



Investigating geometrically nonlinear vibrations of laminated shallow shells with layers of variable thickness via the R-functions theory



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ABSTRACT

A novel numerical/analytical approach to study geometrically nonlinear vibrations of shells with variable thickness of layers is proposed. It enables investigation of shallow shells with complex forms and different boundary conditions. The proposed method combines application of the R-functions theory, variational Ritz's method, as well as hybrid Bubnov–Galerkin method and the fourth-order Runge–Kutta method. Mainly two approaches, classical and first-order shear deformation theories of shells are used. An original scheme of discretization regarding time reduces the initial problem to the solution of a sequence of linear problems including those related to linear vibrations with a special type of elasticity, as well as problems governed by non-linear system of ordinary differential equations. The proposed method is validated by the investigation of test problems for shallow shells with rectangular planform and applied to new vibration problems for shallow shells with complex planforms and variable thickness of layers.

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1. Introduction

The theory of multi-layer shallow shells and plates is widely used for mathematical modeling of thin-walled designs of elements made of advanced composite materials. Many of the constructive and nonlinear factors should be taken into account in order to provide a more accurate mathematical description of the real physical and mechanical processes. These features include geometric nonlinearity, mechanical properties of composite materials, ways of stacking, variable thickness of shell or variable thickness of layers, etc.

The vast literature on the nonlinear vibration of multilayer plates and shallow shells having uniform thickness of layers is available. This problem is a subject of many publications [1,2,6,13,16,18,19]. A review of achievements in this field is presented in works [2,22,24,29]. As follows from the review, there is a small number of papers dedicated to the problem of the nonlinear vibrations of multi-layer plates and shells with variable thickness of layers [4,5,8,28]. A more complex dynamic behavior of the

plates and shells caused by the interactive influence of layers and variable thickness is expected. This requires both more advanced modeling of governing equations and novel proposals to find their solutions. In order to solve this problem appropriately we need first to solve a linear problem associated with multi-layer plates and shells with variable thickness of the layers. Some results for multi-layered annular circular plate have been reported in Ref. [28]. In that work a survey concerning this problem is presented first for circular and annular plates. It should be noted that even linear vibrations of multi-layer plates and shells with variable thickness of layers have not been sufficiently studied [8].

In this paper the algorithm of meshless discretization, based on a combination of the classical approaches and modern constructive tools of the R-functions theory [23] is proposed. Application of the R-functions theory allows us to study geometrically nonlinear dynamic response of the laminated plates and shallow shells with complex shapes, different boundary conditions and non-constant thickness of the layers.

Recently many engineering oriented researchers are focused on modeling, analysis and manufacturing of the tapered laminated structures. Taper configurations may include external plies of rectangular shapes and mid-plane plies of triangle shapes, whereas the internal tapered structure parts may consist of basic triangle shape plies as well as the plies arrange in a staircase way, the overlapping

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dropped plies and continuous plies interspersed. The tapered laminated structures have found wide applications in mechanical and civil engineering as well as in aerospace and aeronautics (wing and tin skin structures, helicopter rotor blades, etc.) because of their weight savings, damage tolerance and structural tailoring capabilities. It means that the interest of such type laminated panels comes from the needs of the mentioned industry branches, and in particular from the aircraft industry. In general, the constant thickness laminates do not allow for stiffness tailoring, and hence tapered layers of variable thickness are required keeping simultaneously the thickness of the shell constant [20,21].

The state-of-art and review of recent developments in the analysis of tapered laminated composite structures taking into account the interlaminar stress analysis, delamination analysis and parametric study has been reported by He et al. [11]. Both analytical and numerical methods have been employed to solve the ply drop-off problem with an emphasis on the stress fields in the presence of interface cracks by Her [12]. The latter knowledge is required to assess the overall strength of the total laminate and to optimize the ply drop-off design.

A modified shear-lag model has been developed and implemented to study tensile stresses and delamination of a composite laminate with drop-off plies by He et al. [10]. In particular, in the case of the laminate without drop-off layers the fiber layers and the resin layers with linearly variable layers thickness have been studied.

The technology of manufacturing the so far described laminates are addressed in Refs. [7,27]. It has been pointed out that components of the composite laminates vary in thickness, and in many cases it is not possible to taper the laminates continuously. The tapering process ends by terminating one or more of the internal plies. Though in our paper we do not exactly study, the mentioned geometrical imperfections, but we analyze in the given examples linear variation of the layers thickness.

Industrial design guidelines for composite structures with ply-drops are discussed by Irisarri et al. [14]. The proposed method of stacking, sequence, table devoted to the optimal design of laminate composite structures with ply drops allowed to solve industrial problems including preventing unwanted coupled behavior, avoiding delamination at ply-drop location, obtaining the ply layouts allowed for manufacturing, as well as to keep ply continuity and smooth load redistribution over the studied structure.

2. Problem statement

A laminated shallow shell of an arbitrary planform composed of M layers of variable thickness is shown in Fig. 1. To investigate geometrically nonlinear vibration of the shell we use both the first-order shear deformation theory (FSDT) and the classical shell theory (CST). According to these theories it is assumed that tangent displacements are the linear functions of coordinate z , whereas transverse displacement w is constant through the shell thickness. While the CST adopts Kirchhoff's hypothesis, FSDT does not adopt it. Observe that in the latter case it is assumed that the mid surface normal to the shell remains straight after deformation, but not necessarily normal to the middle surface. Further, we consider symmetric composite-laminated shallow shells.

According to the considered theories of shells, in-plane displacements u , v and transverse displacement w may be given in the following form [1,9]:

$$u = u_0 + z\psi_x, \quad v = v_0 + z\psi_y, \quad w = w_0, \quad (1)$$

where u_0 , v_0 and w_0 are the displacements at the midsurface, ψ_x and ψ_y are the rotations perpendicular to the midsurface about the y - and x -axes, respectively. The non-linear strain-displacement relations can be written as follows

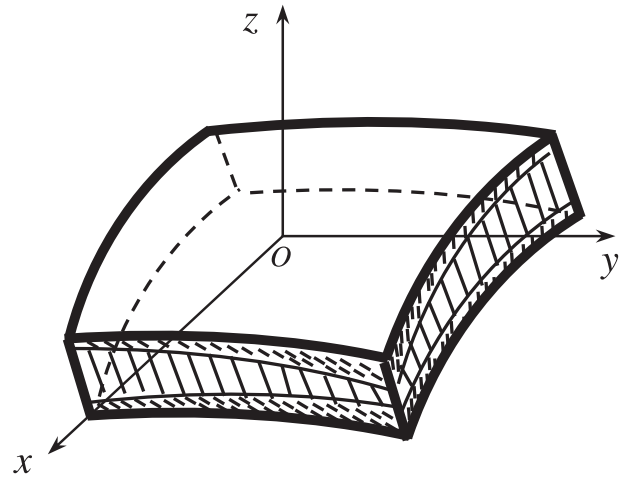


Fig. 1. Shallow shell with layers of variable thickness.

$$\varepsilon_{11} = \varepsilon_{11}^0 + z\chi_{11}, \quad \varepsilon_{22} = \varepsilon_{22}^0 + z\chi_{22}, \quad \varepsilon_{33} = 0, \quad \varepsilon_{12} = \varepsilon_{12}^0 + z\chi_{11}, \quad (2)$$

where

$$\varepsilon_{11}^0 = u_{0,x} + \frac{w_0}{R_x} + \frac{1}{2}w_{0,x}^2, \quad \varepsilon_{22}^0 = v_{0,y} + \frac{w_0}{R_y} + \frac{1}{2}w_{0,y}^2, \quad (3)$$

$$\varepsilon_{12}^0 = u_{0,y} + v_{0,x} + w_{0,x}w_{0,y}, \quad (4)$$

$$\chi_{11} = \delta\psi_{x,x} - (1 - \delta)w_{xx}, \quad \chi_{22} = \delta\psi_{y,y} - (1 - \delta)w_{yy},$$

$$\chi_{12} = \delta(\psi_{x,y} + \psi_{y,x}) - 2(1 - \delta)w_{xy}. \quad (4)$$

Indicator δ is the tracing constant which takes values 1 and 0 for the FSDT and CST, respectively, and subscripts following a comma stand for partial differentiation.

Let us present strain and moment resultants $F = \{N_{11}, N_{22}, N_{12}, M_{11}, M_{22}, M_{12}\}^T$ in the matrix form $F = [A] \cdot \varepsilon^0$, (5)

in which components of vector $\varepsilon^0 = \{\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0, \chi_{11}, \chi_{22}, \chi_{12}\}^T$ are defined by formulas (3) and (4). Matrix A takes the following form

$$[A] = \begin{bmatrix} [C] & 0 \\ 0 & [D] \end{bmatrix}, \quad (6)$$

whereas matrices C and D are defined as follows

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}, \quad [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}. \quad (7)$$

The shear stress resultants of the composite shallow shell can be expressed in the following way

$$Q = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} C_{55} & C_{54} \\ C_{45} & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix}, \quad (8)$$

$$\varepsilon_{13} = \delta \left(w_{0,x} + \psi_x - \frac{u_0}{R_x} \right), \quad \varepsilon_{23} = \delta \left(w_{0,y} + \psi_y - \frac{v_0}{R_y} \right) \quad (9)$$

Elements C_{ij} and D_{ij} in the above given equations are the stiffness coefficients of the shell which are defined by the following formulas [3]:

$$(C_{ij}(x, y), D_{ij}(x, y)) = \sum_{m=1}^M \int_{h_m(x, y)}^{h_{m+1}(x, y)} B_{ij}^{(m)}(1, z^2) dz, \quad i, j = 1, 2, 3, \quad (10)$$

$$C_{ij}(x, y) = K^2 \sum_{m=1}^M \int_{h_m(x, y)}^{h_{m+1}(x, y)} B_{ij}^{(m)} dz, \quad i, j = 4, 5. \quad (11)$$

Here, stiffness matrix elements $B_{ij}^{(m)}$ express the stress-strain relation in the m -th layer, K^2 is the shear correction factor, and h_m is the distance from the midsurface to the upper surface of

the m th layer. While solving the concrete problems, the shear correction K^2 is taken equal to $\frac{5}{6}$. It should be emphasized that for the given class of problems the stiffness coefficients C_{ij} and D_{ij} are the functions of (x, y) .

In what follows we apply the method proposed in Ref. [18]. According to this approach, the aim of the first step is to study a linear problem in order to find the natural frequencies and eigenfunctions $U^{(c)} = \{u^{(c)}, v^{(c)}, w^{(c)}, \psi_x^{(c)}, \psi_y^{(c)}\}^T$ that satisfy the given boundary conditions.

3. Solution to linear problem

We are going to find a solution to linear problems for laminated shells with variable thickness of layers using RFM [17,23]. Note that while solving the linear problem, all inertial forces, including membrane and rotary ones, are taken into account. Since we deal with linear vibrations, the problem can be reduced to a variational problem oriented towards finding a stationary value of the following Lagrange functional

$$J = U_{\max} - T_{\max}, \tag{12}$$

where U_{\max} and T_{\max} are the maximum strain and kinetic energies of the shell, respectively. In the case of the linear vibration analysis, both energies can be found using the standard approaches described in Refs. [1,9]:

$$U_{\max} = \frac{1}{2} \iint_{\Omega} [N_{11} \varepsilon_{11}^0 + N_{22} \varepsilon_{22}^0 + N_{12}^0 \varepsilon_{12} + M_{11} \chi_{11} + M_{22} \chi_{22} + M_{12} \chi_{12} + \delta(Q_x \varepsilon_{13} + Q_y \varepsilon_{23})] d\Omega, \tag{13}$$

$$T_{\max} = \frac{\lambda^2}{2} \iint_{\Omega} \rho(x, y) (h(x, y) (u_0^2 + v_0^2 + w_0^2) + \delta \frac{h^3(x, y)}{12} (\psi_x^2 + \psi_y^2)) d\Omega. \tag{14}$$

In order to minimize functional (12), Ritz's method is used provided that a system of basic functions is built by R-functions theory. However, in order to get the basic functions, the corresponding solution structures should be proposed [23]. Examples, how to define the mentioned solution structures are presented, illustrated and widely discussed in Refs. [4,17–19,23], and therefore this description is omitted in this work.

4. Solution to nonlinear problem

In the case of geometrically nonlinear problem, the approach proposed in earlier works is applied [4,18,19]. Namely, the unknown functions are presented in the following way:

$$\begin{aligned} w &= \sum_{i=1}^n y_i(t) w_i^{(c)}(x, y), \quad \psi_x = \delta \sum_{i=1}^n y_i(t) \psi_{xi}^{(c)}(x, y), \\ \psi_y &= \delta \sum_{i=1}^n y_i(t) \psi_{yi}^{(c)}(x, y), \\ u &= \sum_{i=1}^n y_i(t) u_i^{(c)}(x, y) + \sum_{i=1}^n \sum_{j=1}^n y_i y_j u_{ij}, \\ v &= \sum_{i=1}^n y_i(t) v_i^{(c)}(x, y) + \sum_{i=1}^n \sum_{j=1}^n y_i y_j v_{ij}, \end{aligned} \tag{15}$$

where $y_k(t)$ are the unknown functions to be defined further, $w_i^{(c)}(x, y), u_i^{(c)}(x, y), v_i^{(c)}(x, y), \psi_{xi}^{(c)}(x, y), \psi_{yi}^{(c)}(x, y)$ are the components of the i -th eigenfunctions.

Note that functions u_{ij}, v_{ij} should satisfy the following relations

$$\begin{aligned} L_{11}(u_{ij}) + L_{12}(v_{ij}) &= -Nl_1^{(2)}(w_i^{(c)}, w_j^{(c)}), \\ L_{21}(u_{ij}) + L_{22}(v_{ij}) &= -Nl_2^{(2)}(w_i^{(c)}, w_j^{(c)}), \end{aligned} \tag{17}$$

where

$$Nl_1^{(2)}(w_i^{(c)}, w_j^{(c)}) = w_i^{(c)}{}_{,x} L_{11} w_j^{(c)} + w_i^{(c)}{}_{,y} L_{12} w_j^{(c)}, \tag{18}$$

$$Nl_2^{(2)}(w_i^{(c)}, w_j^{(c)}) = w_i^{(c)}{}_{,x} L_{12} w_j^{(c)} + w_i^{(c)}{}_{,y} L_{22} w_j^{(c)}. \tag{19}$$

Operators $L_{11}, L_{22}, L_{12}, L_{21}$ are defined as follows

$$\begin{aligned} L_{11} &= C_{11}(\cdot)_{,xx} + 2C_{16}(\cdot)_{,xy} + C_{66}(\cdot)_{,yy}, \\ L_{22} &= C_{66}(\cdot)_{,xx} + 2C_{26}(\cdot)_{,xy} + C_{22}(\cdot)_{,yy}, \\ L_{12} &= L_{21} = C_{16}(\cdot)_{,xx} + (C_{12} + C_{66})(\cdot)_{,xy} + C_{26}(\cdot)_{,yy}. \end{aligned} \tag{20}$$

The system of Eq. (17) is solved by RFM. Substituting formulas (16) and (17) for functions u, v, w, ψ_x, ψ_y into equations of motion, and applying the Bubnov–Galerkin procedure, we obtain the following nonlinear system of ODEs regarding the unknown functions $y_j(t)$:

$$y_r''(t) + \omega_{Lr}^2 y_r(t) + \sum_{ij=1}^n \beta_{ij}^{(r)} y_i(t) y_j(t) + \sum_{ij,k=1}^n \gamma_{ijk}^{(r)} y_i(t) y_j(t) y_k(t) = 0 \tag{21}$$

Coefficients $\beta_{ij}^{(r)}, \gamma_{ijk}^{(r)}$ have the following form:

$$\begin{aligned} \beta_{ij}^{(r)} &= \frac{-1}{m_1 \|w_r^{(c)}\|^2} \iint_{\Omega} (N_{11}^{(L)}(w_j^{(c)})_{,xx} + N_{22}^{(L)}(w_j^{(c)})_{,yy} + 2N_{12}^{(L)}(w_j^{(c)})_{,xy} \\ &\quad - k_1 N_{11}^{(Np)} - k_2 N_{22}^{(Np)}) w_r^{(c)} d\Omega, \end{aligned}$$

$$\begin{aligned} \gamma_{ijk}^{(r)} &= -\frac{1}{m_1 \|w_r^{(c)}\|^2} \iint_{\Omega} (N_{11}^{(Np)}(u_{ij}, v_{ij}, w_i^{(c)} w_j^{(c)}) (w_k^{(c)})_{,xx} \\ &\quad + N_{22}^{(Np)}(u_{ij}, v_{ij}, w_i^{(c)} w_j^{(c)}) (w_k^{(c)})_{,yy} \\ &\quad + 2N_{12}^{(Np)}(u_{ij}, v_{ij}, w_i^{(c)} w_j^{(c)}) (w_k^{(c)})_{,xy}) w_r^{(c)} d\Omega \end{aligned} \tag{22}$$

$$\begin{aligned} N^{(L)} &= \{N_{11}^{(L)}; N_{22}^{(L)}; N_{12}^{(L)}\}^T = [C] \varepsilon^{(L)}, \quad \{N^{(Np)}\} \\ &= \{N_{11}^{(Np)}; N_{22}^{(Np)}; N_{12}^{(Np)}\}^T = [C] \{\varepsilon^{(Np)}\}, \end{aligned} \tag{23}$$

$$\begin{aligned} \varepsilon^{(L)} &= \varepsilon^{(L)}(u_i^{(c)}, v_i^{(c)}, w_i^{(c)}) \\ &= \left\{ (u_i^{(c)})_{,x} + k_1 w_i^{(c)}; (v_i^{(c)})_{,y} + k_2 w_i^{(c)}; \left((u_i^{(c)})_{,y} + (v_i^{(c)})_{,x} \right) \right\}^T, \end{aligned} \tag{24}$$

$$\begin{aligned} \varepsilon^{(Np)} &= \varepsilon^{(Np)}(u_{ij}, v_{ij}, w_i^{(c)} w_j^{(c)}) = \left\{ (u_{ij})_{,x} + \frac{1}{2} (w_i^{(c)})_{,x} \cdot (w_j^{(c)})_{,x}; (v_{ij})_{,y} \right. \\ &\quad \left. + \frac{1}{2} (w_i^{(c)})_{,y} \cdot (w_j^{(c)})_{,y}; (u_{ij})_{,y} + (v_{ij})_{,x} + (w_i^{(c)})_{,x} (w_j^{(c)})_{,y} \right\}^T. \end{aligned} \tag{25}$$

Solution to the system of second-order ODEs (21) can be found by different analytical methods, such as the harmonic balance method (HBM), multiple scale, method and the Bubnov–Galerkin techniques or via a direct numerical integration using the Runge–Kutta methods. In this paper we apply the Runge–Kutta method.

5. Numerical results and test problems

The developed theoretical approach and worked out software are validated regarding some tested problems. In order to obtain the solutions to the tested problems, we have applied numerical

algorithms suitable for the analysis of laminated shallow shells with a variable thickness of layers. Note that in the present paper we use only a single mode approximation to solve nonlinear problems. A few of the studied problems regarding constant layers are given below.

Problem 1. A test was carried out to compare isotropic cylindrical and spherical shallow shells of square planform studied earlier by Kobayashi and Leissa [15] using the FSDT formulation. The boundary conditions for a thin shell are as follows

$$v = w = M_x = \psi_y = N_x = 0 \quad \text{at} \quad x = \pm \frac{a}{2}, \quad (26)$$

$$u = w = M_y = \psi_x = N_y = 0 \quad \text{at} \quad y = \pm \frac{a}{2}. \quad (27)$$

Poisson's coefficient is $\nu = 0.3$, whereas shell geometric characteristics satisfy the following relations: $h/a = 0.01$; $r_x = R_x/a = 10$; $r_y = R_y/a = (10, \infty)$. Fig. 2 presents the frequency ratio λ_N/λ_L as a function of the non-dimensional maximum positive deflection w_{max}/h for cylindrical and spherical shallow shells. Natural frequency λ_L corresponds to the first bi-symmetric linear mode. In order to solve Eq. (4), the fourth-order Runge–Kutta method has been used. A comparison of the results regarding λ_N/λ_L versus w_{max}/h validates our approach.

Problem 2. We study nonlinear vibrations of thin laminated composite rectangular plates using the CST taking into account the following material properties:

$$E_1/E_2 = 40, \quad G_{12}/E_2 = 0.6, \quad G_{13}/E_2 = 0.5, \quad \nu_{12} = 0.25. \quad (28)$$

All plate layers are of equal thickness and the fiber orientation is measured from X-axis. The following boundary conditions are considered in the present analysis:

- (i) Immovable simply supported case (SSI): $w = 0, M_n = 0, v = 0, u = 0$ at $x = \pm \frac{a}{2}, y = \pm \frac{b}{2}$;
- (ii) Immovable clamped case (CCI): $w = 0, w_{,n} = 0, v = 0, u = 0$ at $x = \pm \frac{a}{2}, y = \pm \frac{b}{2}$.

Table 1

Comparison of nonlinear frequency ratio λ_N/λ_L of laminated cross-ply plates obtained (our method (RFM) and Ref. [26] method).

$\frac{w_{max}}{h}$	$(0^\circ/90^\circ/0^\circ/90^\circ/0^\circ)$			
	SSI		CCI	
	[26]	RFM	[26]	RFM
0.2	1.032	1.031	1.008	1.008
0.4	1.121	1.120	1.034	1.031
0.6	1.257	1.254	1.074	1.068
0.8	1.428	1.420	1.128	1.118
1.0	1.624	1.609	1.195	1.180
1.2	1.837	1.813	1.271	1.251

Table 2

Comparison of nonlinear frequency ratio λ_N/λ_L of laminated angle-ply plates obtained (our method (RFM) vs. that presented in Ref. [26]).

$\frac{w_{max}}{h}$	$(45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ)$			
	SSI		CCI	
	[26]	RFM	[26]	RFM
0.2	1.015	1.015	1.007	1.007
0.4	1.058	1.057	1.028	1.027
0.6	1.126	1.124	1.063	1.061
0.8	1.215	1.212	1.109	1.106
1.0	1.322	1.317	1.165	1.162
1.2	1.442	1.434	1.227	1.225

The relationship between the nonlinear frequency ratio λ_N/λ_L and non-dimensional maximum amplitude w_{max}/h regarding cross-ply $[0^\circ/90^\circ/0^\circ/90^\circ/0^\circ]$ and angle-ply $[45^\circ/-45^\circ/45^\circ/-45^\circ/45^\circ]$ of thin square plates ($a/b = 1, h/a = 0.01$) are reported in Tables 1 and 2, respectively.

Again, the comparison of results reported in Tables 1 and 2 regarding nonlinear frequency of the considered vibrated plates validates our approach.

Problem 3. Here, non-linear free vibrations of laminated shallow shells with rigidly clamped edges are studied. It is assumed that the shells consist of graphite-epoxy layers and each layer has the same thickness and the following material properties hold:

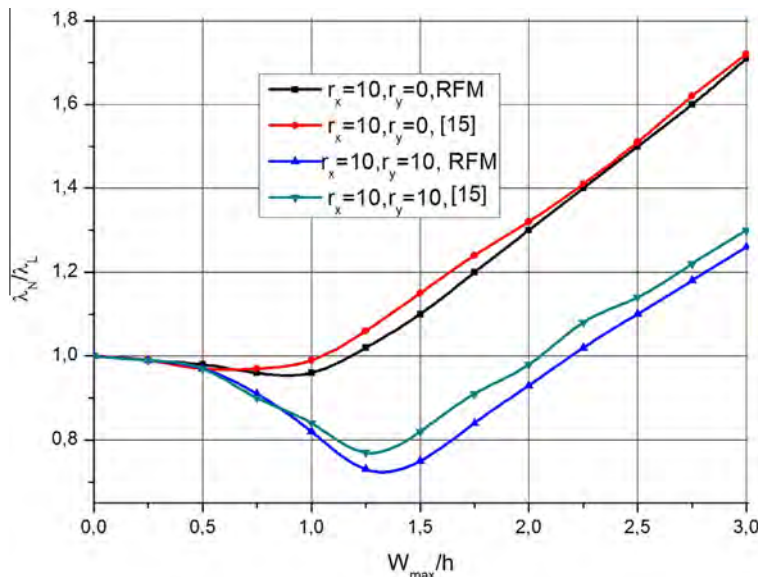


Fig. 2. Non-dimensional frequency λ_N/λ_L versus w_{max}/h of non-linear vibration cylindrical ($r_x = R_x/a = 10$; $r_y = R_y/a = \infty, 1/r_y = 0$) and spherical ($r_x = R_x/a = 10$; $r_y = R_y/a = 10$) isotropic panels (our results (RFM) are compared to those reported in Ref. [15]).

$$E_1 = 138 \text{ GPa}, \quad E_2 = 8.96 \text{ GPa}, \quad G_{12} = 7.1 \text{ GPa},$$

$$G_{13} = G_{23} = E_1/2, \quad \nu_{12} = 0.3. \quad (29)$$

The computations are carried out for a symmetric laminated angle-ply shallow shell ($\theta = 45^\circ / -45^\circ / 45^\circ$) with square planform. Shear correction factor is taken as $K^2 = 5/6$ and $h/a = 0.01$. A comparison regarding linear natural frequency for the first vibration mode of the considered shells obtained through our method and that reported in Ref. [1] is presented in Table 3.

Fig. 3 shows a comparison of the backbone curves obtained through our approach versus results reported in Ref. [1] for angle-ply laminated shallow shell ($\theta = 45^\circ / -45^\circ / 45^\circ$) with curvatures $r_x = R_x/a = 25$; $r_y = R_y/a = 25$ and thickness $h/a = 0.01$.

Since the divergence of results presented in Fig. 3 does not exceed 5%, our results are reliable and validated.

Problem 4. Let us consider the angle-ply laminated shell on a trapezoidal plane-form as shown in Fig. 4.

The studied shell consists of four fiber reinforced layers. Suppose that orientation on θ of the fibers is defined from the positive x -axis as $(\theta / -\theta / \theta / -\theta)$. The material properties, along the principal directions of each layer are:

$$E_{22}/E_{11} = 0.4086; \quad G_{12}/E_{11} = 0.198; \quad G_{13}/E_{11} = 0.198;$$

$$G_{23}/E_{11} = 0.198, \quad \nu_{12} = 0.23; \quad \nu_{13} = 0.23; \quad \rho/\rho_0 = 1, \quad \theta = 30^\circ. \quad (30)$$

The obtained results are compared with results reported in Ref. [25] for shells with following geometric parameters:

$$b/a = 1; \quad R_x/a = r_x = 0; \quad R_y/a = r_y = 2;$$

$$h/a = 0.05; \quad c/a = 0; \quad 0.5. \quad (31)$$

Two opposite edges AD and BC are clamped and the remaining two are free. Following the Ref. [25], the middle point of the edge CD in Fig. 4 is chosen as the reference point, and the deflection at this point is denoted by $W_0 = w(a/2; (b+c)/4)$.

Fig. 5 shows a comparison of frequency ratio (λ_N/λ_L) against the ratio of the deformation parameters (w/h) for trapezoidal and triangular shells obtained using our approach with the results reported in Ref. [25]. Let us note that divergence of our results does not exceed 1% for trapezoidal and 4% for triangular shell, which also validates our method.

6. Nonlinear vibration of shallow shells with layers of variable thickness

Let us now apply our numerical software to investigate nonlinear vibration of shallow shells with layers of variable thickness.

Problem 5. Consider a three-layer clamped shallow shell with square planform of side a and thickness $h = 0.01a$. Suppose that the face layers are isotropic, but a middle layer is orthotropic with the following parameters:

$$E_1/E_0 = 0.25, \quad E_2/E_0 = 0.077, \quad G_{12}/E_0 = 0.029, \quad \nu_1 = 0.24. \quad (32)$$

Table 3
Non-dimensional natural frequency Λ_1 (first vibration mode) of symmetric angle-ply laminated shallow shells obtained via our (RFM) and Ref. [1] approach.

$(45^\circ / -45^\circ / 45^\circ)$			
$\Lambda_1 = \frac{\lambda_1 a^2}{h} \sqrt{\frac{12(1-\nu_{12}\nu_{21})\rho}{E_2}}$			
r_x/a	r_y/r_x	[1]	RFM
10	1	140.1	140.1
25	1	93.01	92.94
50	1	83.97	83.89
100	1	81.54	81.46
10	0	103.2	103.1
10	-1	98.81	98.76

Here E_0 is the elastic modulus of isotropic layers, Poisson's ratio for isotropic layers is $\nu_0 = 0.3$, and density of all layers is the same equal to $\rho = \rho_0$. We take plane $z = 0$ as the middle surface. Assume that thickness of the layers varies linearly (Fig. 6), but the general thickness has a constant value defined as follows

$$\sum_{s=1}^3 h_s = h. \quad (33)$$

Equations of surfaces which bound the inner layer, are formulated in the following way (see Fig. 6):

$$h_1(x,y) = -\frac{h}{2} \left(m + \frac{x}{a} (1-2m) \right), \quad h_2(x,y) = \frac{h}{2} \left(m + \frac{x}{a} (1-2m) \right). \quad (34)$$

Observe that the maximum value of parameter $m = 0.5$ corresponds to constant thickness of layers. For the given case, rigid coefficients C_{ij} and D_{ij} are governed by the following relations

$$C_{ij}(x,y) = h \left(E_0 + (E_1 - E_0) \left(m + \frac{x}{a} (1-2m) \right) \right), \quad (35)$$

$$D_{ij}(x,y) = \frac{h^3}{12} \left(E_0 + (E_1 - E_0) \left(m + \frac{x}{a} (1-2m) \right) \right)^3. \quad (36)$$

In Table 4 the values of non-dimensional frequencies $\Lambda_i = \lambda_i^2 a^2 \rho_0 / (E_0 h^2)$, $i = 1, 2, 3$, for the investigated clamped square plate obtained by our proposed method are compared to similar results reported in Ref. [8]. In Ref. [8] this problem is solved using the classical theory via a numerically-analytical method applying both the spline-approximation and the discrete-orthogonalization method. The results have been obtained only for free vibrations of the plates and they are presented graphically. A comparison of both results with available ones confirms the validation of the proposed method.

Below we solve this problem for cylindrical and spherical shells. In Tables 5 and 6 the values of non-dimensional frequencies $\Lambda_i = \lambda_i a^2 \sqrt{\rho_0 / E_0 h^2}$ for cylindrical and spherical shells with square planform and non-dimensional parameters of curvature k_1, k_2 are reported. These parameters are defined as follows:

$$k_1 = \frac{a}{R_x}, \quad k_2 = \frac{a}{R_y}. \quad (37)$$

Analysis of Tables 5 and 6 shows that difference in the estimation of frequencies for the shells with constant thickness of layers ($m = 0.5$) in comparison to the shells with uniform inner layer ($m = 0$) may achieve the level of 14.7% depending on the number of frequency. Note that for the plates this difference reaches 20% (see Table 4).

In Table 7 we present modes corresponding to the first, second and third frequencies of the studied spherical shells. Note that for a shell with constant thickness of layers the maximum amplitude of the first mode is achieved in the center of the plate. But for spherical shells with a variable thickness of layers the maximum values of the mode is shifted along the Ox axis. Similarly, the nodal lines of the second ($m = 0.25$) and third ($m = 0.25$) modes are also displaced along the Ox axis. In addition, when $m = 0$, the second and third forms are swapped.

The backbone curves for clamped spherical panels with square planform and curvatures $k_1 = k_2 = 0.25$ are shown in Fig. 7. These curves have been obtained assuming that we keep only single mode, that is, we put $n = 1$ in formulas (21). Consequently instead of system ODEs (21) we have only one Duffing-type equation

$$y''(t) + \omega_2^2 y_r(t) + \beta y^2(t) + \gamma y^3(t) = 0. \quad (38)$$

In this case we apply the fourth-order Runge-Kutta method to obtain relation between the amplitude $A = w_{\max}/h$ and ratio λ_N/λ_L .

It follows from Fig. 7 that as parameter m increases, the soft spring response becomes more essential.

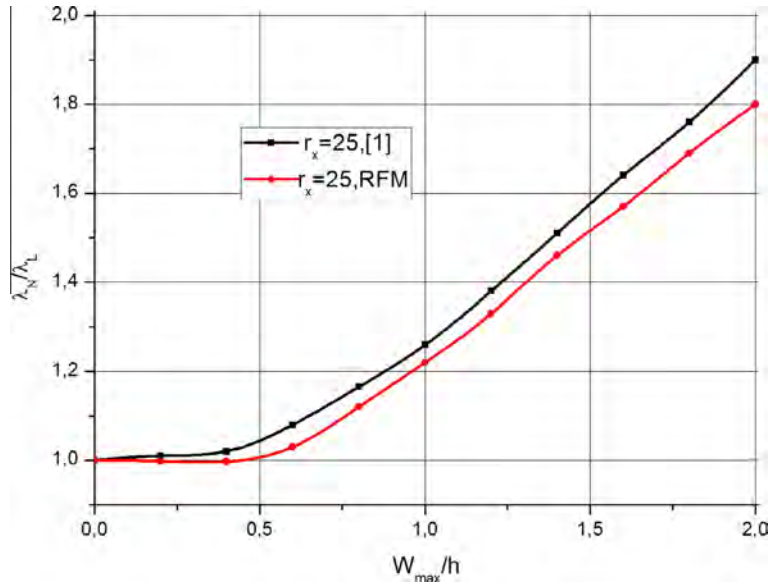


Fig. 3. Comparison of functions $\lambda_N/\lambda_L(W_{max}/h)$ of non-linear vibration clamped spherical angle-ply laminated shallow shell ($\theta = 45^\circ / -45^\circ / 45^\circ$) with curvatures $r_x = 25$; $r_y = 25$ obtained via our method against results of Ref. [1].

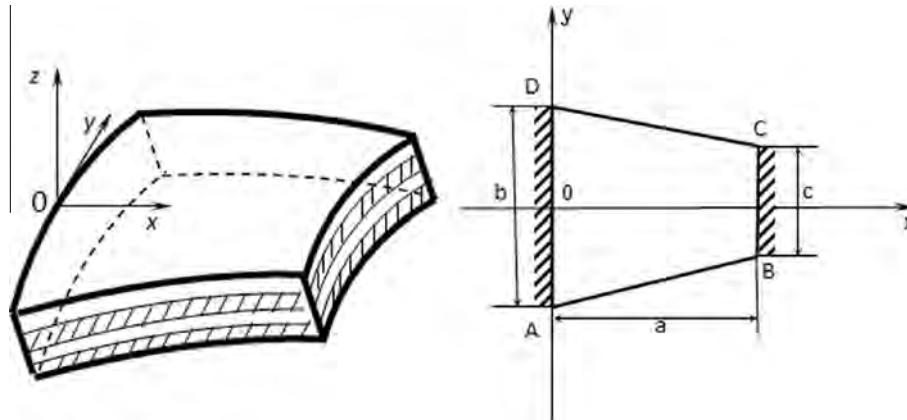


Fig. 4. Shallow shells on trapezoidal planform.

In order to illustrate the possibilities of the proposed approach, we have investigated linear and geometrically nonlinear vibrations of the shells with complex planform and linearly varied thickness of layers. Note that our software allows us to consider the different laws of changing thickness of layers and different boundary conditions. However, in this paper we have considered only the linear law and clamped boundary conditions for shells with complex shape of their plan.

Problem 6. Let us consider three-layer shallow shells with complex planform presented in Fig. 8.

Suppose that the boundary conditions and mechanical characteristics of the layers are the same as in Ref. [1] and defined by relations (29). The shell is a laminated angle-ply shallow shell ($\theta = 45^\circ / -45^\circ / 45^\circ$) with layers of variable thickness. Suppose that thickness of the inner layer varies linearly.

The equations of surfaces which bound the inside layer is as follows

$$h_{1,2}(x,y) = \pm \frac{h}{2} \left(m + \frac{y-b}{a} (1-2m) \right) \quad (39)$$

Some particular cases ($m = 0$; $m = 0.25$; $m = 0.5$) of the behavior thickness of the inside layer are presented in Fig. 8. The stiffness coefficients are defined as follows

$$C_{ij}(x,y) = h \left(E_0 + (E_1 - E_0) \left(m + \frac{y-b}{a} (1-2m) \right) \right), \quad (40)$$

$$D_{ij}(x,y) = \frac{h^3}{12} \left(E_0 + (E_1 - E_0) \left(m + \frac{y-b}{a} (1-2m) \right)^3 \right) \quad (41)$$

Let us choose the solution structure [17,23] in the following way $u = \omega\Phi_1, v = \omega\Phi_2, w = \omega^2\Phi_3, \psi_x = \omega\Phi_4, \psi_y = \omega\Phi_5.$ (42)

Function $\omega(x,y)$ in formula (42) satisfies the following conditions $\omega(x,y) = 0, \forall (x,y) \in \partial\Omega, \omega(x,y) > 0, \forall (x,y) \in \Omega,$ (43)

and it will be constructed using the R-functions theory [23]. In the case of the planform shown in Fig. 8 we take

$$\omega(x,y) = (f_1 \wedge f_2) \vee_0 (f_3 \vee_0 f_4) \vee_0 (f_5 \wedge f_6), \quad (44)$$

where

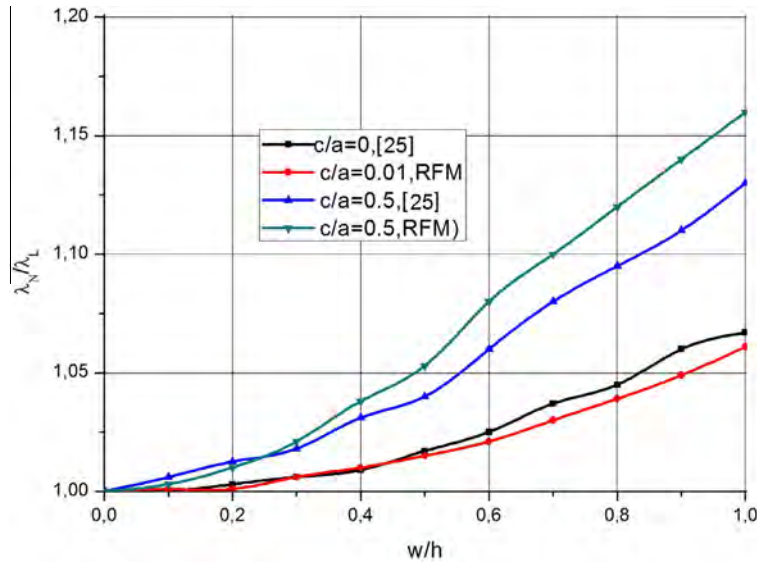


Fig. 5. Comparison of function $\lambda_N/\lambda_L(w/h)$ characterizing non-linear vibrations of cylindrical four-layer laminated shallow shell ($30^\circ/-30^\circ/30^\circ/-30^\circ$) of trapezoidal ($c/a = 0.5$) and triangular planforms ($c/a = 0$ and $c/a = 0.01$) with the results of Ref. [25].

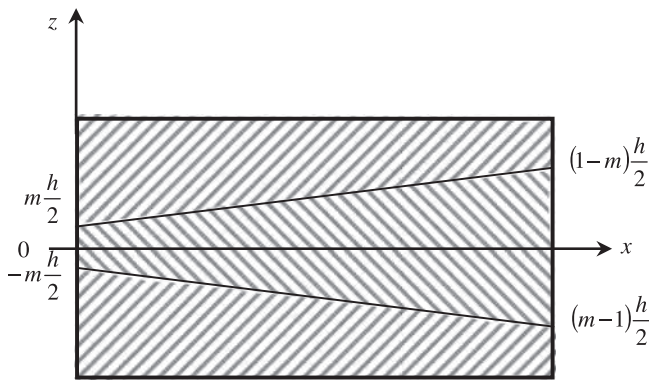


Fig. 6. Thickness of layers varies linearly.

Table 4
Comparison of non-dimensional frequencies $\Lambda_i = \lambda_i^2 a^2 \rho_0 / (E_0 h^2)$, $i = 1, 2, 3$ for three-layer square clamped plate estimated by our (RFM) method and that reported in Ref. [8].

Λ_i	Method	$m = 0$		$m = 0.25$		$m = 0.5$	
		CST	FSDT	CST	FSDT	CST	FSDT
Λ_1	RFM	0.886	0.882	1.023	1.019	1.057	1.053
	[8]	0.88	-	1.02	-	1.06	-
Λ_2	RFM	3.608	3.578	4.259	4.231	4.369	4.341
	[8]	3.60	-	4.25	-	4.35	-
Λ_3	RFM	3.781	3.756	4.264	4.235	4.430	4.401
	[8]	3.80	-	4.26	-	4.40	-

$$f_1 = \frac{1}{2a}(a^2 - x^2) \geq 0, \quad f_2 = \frac{1}{2b}(b^2 - y^2) \geq 0, \quad (45)$$

$$f_3 = (y - b_1) \geq 0, \quad f_4 = \frac{1}{2c}(c^2 - x^2) \geq 0, \quad (46)$$

$$f_5 = (y(a - c) - b_2(x - c)) / \sqrt{(a - c)^2 + b_2^2} \geq 0, \quad (47)$$

$$f_6 = ((y - b_2)(a - c) + b_2(x + a)) / \sqrt{(a - c)^2 + b_2^2} \geq 0. \quad (48)$$

Here, \wedge_0 and \vee_0 are the symbols of R-conjunction and R-disjunction (see [23] for more details), respectively.

In order to construct the basic functions, indefinite components Φ_i , $i = 1, \dots, 5$ (formulas (42)) should be taken in the form of the following truncated series

$$\Phi_i = \sum_{k=1}^{k=N_i} a_k^{(i)} \varphi_k^{(i)}, \quad (49)$$

where $\{\varphi_k^{(i)}\}$ are the known complete systems of functions, for instance, power or Chebyshev's polynomials, trigonometric functions, splines or other. In the present study, power polynomials have been used in Eq. (49) to get numerical results. Substituting (49) into Eq. (39) yields

$$u = \sum_{i=1}^{R_1} a_i u_i, \quad v = \sum_{i=1}^{R_2} a_i v_i, \quad w = \sum_{i=1}^{R_3} a_i w_i, \quad (50)$$

$$\psi_x = \sum_{i=N_3+1}^{R_4} a_i \psi_{xi}, \quad \psi_y = \sum_{i=N_4+1}^{R_5} a_i \psi_{yi}$$

where $u_i = \omega \varphi_i^{(1)}$, $v_i = \omega \varphi_i^{(2)}$, $w_i = \omega \varphi_i^{(3)}$, $\psi_{xi} = \omega \varphi_i^{(4)}$, $\psi_{yi} = \omega \varphi_i^{(5)}$ are the admissible functions that satisfy boundary conditions. Unknown coefficients a_i , $i = \sum_{j=1}^5 R_j$ are calculated by minimizing functional (12). In order to determine the needed numbers of coordinate functions the computational experiment for plates with the following geometrical and mechanical parameters is carried out:

$$h/2a = 0.01, \quad c/2a = 0.3, \quad b_1/2a = 0.2, \quad b_2/2a = 0.2, \quad a/b = 1, \quad (51)$$

$$E_1 = 138 \text{ GPa}, \quad E_2 = 8.96 \text{ GPa}, \quad G_{12} = 7.1 \text{ GPa}, \quad (52)$$

$$G_{13} = G_{23} = E_1/2, \quad \nu_{12} = 0.3.$$

Table 5
Effect of parameter m on the values of non-dimensional frequencies $\Lambda_i = \lambda_i a^2 \sqrt{\rho_0/E_0 h^2}$ of the clamped cylindrical shells.

Λ_i	Cylindrical shells ($k_1 = 0.25, k_2 = 0$)					
	CST			FSDT		
	$m = 0$	$m = 0.25$	$m = 0.5$	$m = 0$	$m = 0.25$	$m = 0.5$
Λ_1	18.28	19.53	19.84	18.27	19.51	19.82
Λ_2	20.63	22.04	22.39	20.55	21.97	22.32
Λ_3	25.75	26.99	27.38	25.21	26.94	27.30
Λ_4	30.98	32.81	33.27	30.85	32.68	33.13

Table 6
Effect of parameter m on the values of non-dimensional frequencies $\Lambda_i = \lambda_i a^2 \sqrt{\rho_0/E_0 h^2}$ of the clamped spherical shells.

Spherical shells ($k_1 = 0.25, k_2 = 0.25$)						
Λ_i	CST			FSDT		
	$m = 0$	$m = 0.25$	$m = 0.5$	$m = 0$	$m = 0.25$	$m = 0.5$
Λ_1	24.38	26.84	27.56	24.34	26.81	27.54
Λ_2	26.94	28.92	28.99	26.87	28.86	28.93
Λ_3	28.02	29.08	29.56	27.95	29.02	29.51
Λ_4	34.42	36.52	37.03	33.29	36.40	36.90

Results of the performed investigations are presented in Table 8.

The analysis of Table 8 shows that convergence of the frequencies is better for the classical theory. Let us explain a reason of the essentially small dimension of Ritz matrices. We have three unknown functions in the case of CLT and five ones in the case of FSDT. It is obvious that here we are faced with the accumulation of errors for the densely filled matrices of high order and complexity of the integration domain and integrand. Therefore, it is better to apply finite functions (for example splines) to approximate the uncertain components. Since we consider thin shallow shells, the CLT may be used. The total number of coordinate functions 234 has been chosen to solve the problem that corresponds to the 12th degree of polynomials Φ_1, Φ_2, Φ_3 . Note that the complete systems $\{\varphi_k^{(i)}\}$ have the following form

$$\{\varphi_k^{(1)}\} \{\varphi_k^{(2)}\}, \{\varphi_k^{(3)}\} : 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^3, y^3, \dots \quad (53)$$

In the case of the application of FSDT we take the total numbers of the admissible functions equal to 362 that corresponds to the 11th degree of polynomials $\Phi_1, \Phi_2, \Phi_4, \Phi_5$ and the 12th degree of polynomial Φ_3 . Next, we discuss the obtained results using the introduced data.

Table 7
Effect of parameter m on modes of the clamped spherical shells with curvatures $k_1 = k_2 = 0.25$.

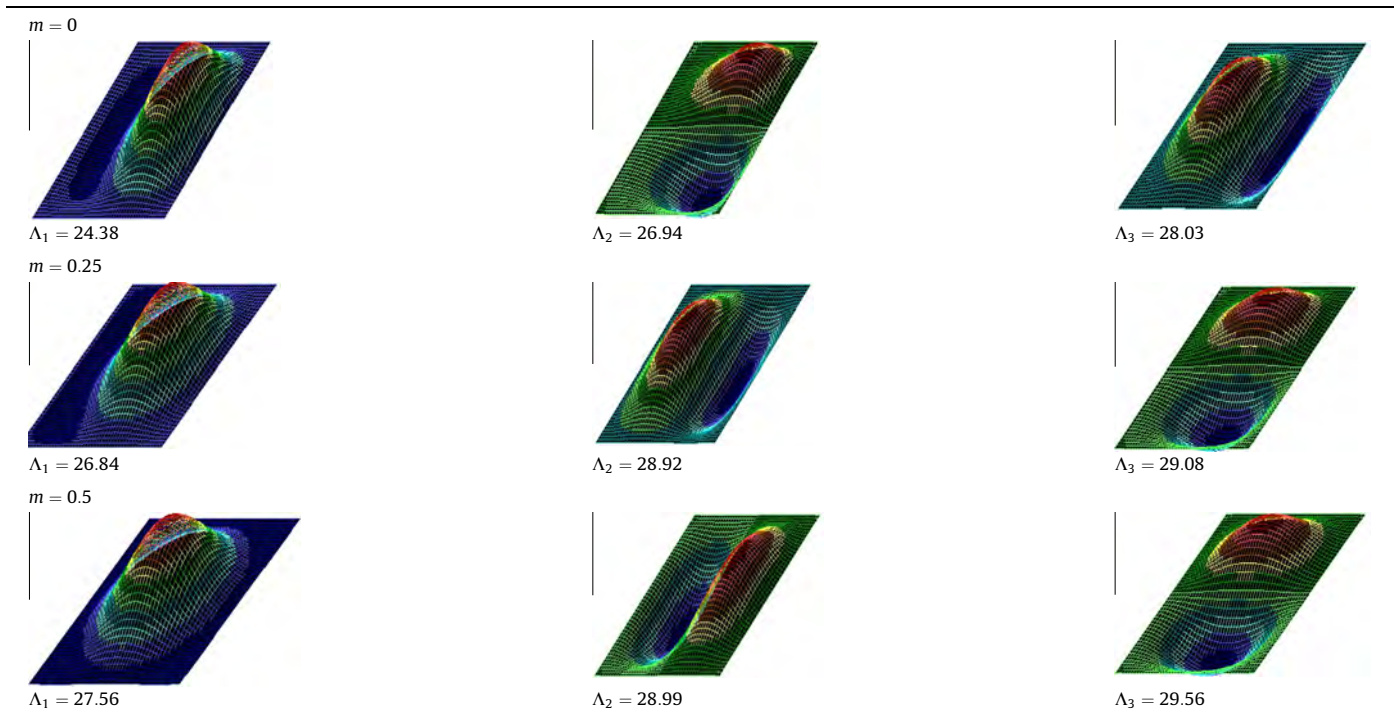


Table 9 reports frequency parameters $\Lambda_i = \frac{\lambda_i a^2}{h} \sqrt{\frac{12(1-\nu_{12}\nu_{21})\rho}{E_2}}$ $i = 1, 2, 3$, for plates, cylindrical and spherical panels for different values of the thickness parameter m . The non-dimensional curvatures equal to 0.25 have been chosen and the geometric parameters are defined by relations (51).

It is seen from Table 9 that the influence of parameter m on the values of natural frequencies for the given material (graphite-epoxy (52)) is not essential if we compare the obtained results with the previous example (Tables 4–6). The natural frequencies of the plate and shells are decreased by 2%, when the value of parameter m increases. This problem shows that corresponding investigations have to be satisfied in every case, because there are many different factors which may influence the behavior of the studied mechanical objects: mechanical characteristics and ways of packing of the layers, geometric form, boundary conditions, curvature, law of variation of the layer thickness, and other factors.

Fig. 9 shows the effect of amplitude on frequency dependencies for the clamped plate, cylindrical and spherical three-layered shallow shells with variable thickness of layers for $m = 0.25$ and for the ratios $h/2a = 0.01, c/2a = 0.3, b_1/2a = 0.2, b_2/2a = 0.2, a/b = 1$. The plate exhibits an entirely hard behavior, but in cylindrical and spherical shells soft spring stiffness type behavior occurs. For a cylindrical shell the soft spring response diminishes as compared to the counterpart spherical shell behavior. The results shown in Fig. 9 are obtained within the framework of CLT.

7. Concluding remarks

In this paper the geometrically nonlinear vibrations of the laminated shallow shells of an arbitrary planform and variable thickness of layers are considered. To solve this problem in the framework of the classical lamination and first-order shear deformation theories a numerical-analytical method is developed. The proposed approach is meshless and utilizes the R-functions theory

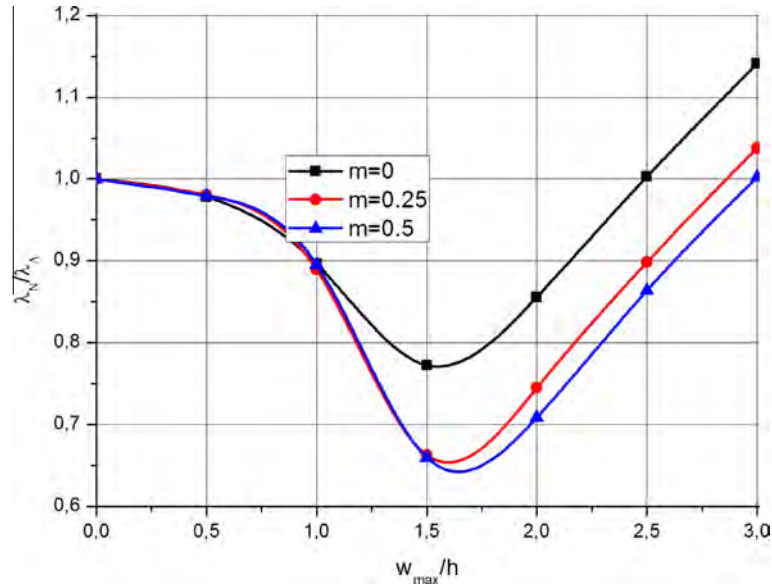


Fig. 7. Amplitude versus frequency for different thickness parameter m of the clamped spherical square shell ($k_1 = k_2 = 0.25$, $h/a = 0.01$, $b/a = 1$).

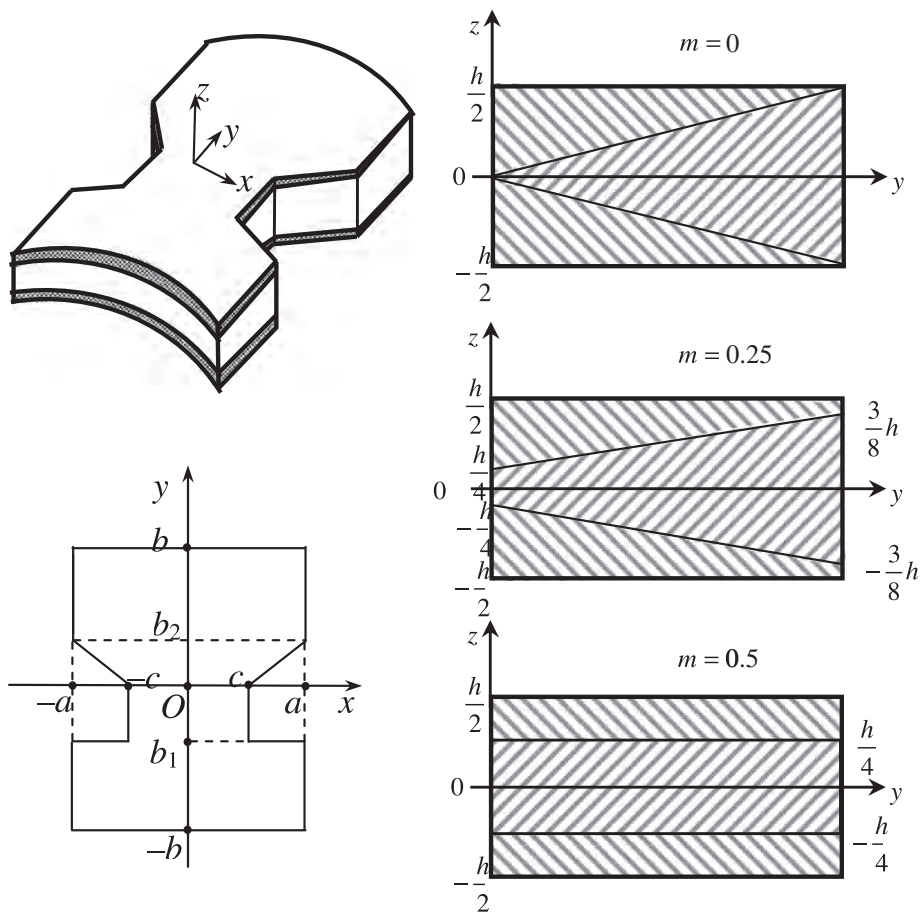


Fig. 8. Thin shallow shell with complex planform and the studied cases of the variable thickness of layers.

as well as the variational and purely numerical methods. One of the main advantages of the proposed approach is its universality, given by the analytical representation of the general solution and solutions of intermediate problems, including the problem of free vibrations and a series of problems of the elasticity theory. The

key achievement is the analytical expressions for the coefficients of the system of non-linear ODEs yielded by a reduction procedure of the PDEs. It essentially simplifies the solution of nonlinear problems including PDEs with variable coefficients (shells and plates of the variable thickness or laminated shells with layers of variable

Table 8
Convergence of frequency parameters of the first vibration mode $\Lambda_1 = \frac{\lambda_1(2a)^2}{h} \sqrt{\frac{12(1-\nu_{12}\nu_{21})\rho}{E_2}}$ of angle-ply laminated plate ($\theta = 45^\circ / -45^\circ / 45^\circ$) with variable thickness of the layers.

$R_1/R_2/R_3$	CLT			$R_1/R_2/R_3/R_4/R_5$	FSDT		
	$m = 0$	$m = 0.25$	$m = 0.5$		$m = 0$	$m = 0.25$	$m = 0.5$
66/66/66	144.73	144.70	143.28	45/45/66/45/45	147.69	146.38	143.76
66/66/78	144.60	144.58	143.09	55/55/66/55/55	145.30	144.65	142.75
78/78/88	144.32	144.28	142.79	66/66/78/66/66	144.57	143.97	142.04
88/88/105	144.32	144.28	142.79	78/78/78/78/78	143.72	143.27	141.49

Table 9
Effect of parameter m on natural frequencies $\Lambda_i = \frac{\lambda_i a^2}{h} \sqrt{\frac{12(1-\nu_{12}\nu_{21})\rho}{E_2}}$ of plates, angle-ply ($\theta = 45^\circ / -45^\circ / 45^\circ$) cylindrical and spherical panels.

CST	FSDT		
	$m = 0$	$m = 0.25$	$m = 0.5$
Λ_1	297.36	302.94	304.00
Λ_2	324.63	328.39	329.42
Λ_3	364.83	367.63	364.66
Λ_4	405.38	405.59	406.33
<i>Spherical shells ($k_1 = 0.25, k_2 = 0.25$)</i>			
Λ_1	297.36	302.94	304.00
Λ_2	324.63	328.39	329.42
Λ_3	364.83	367.63	364.66
Λ_4	405.38	405.59	406.33
<i>Cylindrical shells ($k_1 = 0.25, k_2 = 0$)</i>			
Λ_1	207.46	209.53	209.14
Λ_2	260.36	263.30	266.19
Λ_3	322.51	318.87	308.91
Λ_4	356.65	361.40	365.78
<i>Plates ($k_1 = 0, k_2 = 0$)</i>			
Λ_1	144.60	144.58	143.09
Λ_2	209.47	212.93	217.34
Λ_3	306.82	300.32	286.88
Λ_4	324.94	331.75	337.69

thickness and complex planforms). In order to construct analytical representation of the solutions the method of structure solutions is applied. The structure solutions is invariant with respect to the geometry of the domain. The equations on the border domain and its sections are presented in analytical forms. They can contain literal parameters, variation of which allows for the investigation of a wide class of shells of an arbitrary shape within the same program. The boundary conditions are satisfied exactly and the thickness of the layers is taken into account analytically.

The developed method is widely tested. First four problems regarding laminated shells of constant layers have been solved, and our results have been compared with those obtained by other researchers in order to validate our approach. New numerical results in the form of backbone curves are presented for a three-layered spherical clamped panel with square planform (outer slices are isotropic and inner slice is orthotropic). The thickness of the layers varies linearly. In order to illustrate the possibilities of the proposed method nonlinear vibrations of three-layered plates, spherical and cylindrical shells with complex shape of plan are studied. The considered shells are angle-ply ($\theta = 45^\circ / -45^\circ / 45^\circ$) orthotropic ones with a variable thickness of layers.

Based on the obtained numerical results it can be concluded that the thickness of the layers can be controlled in order to reduce the weight of the projected object to increase its strength, and to change appropriately the stress–strain state, etc.

Note that like many other numerical methods, the proposed approach has its own drawbacks. One disadvantage of the RFM is that when using polynomial approximations of the undefined components in the structural formulas we obtain a densely filled matrix. In problems concerning vibrations of the laminated shallow shells such components may vary from three up to seven. It depends on the order of the used theory and the type of boundary conditions. Consequently, the order of the Ritz matrices is large. Therefore, when solving the eigenvalue problem we have poor convergence due to bad conditions of the matrices. However, this problem can be solved by joint use of the R-functions method and the approximation of uncertain components via finite functions such as splines.

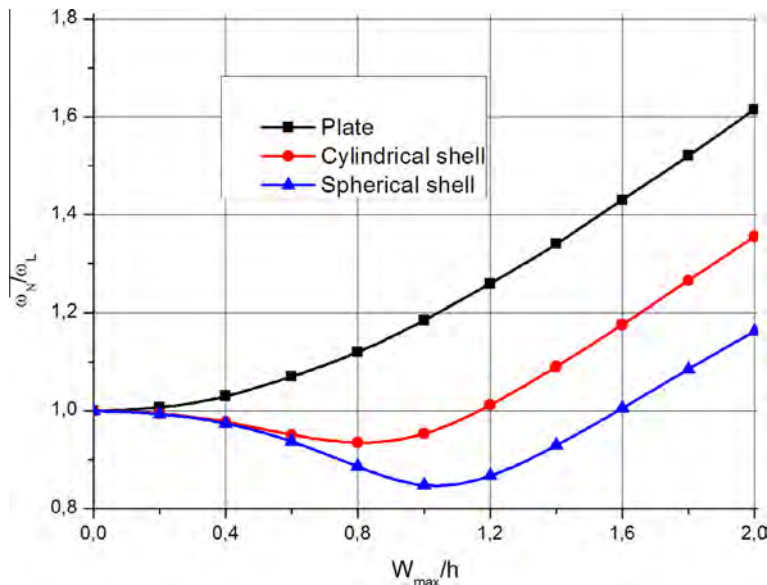


Fig. 9. Amplitude versus frequency for various curvature ratios for angle-ply ($\theta = 45^\circ / -45^\circ / 45^\circ$) spherical shell ($k_1 = k_2 = 0.25$), cylindrical shell ($k_1 = 0.25, k_2 = 0$) and plates ($k_1 = k_2 = 0$) and for $m = 0.25$.

The multi-modes approximation of unknown functions is tempting, but taking into account our earlier remarks it greatly complicates the problem, and accordingly the development of a software. The presented approach and the new results will be useful for researchers dealing with this type of problems.

Acknowledgments

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