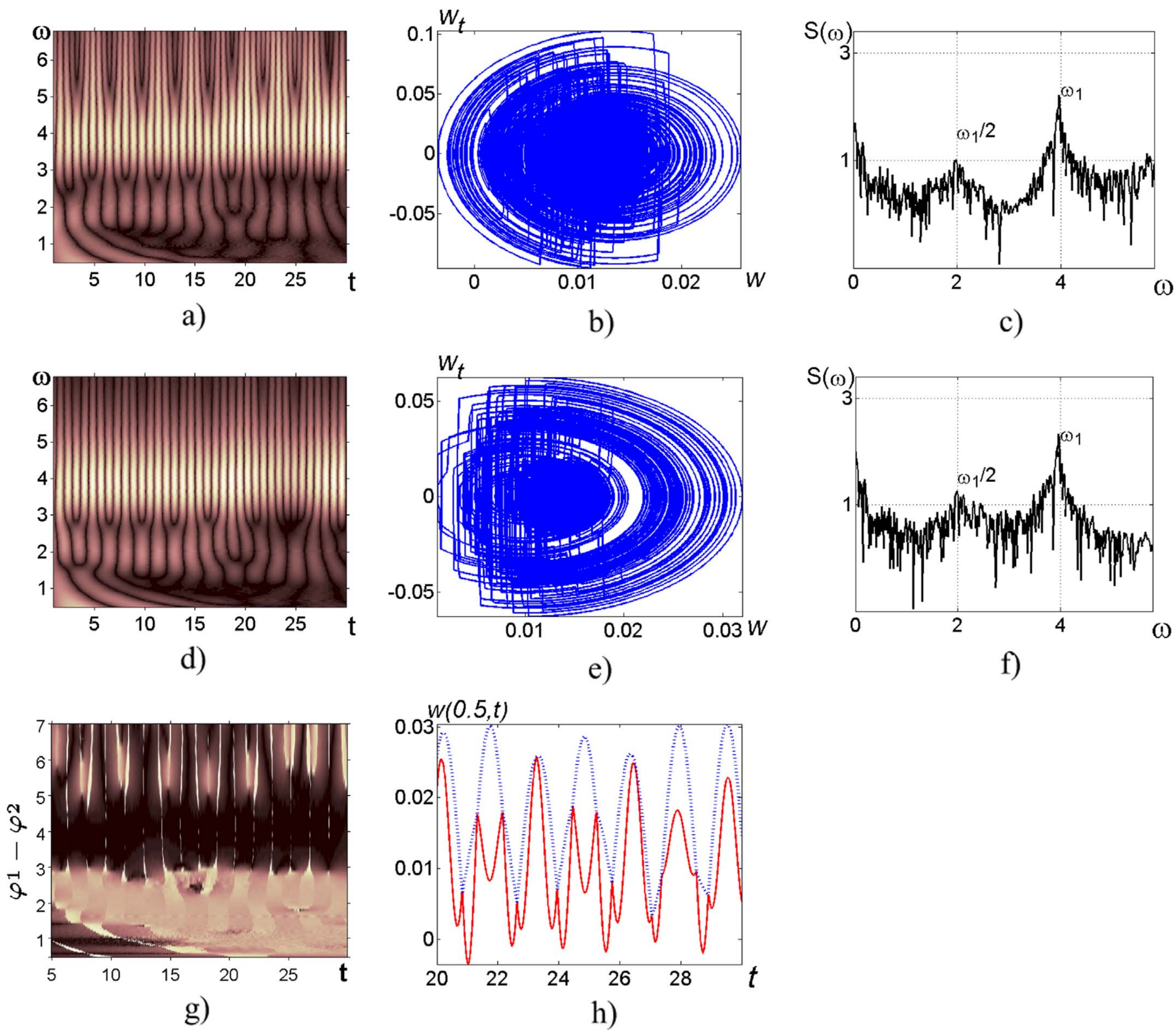

Fig. 3. Fully developed synchronization exhibited by plate vibrations ($q = const = 0.3$).

with the main contribution of frequency $\omega_p = 5$. An increase in the excitation amplitude up to $q_0 = 0.2$ implies the phase loop broadening (Fig. 4), and the plate and beam vibrate in a chaotic manner with dominating frequency of excitation $\omega_p = 5$, which is approved by the Fourier power spectrum and the 2D wavelet-spectrum.

4. Chaotic and synchronized vibrations of a plate and two beams

In this section we study synchronous and chaotic dynamics of the structure composed of the plate and two supporting beams including a clearance (see Fig. 5).

The complex vibrations of the design-type structure consisting of a non-linear plate and two parallel beams are governed by the following system of PDEs:

$$\frac{1}{12(1-\mu^2)}\left(\frac{1}{\lambda^2}\frac{\partial^4 w_1}{\partial x^4} + \lambda^2\frac{\partial^4 w_1}{\partial y^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2}\right) + \frac{\partial^2 w_1}{\partial t^2} + \varepsilon\frac{\partial w_1}{\partial t} - q_1(x, y, t) +$$
$$+ P_{1x}\frac{\partial^2 w_1}{\partial x^2} + P_{1y}\frac{\partial^2 w_1}{\partial y^2} - K(w_1 - w_2 - h_k)\Psi = 0,$$
$$\frac{1}{12}\frac{\partial^4 w_2}{\partial x^4} + \frac{\partial^2 w_2}{\partial t^2} + \varepsilon\frac{\partial w_2}{\partial t} - q_2(x, t) + P_2\frac{\partial^2 w_2}{\partial x^2} + K(w_1 - w_2 - h_k)\Psi = 0,$$
$$\frac{1}{12}\frac{\partial^4 w_3}{\partial x^4} + \frac{\partial^2 w_3}{\partial t^2} + \varepsilon\frac{\partial w_3}{\partial t} - q_3(x, t) + P_3\frac{\partial^2 w_2}{\partial x^2} + K(w_1 - w_3 - h_k)\Psi = 0. \quad (7)$$

The first Equation of the system (7) describes the plate vibrations, whereas the remaining two govern the beams' vibrations. The attached boundary conditions (8) and the zero initial conditions (9) have the following form:

$$w_1 = 0;\ \ \frac{\partial^2 w_1}{\partial x^2} = 0;\ \ \text{for } x = 0; 1;\ \ w_1 = 0;\ \ \frac{\partial^2 w_1}{\partial y^2} = 0;\ \ \text{for } y = 0; 1;$$
$$w_2 = 0;\ \ \frac{\partial^2 w_2}{\partial x^2} = 0;\ \ \text{for } x = 0; 1;\ \ w_2 = 0;\ \ \text{for } y = 0; 1; \quad (8)$$

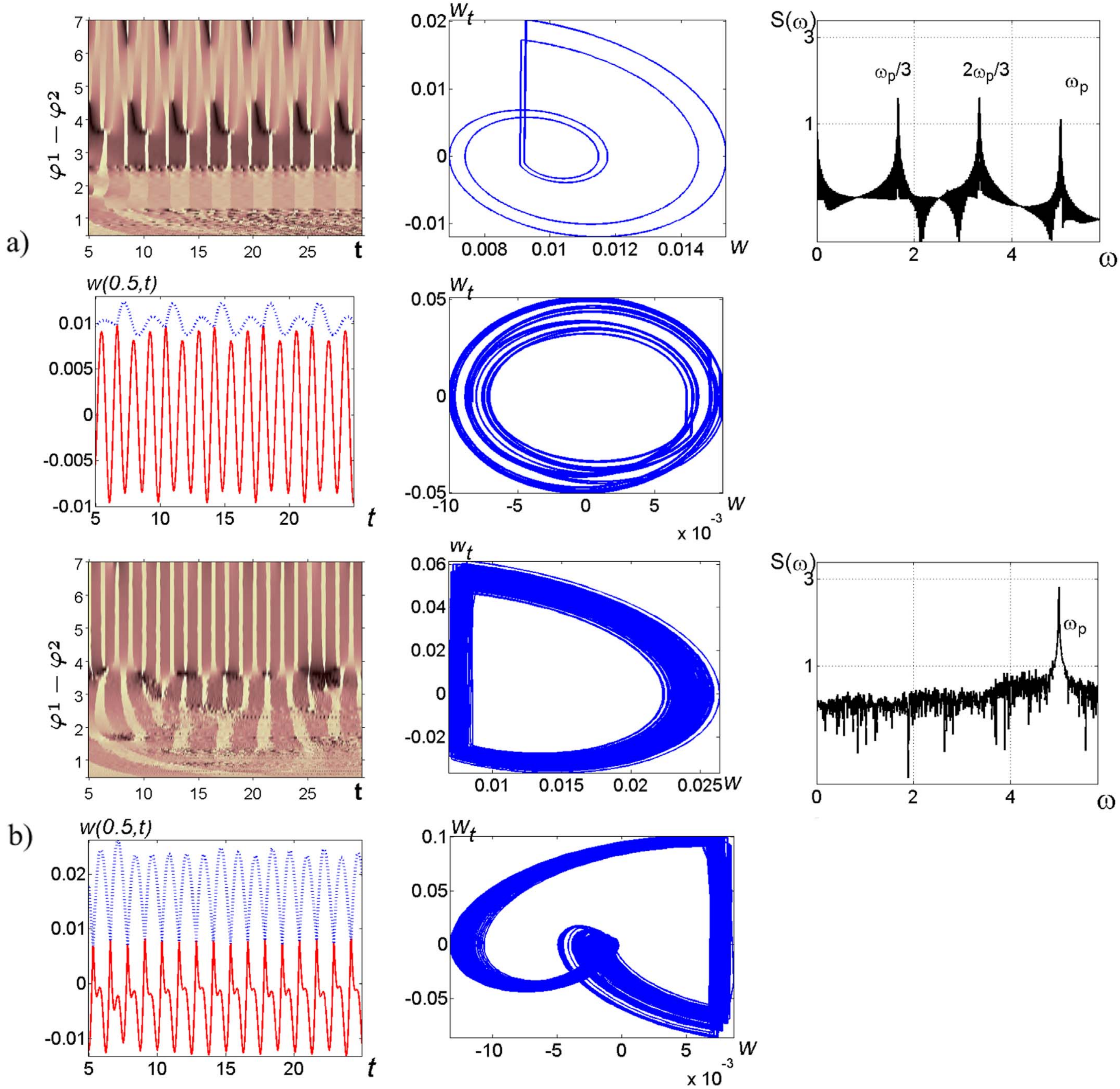

Fig. 4. Contact interaction of the plate and beam located in the beam centre for $q_0 = 0.07$ (a) and $q_0 = 0.2$(b). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

$$w_3 = 0; \quad \frac{\partial^2 w_3}{\partial x^2} = 0; \quad \text{for } x = 0;\ 1; \quad w_3 = 0; \quad \text{for } y = 0;\ 1;$$
$$w_1(x, y)|_{t=0} = 0, \quad w_2(x)|_{t=0} = 0, \quad w_3(x)|_{t=0} = 0, \quad \frac{\partial w_m}{\partial t}\Big|_{t=0} = 0,$$
$$m = 1, 2, 3. \tag{9}$$

In order to present novel non-linear phenomena of the two-layer structure consisting of the plate and two beams and subjected to the action of transversal continuous load $q = q_0 \sin(\omega_p t)$, the following fixed parameters are taken: $h_k = 0.01$, $\varepsilon = 1$. In this case we may detect phase synchronization of the mentioned structural members with the chaotic vibrations.

Two parallel beams are asymmetrically located with respect to the plate centre ($y = 0.2$ and $y = 0.6$) and there is a clearance between each of the beam and the plate. In the initial contact ($q_0 = 0.065$) of the plate and beams, the plate vibrations exhibit the excitation frequency $\omega_p = 5$, whereas the beam vibrations are damped. There is a phase difference between the vibrating objects. An increase in the excitation amplitude ($q_0 = 0.07$) changes qualitatively the vibrational regimes. Namely, the plate vibrations after period tripling bifurcations become chaotic, whereas vibrations of the second beam consisting the natural frequency ω_0 are damped (not shown in Fig. 6). In the mentioned Fig. 6 the time histories (a1, a2), phase portraits (b1, b2), Fourier power spectra (c1, c2), as well as 2D (d1, d2) wavelet-spectra are presented for the first beam and the plate, respectively. In addition, the phase difference of the plate and the first beam (e1, e2) as well as time histories of structural members vibration (f) are shown.

For $q_0 = 0.2$ the plate and the second beam exhibit chaotic vibrations with the excitation frequency $\omega_p = 5$, and the Fourier frequency

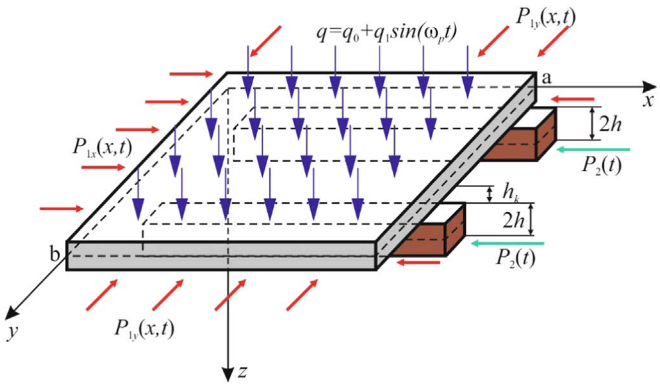

Fig. 5. The plate and two supporting beams with a clearance.

power spectra have broad band bases, the phase portraits exhibit loops, whereas vibrations of the first beam are damped. Here, we deal with a qualitatively different contact/no-contact dynamical behaviour. On the contrary to the earlier example, the first beam exhibits damped oscillations, whereas two other structural members exhibit chaotic dynamics. It means that the beams are now separated with respect to non-linear dynamics (one vibrates in a chaotic manner, whereas the second one exhibits damped vibrations).

Next, we have studied vibrations of the structure, when beams are located symmetrically with respect to the plate centre (not reported here in the graphical form). When the first contact appears between the plate and the beams ($q_0 = 0.065$), each of the members vibrate in a periodic manner. Increasing the external amplitude up to $q_0 = 0.07$, the qualitative reconstruction of the system vibrations takes place, and chaotic vibrations occur followed by the period tripling bifurcations. Beams vibrate in a synchronous way for the same subjected loads and the phase difference implies that the synchronization is realized at the beam excitation frequency, i.e. harmonic and subharmonic parts of ω_p play a crucial role in the chaotic dynamics.

Here, another example of an interesting non-linear phenomenon is given. Both beams are synchronized (see all reported characteristics including the Morlet wavelets), whereas the plate is not synchronized with dynamics of two beams.

A further increase in the excitation load amplitude implies the qualitative change of dynamics ($q_0 = 0.2$), i.e. a loop is seen on the phase portrait regarding plate, whereas the broad band bases on the Fourier spectra are visible, and the phase portraits of two beams vibrations are different. Therefore, for $q_0 = 0.2$ we have asymmetric forms of vibrations of two beams.

5. Chaos and switching dynamics of the plate and three beams

In this section we study the plate supported by three parallel beams. The structure is governed by the system of PDEs similar to that already studied in the previous sections. We consider the case when the structure consists of the plate and three beams located asymmetrically with respect to the plate centre ($y = 0.2, y = 0.4$ and $y = 0.7$), where the gap $h_k = 0.01$ (see Fig. 7).

The governing equations have the following form

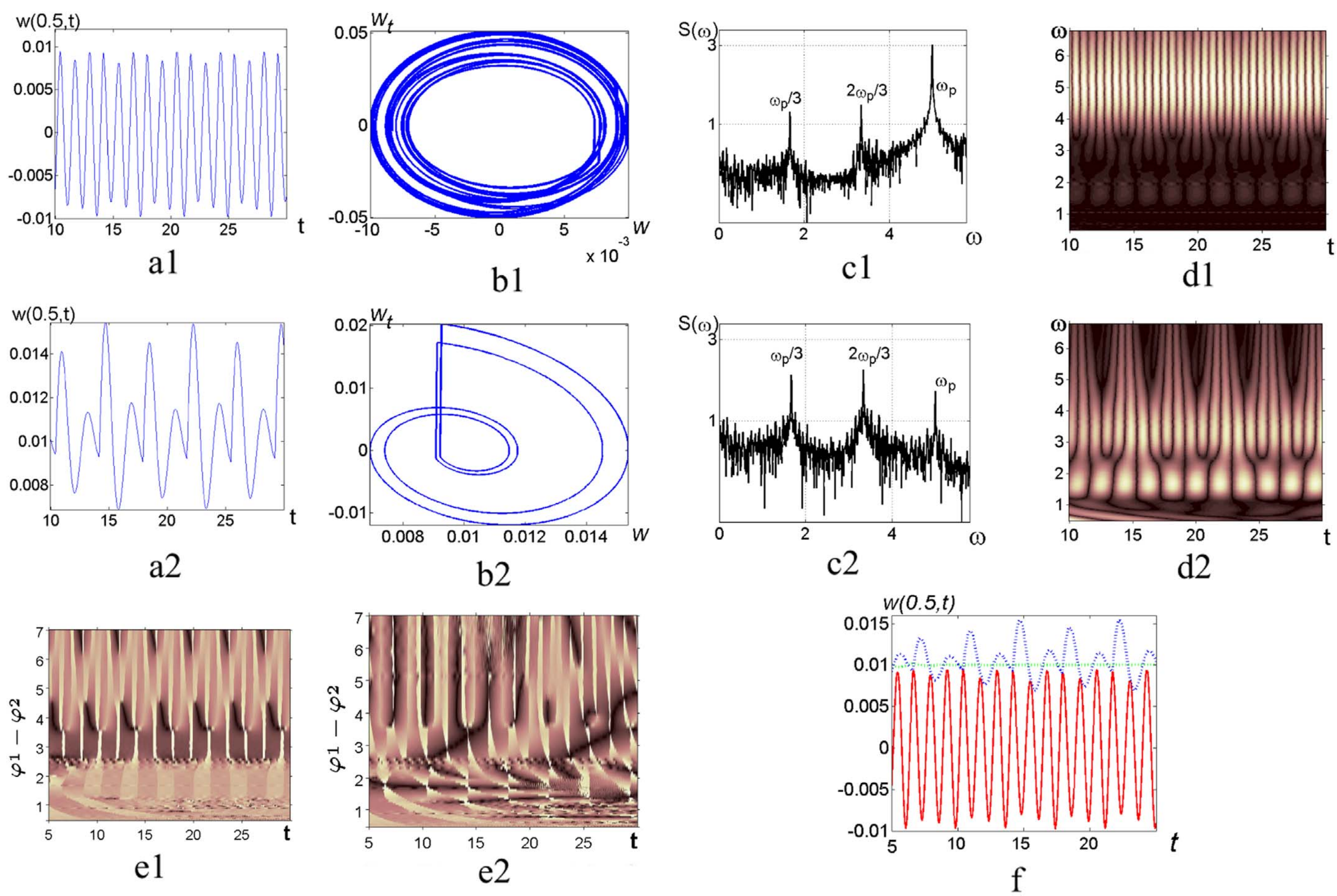

Fig. 6. Contact interaction of the plate and two asymmetrically located beams for $q_0 = 0.07$ (see text for more details).

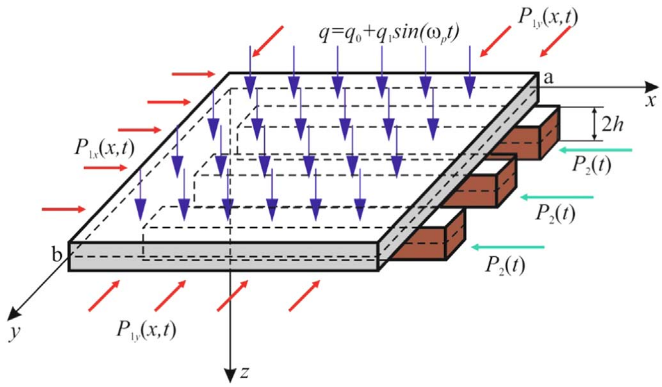

Fig. 7. A multi-layer structure: a plate supported by three beams.

$$\frac{1}{12(1-\mu^2)}\left(\frac{1}{\lambda^2}\frac{\partial^4 w_1}{\partial x^4}+\lambda^2\frac{\partial^4 w_1}{\partial y^4}+2\frac{\partial^4 w_1}{\partial x^2\partial y^2}\right)+\frac{\partial^2 w_1}{\partial t^2}+\varepsilon\frac{\partial w_1}{\partial t}-q_1(x,y,t)+P_{1x}\frac{\partial^2 w_1}{\partial x^2}+P_{1y}\frac{\partial^2 w_1}{\partial y^2}-$$
$$-K(w_1-w_2-h_k)\Psi_1-K(w_1-w_3-h_k)\Psi_2-K(w_1-w_4-h_k)\Psi_3=0,$$
$$\frac{1}{12}\frac{\partial^4 w_2}{\partial x^4}+\frac{\partial^2 w_2}{\partial t^2}+\varepsilon\frac{\partial w_2}{\partial t}-q_2(x,t)+P_2\frac{\partial^2 w_2}{\partial x^2}+K(w_1-w_2-h_k)\Psi_1=0,$$
$$\frac{1}{12}\frac{\partial^4 w_3}{\partial x^4}+\frac{\partial^2 w_3}{\partial t^2}+\varepsilon\frac{\partial w_3}{\partial t}-q_3(x,t)+P_3\frac{\partial^2 w_2}{\partial x^2}+K(w_1-w_3-h_k)\Psi_2=0,$$
$$\frac{1}{12}\frac{\partial^4 w_4}{\partial x^4}+\frac{\partial^2 w_4}{\partial t^2}+\varepsilon\frac{\partial w_4}{\partial t}-q_4(x,t)+P_4\frac{\partial^2 w_2}{\partial x^2}+K(w_1-w_4-h_k)\Psi_3=0, \quad (10)$$

and the functions describing the contact between structural members are as follows:

$$\psi_1=\frac{1}{2}[1+sign(w_1-h_k-w_2)],\quad \psi_2=\frac{1}{2}[1+sign(w_1-h_k-w_3)],$$
$$\psi_3=\frac{1}{2}[1+sign(w_1-h_k-w_4)].$$

We apply simple support (11) and zero initial conditions (12):

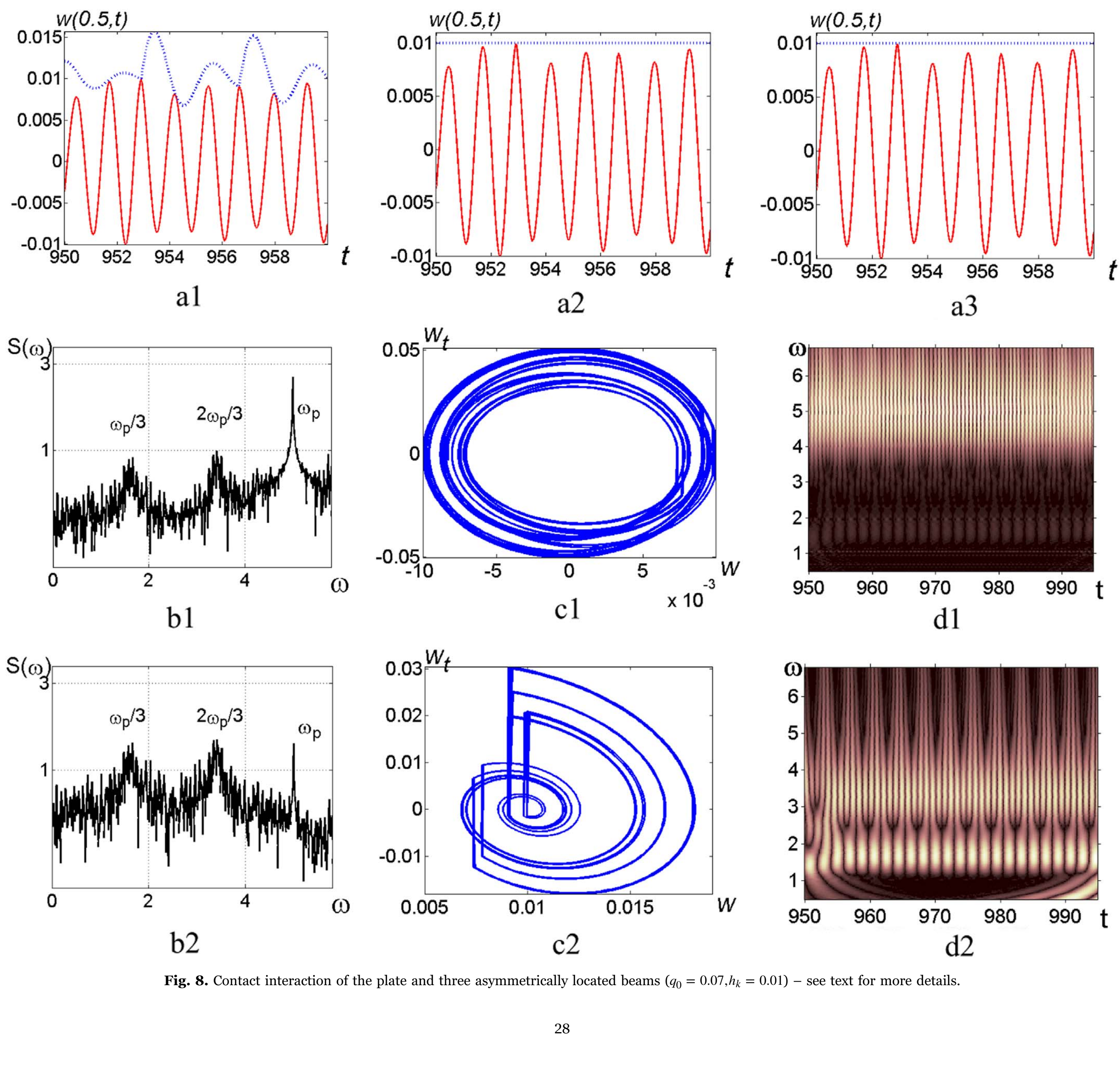

Fig. 8. Contact interaction of the plate and three asymmetrically located beams ($q_0=0.07, h_k=0.01$) – see text for more details.

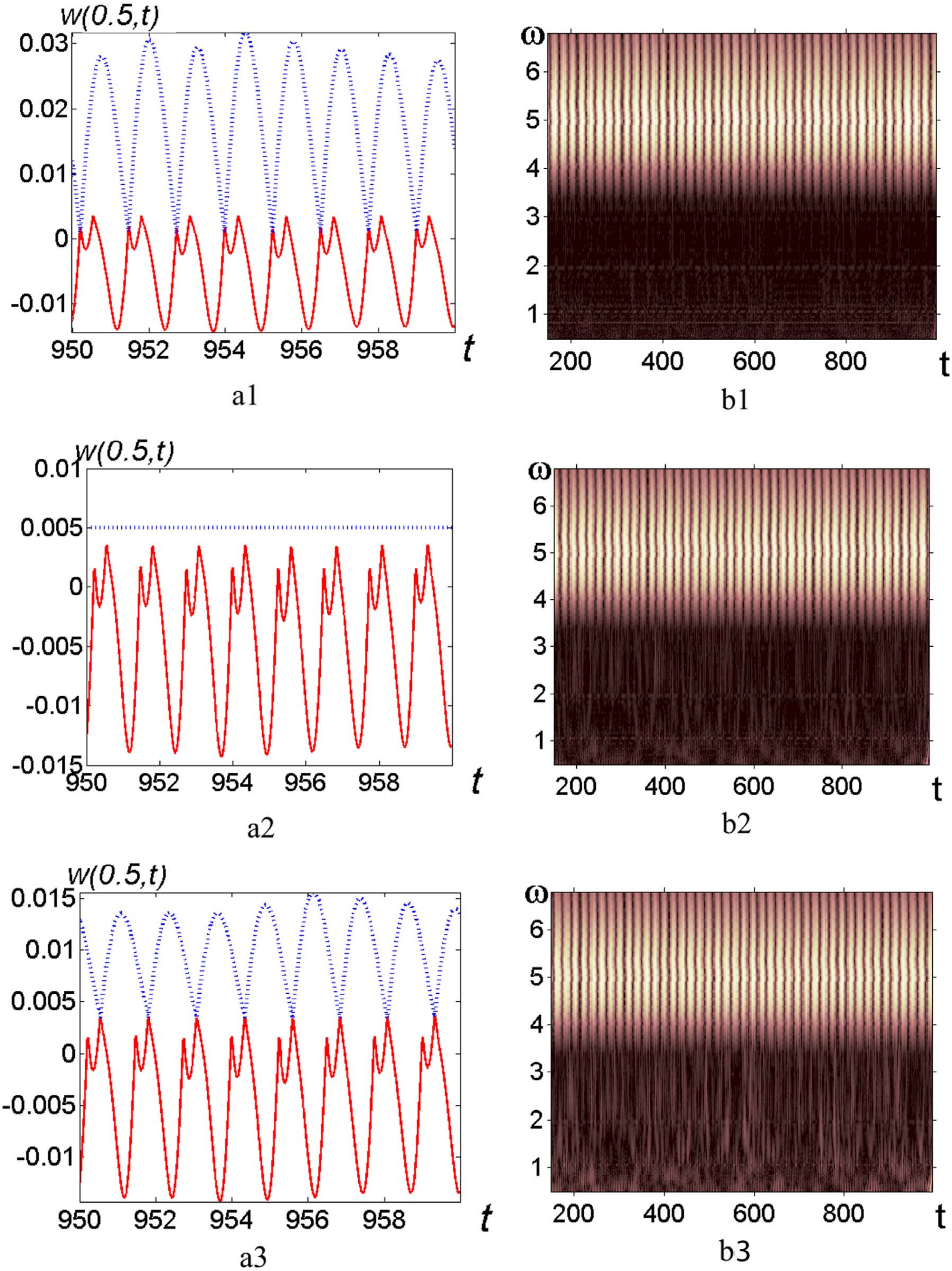

Fig. 9. Contact interaction of the plate and three asymmetrically located beams ($q_0 = 0.195, h_k = 0.005$).

$$w_m = 0; \quad \frac{\partial^2 w_m}{\partial x^2} = 0; \ \text{for } x = 0;\ 1; \ w_m = 0; \quad \frac{\partial^2 w_m}{\partial y^2} = 0; \quad \text{for } \ y = 0;\ 1;$$
$$m = 1,\ 2,\ 3,\ 4, \tag{11}$$

$$w_1(x,\ y)|_{t=0} = 0, \quad w_2(x)|_{t=0} = 0, \quad w_3(x)|_{t=0} = 0, \quad \left.\frac{\partial w_m}{\partial t}\right|_{t=0} = 0,$$
$$m = 1,\ 2,\ 3,\ 4, \tag{12}$$

corresponding to the plate and beams. For $q_0 = 0.065$ a contact between the plate and beams occurs, vibrations take place at the excitation frequency $\omega_p = 5$, whereas vibrations of the beams are damped. For $q_0 = 0.07$ a change in the vibrations of the plate and the first beam appears: the structure dynamics, after period tripling (Fig. 8 (b1, b2)) is transited into chaos (the second and third beams are in equilibrium states). In this case a different dynamical state is illustrated. Namely, either of the two beams may behave in a qualitatively different manner (the second and third beams exhibit periodic oscillations) with respect to the remaining structural members vibration (the first beam and the plate vibrate chaotically). Fig. 8 shows the Fourier power spectra (b1, b2), phase portraits (c1, c2), 2D wavelet-spectra (d1, d2) for the plate and first beam, respectively. The picture of common vibrations of the plate with each of the beams: the first $y = 0.2$ (a1), the second $y = 0.4$ (a2) and the third $y = 0.7$ (a3), are reported (solid curve–plate, dotted curve–beam).

There are also further interesting dynamical regimes of the studied plate-beams vibrations. An increase in the excitation load amplitude yields periodic vibrations of the plate with each of the edge-located beams in an alternate manner. For $q_0 \in [0.074; 0.09]$ each layer of the mechanical structure vibrates (plate and three beams). The 2D Morlet wavelet-spectra exhibit intermittency zones as well a window of *switching on/switching off* of the frequencies depending on the chosen various time intervals (not reported here). Next, further increase in the load again exhibits the *alternate contact dynamics* of the plate vibrations with one of the three beams.

For $q_0 = 0.2$ the plate and the second beam vibrate chaotically with the dominant frequency $\omega_p = 5$, the Fourier spectra have broad band bases, the phase portrait of the plate presents a broadening loop, whereas vibrations of the first beam are damped. Beginning with $q_0 = 0.31$, the plate interacts with two external beams simultaneously but in the alternate manner ($y = 0.2$ $y = 0.7$).

Next, we are going to study the influence of the size of the clearance between the layers, and we take fixed value of $h_k = 0.005$. For the load amplitude $q_0 \in [0.06; 0.073]$ periodic vibrations are exhibited by the plate and one of the external beams ($y = 0.2$). Contact interaction appears. For load amplitude $q_0 \in [0.074; 0.096]$ the plate and middle beam ($y = 0.4$) vibrate, whereas two remaining beams are at rest.

Then, we report the contact/no-contact dynamical states, i.e. the states with switching of the subsets of interacting objects. A further increase in the load amplitude yields the occurrence of contact interactions of the plate with each beam successively from the left ($y = 0.2$) to the right ($y = 0.7$), and vice versa. In the interval of $q_0 \in [0.102; 0.148]$ vibrations of the external beams appear alternatively between external beams ($y = 0.2$ and $y = 0.7$). Beginning with$q_0 = 0.149$, a simultaneous contact interaction of the plate and two beams takes place. Fig. 9 shows the 2D Morlet wavelet-spectra for the plate, the first and the third beams, respectively (b1–b3). Time histories of the common vibrations of the plate and each of the beams: the first $y = 0.2$ (a1), the second $y = 0.4$ (a2), the third $y = 0.7$ (a3), are also shown (solid curve-plate, dotted curve-beam).

6. Concluding remarks

We have proposed a general approach/algorithm to investigate the previously introduced mathematical models of complex/chaotic vibrations of plates supported by flexible beams taking into account clearances between interacting structural members (here one plate and beams). A few problems important for both theoretical and applied mechanics have been studied. They include the contact interaction of structural members when the upper layer is subjected to harmonic continuous and uniformly distributed load, and the occurrence of phase synchronization of the multi-layer structure where the layers are coupled with each other only via boundary conditions. In the case of the contact interaction of a beam and plate with a small clearance the period tripling bifurcation scenario following Sharkovskiy's order, and yielding eventually chaotic dynamics has been detected and illustrated. We have also discussed and discovered an important property of non-linear dynamics of the plate and two parallel beams with small clearances between contacting layers. Namely, we have demonstrated that for different amplitudes of the exciting plate loads the beams vibrate in the alternate manner.

Acknowledgements

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