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Optimal design of a functionally graded corrugated rods subjected to longitudinal deformation

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Abstract In this paper, first, we consider corrugated rods, i.e., those in which the values of amplitude and step corrugations are varied continuously as a function of position along a central axis direction to achieve a required function. The latter corrugation type is called the functionally gradient corrugation (FGC). State equations obtained via the homogenization procedure are projected onto the central direction. Second, we study problems regarding the optimization of the FGC rods with variable amplitude and variable step of corrugation. In result of our approach, the optimal profiles of corrugated rods are obtained guaranteeing the largest stiffness of the FGC rods regarding a longitudinal deformation. We have applied the most important, from an engineering point of view, constraints of keeping constant length of the curvilinear rod axis and both linear dimension (projection onto a straight line axis of symmetry) and number of waves of corrugations.

Keywords Homogenization · Optimal design · Corrugated rod · Functionally graded structure

1 Introduction

Corrugated rods (Fig. 1) are widely used as structural elements in civil engineering, reinforced concretes and flat springs fabrication. The corrugated rod model is also used while investigating elastic properties of reinforced composites [1], and they are also applied in reinforced glass–plastic materials [2]. In the so-far-mentioned areas of applications, the longitudinal deformation of corrugated rods plays a dominant role. The problem related to a proper choice of the corrugation profile to preserve the largest longitudinal stiffness is most important while designing corrugated rods. It has been shown in reference [3] that in the case of a regularly fabricated corrugation profile (Fig. 1), a profile with the elliptic sinus shape is optimal one. On the other hand, optimization of the regular corrugated rods yields a rather negligible benefit, i.e., the change of a sinusoidal profile into an optimized one increases the longitudinal rod stiffness only about 1 % [3]. The so-far-mentioned optimization procedure does not include important factors as a way of external load distribution along the

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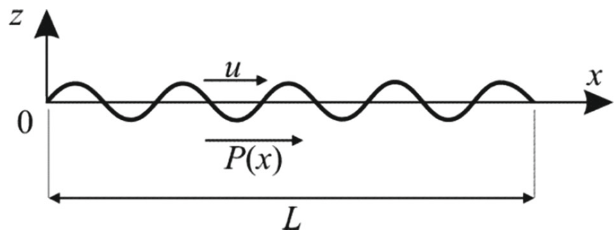

Fig. 1 Regular corrugated rod

Fig. 2 FGC with squared step gradient (10)

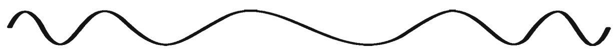

Fig. 3 FGC with a cubic step gradient (11)

rod length as well as the influence of applied boundary conditions. The mentioned factors can be taken into account while constructing of the so called functionally graded corrugated (FGC) rod, which characteristics (amplitude and step) are changed along its length through a given rule. In this work, we study separately the corrugations with variable amplitude (Fig. 2) and variable step (Fig. 3).

2 State equation

The conventional investigation of the stress–strain state of a corrugated rod requires solution to the ODEs with variable coefficients. When the number of corrugation waves is high, the straightforward numerical solution to these equations using FEM [3–5] or the sweep method [6] causes significant computational difficulties. Besides, this way of calculation does not allow setting the problem of optimal design of structures.

In reference [7], the computational scheme of the stress–strain state of regularly corrugated plates using projections of stress, moments and displacements onto the symmetry axis with a help of the homogenization method has been proposed. In reference [3], the same scheme is carried out proving the high efficiency of the applied method. The so-far-mentioned approach allows studying the FGC with either varied amplitude [8] or varied stepping [9,10].

2.1 Equilibrium equation for varied amplitude

We consider the FGC rod, whose curvilinear axes are governed by the equation

$$z_1 = h_1(x)\tilde{z}(nx), \tag{1}$$

where n denotes corrugation waves number; $\tilde{z}(nx)$ is periodic function with period $l_1 = \frac{L}{n}$; L denotes length of the rod axis, whereas $h_1(x)$ defines the corrugations amplitude gradient (small gradient is further considered, $\frac{dh_1}{\mathrm{d}x} \sim 1$).

The homogenized equation of the longitudinal FGC rod with a varied step coincides with the equilibrium equation of the regularly corrugated rod [3] subjected to the longitudinal load (see Fig. 3)

$$\frac{d}{\mathrm{d}x}\left(\frac{1}{k_1}\frac{\mathrm{d}u}{\mathrm{d}x}\right) = A_1 \cdot P(x), \tag{2}$$

where u denotes projection of the rod displacements along the neutral axis ox; k_1 is the averaged coefficient of the longitudinal stiffness, and A_1 is the averaged squared form.

For the FG of profile (1), we have

$$k_1 = \frac{1}{EI}\int_0^L \left(\frac{1}{\tilde{A}F} + \frac{h^2(x)\tilde{z}^2(\xi)\tilde{A}}{I}\right)\mathrm{d}\xi, \tag{3}$$

$$A_1 = \frac{1}{L}\int_0^L \sqrt{1 + (h_1'(x)\tilde{z}(\xi) + nh_1(x)z'(\xi))^2}\mathrm{d}\xi, \tag{4}$$

where $\xi = nx$, x and ξ are treated as independent quantities; $\tilde{A} = \sqrt{1 + (h_1(x)\tilde{z}(nz))\prime^2}$; I is the moment of inertia of the transversal rod cross section regarding its neutral axis, and F denotes the rod cross-sectional area.

Assuming that a corrugation has an amplitude larger than the characteristic cross-sectional size, first term standing under integral in (3) can be neglected. Analogously, for large number of corrugations and a small gradient, one may neglect $h'z$ in formulas (3), (4). Then, formulas (3) and (4) are simplified and cast to the form

$$k_1 = \frac{y_1^2}{EIn^2}\frac{1}{L}\int_0^L \sqrt{1 + y_1^2\tilde{z}'^2}\mathrm{d}\xi, \tag{5}$$

$$A_1 = \frac{1}{L}\int_0^L \sqrt{1 + y_1^2\tilde{z}'^2}\mathrm{d}\xi, \tag{6}$$

where $y_1 = nh_1(x)$.

2.2 Equilibrium equation for varied step

In some applications, for instance, when corrugated elements are used as a middle layer in a three-layer composite material [11], direct fabrication of the amplitude-graded corrugation is not possible due to technological reasons. Equation describing a curvilinear axis of this corrugation can be presented in the following form

$$z_2 = h_2\tilde{z}(nf(x)), \tag{7}$$

where $h_2 = \text{const}$, and $f(x)$ denotes the change of a corrugation step.

Conditions for the function $f(x)$, where a number of corrugations waves is constant, follow

$$f(0) = 0, \quad f(L) = L, \quad f(x) > 0, \quad f'(x) > 0, \tag{8}$$

and the approximate formula for the changeable corrugation step is as follows

$$\tilde{l}_2(x) \approx \frac{l}{f'(x)}. \tag{9}$$

In what follows, we consider two characteristic cases of the varied step for sinusoidal corrugation. In the first case, the corrugation profile monotonously either decreases (Fig. 2) or increases along the rod length. Such case can be described by a quadratic function, which taking into account (8) yields

$$f(x) = \beta x^2 + (1 - 2\pi\beta)x, \tag{10}$$

where β parameter characterizing the magnitude and direction of the gradient vector should satisfy the inequality $|\beta| < \frac{1}{2\pi^2}$. For $\beta > 0$ the corrugation step monotonously decreases, whereas for $\beta < 0$ it increases. In other words, large values of $|\beta|$ correspond to large values of gradient step changes.

The second characteristic case is associated with the increase (decrease) of the varied step into direction of the rod center and with decrease (increase) of this step in the neighborhood of the rod edges (Fig. 3). The so-far-described symmetric varied step can be modeled via a cubic function, which takes the following form:

$$f(x) = \beta x^3 - 3\beta\pi x^2 + (2\beta\pi^2 + 1)x, \tag{11}$$

and for $|\beta| < \frac{1}{2\pi^2}$, the conditions (8) are satisfied.

Equilibrium equation of the FGC rod with the varied step (7) subjected to the longitudinal deformation can be obtained with a help of the modified homogenization approach [10,11]. This modification relies on introduction of the variable $\eta = nf(x)$, which is treated as independent, and hence, the differentiation operator takes the following form

$$\frac{d}{dx} = \frac{\partial}{\partial x} + nf'(x)\frac{\partial}{\partial \eta}. \tag{12}$$

In result, the following homogenized equation is obtained

$$\frac{d}{\mathrm{d}x}\left(\frac{1}{k_2}\frac{\mathrm{d}u}{\mathrm{d}x}\right) = A_2 \cdot P(x), \tag{13}$$

and for a small gradient and non-shallow corrugation, we get

$$k_2 = \frac{h_2^2}{EI}\frac{1}{L}\int_0^L \tilde{z}^2(\eta)A_2\mathrm{d}\eta, \quad A_2 = \sqrt{1 + y_2^2\tilde{z}\prime^2(\eta)}, \tag{14}$$

where $y_2 = nh_2 f\prime(x)$.

3 Optimization procedure

For both cases of the FGC, we take $y_i(x), i = 1, 2$ as the control function. On the other hand, we take deformation energy as the minimized functional

$$\int_0^L uA_i q\mathrm{d}x \to \min_{y_i} \tag{15}$$

by keeping the isoperimetric condition, which guarantees a constant length of the curvilinear rod:

$$\int_0^L A_i\mathrm{d}x = S - \mathrm{const.} \tag{16}$$

Boundary conditions for equations (2) and (13) are taken as follows

$$u(0) = 0, \quad u'(L) = 0. \tag{17}$$

Similarly, we may carry out the optimization procedure for a sinusoidal FG corrugation of the form

$$z_1 = h_1(x)\sin(nx), \tag{18}$$

$$z_2 = h_2 \sin(nf(x)). \tag{19}$$

Further, we consider the often met in practice case, when the corrugation amplitude is essentially less than the rod axis length ($h_i << L$), and hence, $y_i < 1$. Further, in order to simplify the analysis, we take $L = 2\pi$.

3.1 Varied amplitude

We consider the BVP (2), (5), (6), (15)–(18) for $i = 1$. We develop coefficients of Eq. (2) as well as the under integral functional (15), (16) into series regarding y_1, and we limit the consideration up to a term of second power. In the latter case, the optimization problem is cast to the following form

$$\int_0^{2\pi} u\left(1 + 0.25y_1^2\right)P\mathrm{d}x \to \min_{y_1}; \tag{20}$$

$$\int_0^{2\pi} \left(1 + 0.25y_1^2\right)\mathrm{d}x = S, \tag{21}$$

$$\frac{d}{\mathrm{d}x}\left(\frac{8 - y_1^2}{4y_1^2}\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \left(1 + 0.25y_1^2\right)rP(x), \tag{22}$$

where $r = \frac{1}{EIn^2}$.

Solving the BVP (17), (20)–(22), and following the classical scheme of variation calculus [12], the following necessary optimality condition is obtained

$$\left(\frac{4}{y_1^4}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 0.5\lambda_1 r\right) y_1 = 0, \tag{23}$$

where the Lagrange constant is defined by the isoperimetric condition (21).

If one compares to zero only the first multiplier appeared in the optimality condition (23)

$$\frac{4}{y_1^4}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 0.5\lambda_1 r = 0, \tag{24}$$

we get a formula for the target function.

Therefore, we are going to determine the target function in the following piecewise form $y_1 = \begin{cases} y_{11}, \; 0 \le x < l_1 \\ y_{12}, \; l_1 \le x \le 2\pi \end{cases}$, which on the interval $[0, l_1)$ it serves as a solution to the equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 0.5\lambda_1 r y_{11}^4 = 0, \tag{25}$$

whereas on the interval $[l_1, 2\pi]$ it satisfies the following condition

$$y_{12} \equiv 0. \tag{26}$$

On the other hand, the value l_1 is defined via the continuity condition of the corrugation profile in the point $x = l_1$

$$\lim_{x\to l_1-0} z_1 = \lim_{x\to l_1+0} z_1, \quad \lim_{x\to l_1-0} \frac{\mathrm{d}z_1}{\mathrm{d}x} = \lim_{x\to l_1+0} \frac{\mathrm{d}z_1}{\mathrm{d}x}. \tag{27}$$

For profile (18), conditions (26) and (27) yield

$$y_{11}(l_1) = 0, \quad \lim_{x\to l_1-0}\left(\frac{\mathrm{d}y_{11}}{\mathrm{d}x}\sin nx\right) = 0, \tag{28}$$

and y_{11} is defined by Eq. (24)

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \pm 0.71\sqrt{\lambda_1 r}\, y_{11}^2. \tag{29}$$

Substituting formula (29) into Eq. (22), the following equation is obtained

$$0.5 y_{11}\frac{\mathrm{d}y_{11}}{\mathrm{d}x} = \pm(1 + 0.25 y_{11}^2)\lambda_1 P(x), \tag{30}$$

where $\lambda_{11} = \sqrt{\frac{2r}{\lambda_1}}$.

Equation (30) yields the following solution

$$y_{11} = \pm 2\sqrt{C_1 e^{\pm\lambda_{11}\int P(x)} - 1}, \tag{31}$$

where C_1 is the integration constant, which can be found from the first condition of continuity (28). Sign $\pm$ appearing before the right part of formula (31) defines direction of convexity of the initial wave of the corrugation and can be neglected due to the symmetry of the stated problem.

3.2 Varied step

In this case, we consider the optimization problem of (2), (5), (6), (15)–(17) for the varied step (19) for $i = 2$. After development of coefficients (14) into series regarding y_2 up to the second power inclusively, the step gradient optimization follows:

$$\int_0^{2\pi} u\left(1 + 0.25 y_2^2\right) P \mathrm{d}x \to \min_{y_2}, \tag{32}$$

$$\int_0^{2\pi} \left(1 + 0.25 y_2^2\right) \mathrm{d}x = S, \tag{33}$$

$$\frac{d}{\mathrm{d}x}\left(\left(1 + 0.125 y_2^2\right) \frac{\mathrm{d}u}{\mathrm{d}x}\right) = \left(1 + 0.25 y_2^2\right) r P(x). \tag{34}$$

We are going to solve this problem with the supplemented condition regarding conservation of the corrugations number, which is governed by condition (8). In order to satisfy condition (8), we introduce the function $\varphi(x)$ using the following equation

$$y_2 = y_{20} + \alpha \sin \varphi, \tag{35}$$

where y_{20}, α are constants a priori given; $|\alpha| < y_{20}$. The minimized functional (32), with respect to a new control function $\varphi(x)$, takes the following form

$$\int_0^{2\pi} u\left(1 + 0.25 y_2^2\right) P \mathrm{d}x \to \min_{\varphi}. \tag{36}$$

Varying $\varphi(x)$ in the optimization problem (33)–(36), (17), with a help of the method of a conjugated variable [12], following optimality conditions are obtained

$$\cos \varphi \left(\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 2 y_2 r \lambda_2\right) = 0, \tag{37}$$

where λ_2 denotes the Lagrange constant. If we compare to zero only the second term of optimality condition (37), the condition (8) cannot be satisfied. Therefore, we proceed in a way similar to the problem of amplitude optimization. Namely, we search y_2 in the following form

$$y_2 = \begin{cases} y_{21}, & 0 \le x < l_2 \\ y_{22}, & l_2 \le x \le 2\pi \end{cases}. \tag{38}$$

It satisfies (in the interval $[0, l_2)$) the following equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 2\lambda_2 r y_{21} = 0, \tag{39}$$

whereas in the interval $[l_2, 2\pi]$ it satisfies the following algebraic condition

$$\cos \varphi = 0. \tag{40}$$

Value l_2 is defined via continuity condition regarding corrugation profile in the point $x = l_2$

$$\lim_{x \to l_2 - 0} z_2 = \lim_{x \to l_2 + 0} z_2, \quad \lim_{x \to l_2 - 0} \frac{\mathrm{d}z_2}{\mathrm{d}x} = \lim_{x \to l_2 + 0} \frac{\mathrm{d}z_2}{\mathrm{d}x}. \tag{41}$$

Conditions (40), (35) and (8) yield

$$y_{22} = y_0 - \alpha. \tag{42}$$

Next, y_{21} is defined by Eq. (39), and hence,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \pm\sqrt{2r\lambda_2 y_2}. \tag{43}$$

In the above sign "+" is associated with the extension problems, whereas sign "-" is associated with compression. Further, we consider only the extension problems. Substituting formula (43) into equilibrium equation (34), the following equation is obtained

$$\frac{5y_{21}^2 + 8}{\sqrt{y_1}\left(4 + y_{21}^2\right)}\mathrm{d}y_{21} = \frac{2.83}{\sqrt{\lambda_{21}}}P(x)\mathrm{d}x, \tag{44}$$

where $\lambda_{21} = \frac{\lambda_2}{r}$.

Integration of (44) yields

$$\begin{aligned} &10\sqrt{y_{21}} - 1.5\ln\left(2 + 2\sqrt{y_{21}} + y_{21}\right) - 3\arctan\left(\sqrt{y_{21}} + 1\right) + 1.5\ln\left(2 - 2\sqrt{y_{21}} + y_{21}\right) \\ &-3\arctan\left(\sqrt{y_{21}} - 1\right) = \frac{2.83}{\sqrt{\lambda_{21}}}\int P(x)\mathrm{d}x + C_2, \end{aligned} \tag{45}$$

where C_2 is the integration constant, which is defined through the first condition of (8).

Developing the l.h.s. of Eq. (45) into series regarding y_{21}, and limiting the considerations to a second power, we get

$$y_{21} = \frac{1}{\lambda_{21}}\left(C_2 + \int P(x)\mathrm{d}x\right)^2. \tag{46}$$

Having in hand the target function $y_2(x)$, the function $f(x)$ defining the varied step is found via integration

$$f(x) = \begin{cases} f_1(x), & 0 \le x < l_2, \\ f_2(x), & l_2 \le x \le 2\pi, \end{cases}$$

where

$$f_1(x) = \frac{1}{nh_2}\int y_{21}\mathrm{d}x + C_{21}, \quad f_2(x) = \frac{1}{nh_2}\int y_{22}\mathrm{d}x + C_{22}, \tag{47}$$

and C_{2i} are integration constants, and C_{22} is defined by the second condition of (8), whereas C_{21} is defined through continuity of the function $f(x)$ in the point $x = l_2$

$$\lim_{x\to l-0} f(x) = \lim_{x\to l+0} f(x). \tag{48}$$

Constant λ_{21} is defined by the isoperimetric condition (33), which in the case of the target function (38) takes the form

$$\int_0^{l2}\left(1 + 0.25y_{21}^2\right)\mathrm{d}x + \int_{l2}^{2\pi}\left(1 + 0.25y_{22}^2\right)\mathrm{d}x = S. \tag{49}$$

4 Computational examples

In order to compare results offered by two choices of FGC's profile, we study the rod with parameters $L = 2\pi$, $S = 3\pi$, $n = 5$ subjected to the following longitudinal load action

$$P(x) = p - \text{const}. \tag{50}$$

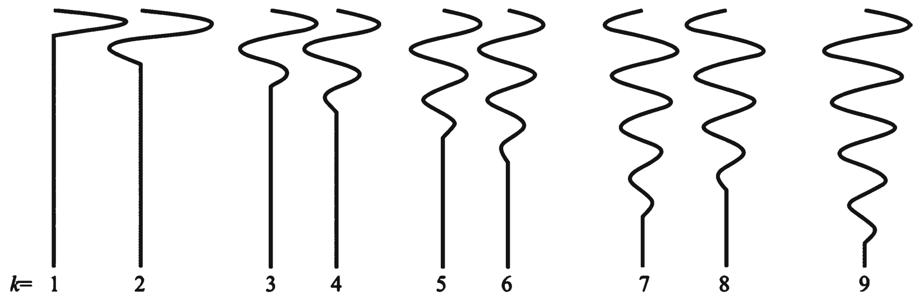

Fig. 4 Schemes of optimal FGC profiles for the amplitude gradient

4.1 Amplitude optimization

For the load (50), formula (31) yields

$$y_{11} = 2\sqrt{\ell^{\lambda_1(l-x)} - 1}. \quad (51)$$

Substituting (51) into the second continuity condition of the profile (28), we obtain nine possible values of l_1

$$l_1 = \frac{k\pi}{n}, \quad k = 1, 2, \ldots, 9. \quad (52)$$

Therefore, a straight part with zeros amplitude should be located in the vicinity of the free rod edge and should be multiple of the corrugation half-wave. Isoperimetric condition (21) yields the following nine values of the Lagrange constant

$$\lambda_{1k} = 4.64; 1.70; 0.91; 0.58; 0.40; 0.29; 0.23; 0.18; 0.15.$$

Therefore, we have obtained nine optimal designs, which are schematically presented in Fig. 4.

If one applies equation (25) to estimate the extension of the optimized rods $\Delta_{1k} = \frac{g_{1k}p}{EI}$, then obtained values are located close to each other, since $g_{1k} = 0.251; 0.250; 0.250; 0.254; 0.251; 0.246;\ \ 0.257;\ \ 0.252; 0.251$, and the largest difference is of 4 %.

However, it should be noted that for small values of k the so-far-proposed design of the corrugated rods is rather of a marginal use, since the corrugation profile is usually applicable only to small parts of the rod. Observe that the input equilibrium equation (2) has been obtained with the help of the homogenization method for large number of corrugations, i.e., for small k an error of proposed optimization procedure increases. Therefore, we take $k = 9$ for further analysis. In what follows, we are going to study the difference between the extension of the regular corrugated profile and the optimal one keeping the same length of their curvilinear axes. The optimal profile is defined by the following formula

$$\tilde{L}_1 = \int_0^{l_1} \sqrt{1 + z_{1x}^2}\mathrm{d}x + 2\pi - l_1, \quad (53)$$

where $z_{1x} = \frac{d}{\mathrm{d}x}\left(\frac{y_{11}}{n}\sin(nx)\right)$.

Substituting l_1 of (52) and y_{11} of (51) into (53) for $k = 9$, and carrying out the integration, we find $\tilde{S} = 9.11$. Amplitude of the regular corrugation (using the same length of the curvilinear axis) is equal to $h_{10} = 0.31$.

Both graphs of the optimal and regular profiles are shown in Fig. 5, whereas in Fig. 6 both of them are shifted with coinciding curvilinear axes.

Let us compare extension of the optimal FGC rod with the gradient Δ_{19} and that of the equivalent regular one with the same length of curvilinear axis Δ_{01} under action of the rod specific gravity (49). For the regular rod, we get $k_1 = \frac{0.012}{EI}$; $A_1 = 1.45$, and Eq. (2) yields $\Delta_{01} = \frac{0.34}{EI}p$. Therefore, the relative decrease of rod extension being loaded by its own weight yields the following relative error

$$\delta_1 = \frac{\Delta_{01} - \Delta_{19}}{\Delta_{01}} \times 100\,\% = 27\,\%. \quad (54)$$

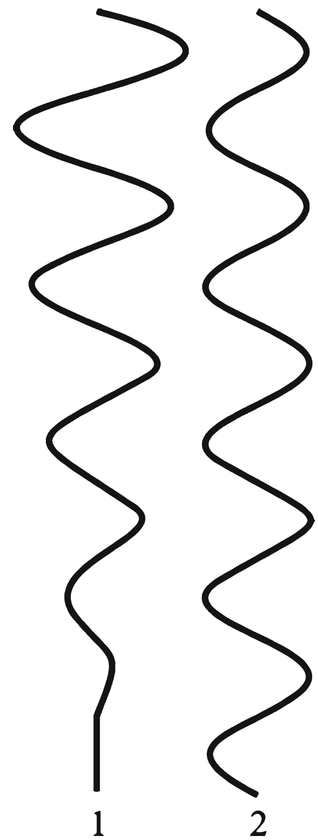

Fig. 5 Optimal FGC profile, with varied amplitude (1) for the load of the rod specific gravity and the regular profile (2) of the same length as in (1)

Fig. 6 Comparison of amplitudes of the optimal FGC and regular corrugations

4.2 Step's optimization

Taking into account (49), formula (46) yields

$$y_{21} = 0.5\lambda_{21}^2 (C_2 + px)^2. \tag{55}$$

Substituting formulas (42) and (55) into the continuity condition (41), we get

$$l_2 = \frac{1}{p}\left(C_2 + \sqrt{2\lambda_{21}(y_0 - \alpha)}\right). \tag{56}$$

Formulas (47), (55) and (42) yield

$$f_1(x) = \frac{1}{6\lambda_{21} y_0 p}(C_2 + px)^3 + C_{21}, \quad f_2(x) = \left(1 - \frac{\alpha}{y_0}\right)x + C_{22}. \tag{57}$$

In order to define constants $C_2, C_{21}, C_{22}, \lambda_{21}$, we use the condition of conservation of the number of corrugation waves (8), coupling condition (48) and isoperimetric condition (49). The obtained equations, after taking into account (56), can be cast to the form

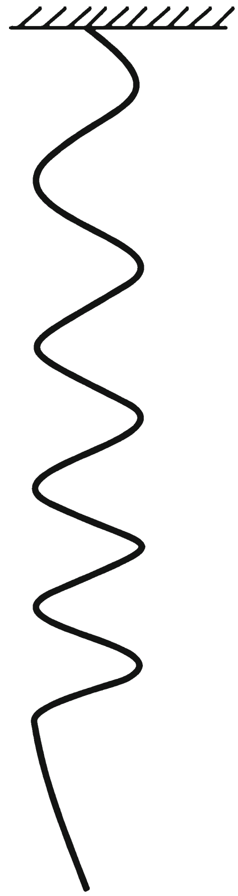

Fig. 7 Optimal FGC profile with the varied step subjected to the rod weight loading

$$C_2 = l_2 p - \sqrt{2\lambda_{21}(y_0 - \alpha)}, \quad C_{21} = \frac{\left(l_2 p - \sqrt{2\lambda_{21}(y_0 - \alpha)}\right)^3}{6\lambda_{21} y_0 p}, \quad C_{22} = \frac{2\pi\alpha}{y_0}, \tag{58}$$

$$7l_2^3 p^2 - 9l_2^2 p\sqrt{2\lambda_{21}(y_0 - \alpha)} - 12\pi\alpha\lambda_{21} = 0, \tag{59}$$

$$\frac{\left(2l_2 p - \sqrt{2\lambda_{21}(y_0 - \alpha)}\right)^5 - \left(l_2 p - \sqrt{2\lambda_{21}(y_0 - \alpha)}\right)^5}{20\lambda_{21}^2 p} + (y_0 - \alpha)^2(2\pi - \alpha) = 4(S - 2\pi). \tag{60}$$

Equations (59) and (60) can be solved numerically, and we report a solution for the following fixed parameters: $y_0 = 2, \alpha = 1, p = 1$. Note that for the given parameters system of equations (59) and (60) has only one solution in the interval $0 < l_2 < 2\pi$:

$$l_2 = 5.05, \qquad \sqrt{\lambda_{21}} = 7.72. \tag{61}$$

Substituting the values (60) into formulas (19), (57) and (58), we find an optimal corrugation profile with the varied step for the longitudinal load $p = 1$

$$z_2 = \begin{cases} z_{21}, & 0 \le x < 5.05 \\ z_{22}, & 5.05 \le x \le 2\pi \end{cases}, \tag{62}$$

where $z_{21} = h_2 \sin(0.011(10.93 + x)^3 - 14.14)$, $z_{22} = h_2 \sin(1.12x + 7.75)$.

Optimal rod profile (62) for $h_2 = 0.31$ is shown in Fig. 7.

Let us compare deformation Δ_2 of an optimal FGC rod (Fig. 5) with Δ_{02} deformation of an equivalent regular rod having same length of the curvilinear axis. General extension of the FGC of the rod $\tilde{\Delta}_s$ is $\Delta_2 = \Delta_{21} + \Delta_{22}$, where Δ_{21} denotes the extension of the FGC rod part $(0 \le x \le l_2)$; Δ_{22} is the extension of its regular part ($l_2 \leq x \leq 2\pi$). Both extensions are yielded by equation (13) for the coefficients k_2, A_2, for the profiles z_{21} and z_{22} of (62), respectively. In result, we get $\Delta_2 = \frac{0.32}{EI}$ for $p = 1$.

In order to define the extension of the equivalent regular rod, we need to estimate its amplitude. However, a direct use of the results obtained for FGC with the varied amplitude cannot be applied. The so-far numerical study of the FGC rods is based on the same averaged length of the curvilinear axis S, but it does not mean that they have the same real length, since different under integral terms (21) and (33) are averaged. Length of the curvilinear axis FG of the rod (62) follows

$$\tilde{S} = \int_0^l \sqrt{1 + \left(\frac{\mathrm{d}z_{21}}{\mathrm{d}x}\right)^2} \mathrm{d}x + \int_0^{2\pi} \sqrt{1 + \left(\frac{\mathrm{d}z_{22}}{\mathrm{d}x}\right)^2} \mathrm{d}x = 9.98. \tag{63}$$

Amplitude of a regular corrugated having the length of its curvilinear axis (62) is $h_{20} = 0.32$, and its extension for $p = 1$is $\Delta_{20} = \frac{0.35}{EI}$. Therefore, relative decrease of the rod extension under action of its weight and the change of the regular FGC with the varied step is of amount of $\delta = \frac{\Delta_{20} - \Delta_2}{\Delta_{20}} \times 100\,\% = 8.6\,\%$.

5 Conclusions

Analysis of the optimization results of FGC with varied amplitude and step has shown that for the given parameters of the weight loading, the varied amplitude is more suitable to increase the rod longitudinal stiffness. Physical modification of the obtained optimal FGC forms of rod profiles (Figs. 5, 7) follows. Longitudinal loading of the corrugated rod results in its extension due to the bending phenomenon, which mainly takes place in the corrugation tops. Its stiffness against compression–extension is defined by the magnitude of the profile curvatures on its tops. This curvature is defined mainly by the corrugations step rather than by its amplitude. Therefore, in the case of the varied amplitude, the role of a key factor of the rod extension decrease plays a shift of the rod mass into its clamping, which is displayed by the optimal rod profile shown in Fig. 5. In the case of the varied step, the source of optimization is given by the change of the longitudinal stiffness, i.e., the curvatures in the corrugation tops. Owing to the corrugation step changes of the optimal rod profile (Fig. 7), the reinforced rod stiffness is transmitted into the upper more uploaded rod part.

Inversed problems related to the general FGC rod with a simultaneous change of varied amplitude and step can be solved in a few successive steps: find an optimal profile, define its gradient amplitude and then solve the problem of the optimal varied step.

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