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ANALYTICAL PERTURBATION METHOD FOR CALCULATION OF SHELLS BASED ON 2D PADÉ APPROXIMANTS*

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Calculations of nonlinear displacements and vibrations of inhomogeneous loaded shells with developable principal surface by means of different analytical methods are represented. It is shown that solutions to these methods are the expansions of exact solution in the Taylor series for an independent variable, and in the particular case — for the powers of a natural parameter. A method that provides a polynomial asymptotic approximation of the exact solution of the general form and its meromorphic continuation based on 1D and 2D Padé approximations is proposed. Calculations of nonlinear deformation and stability of elastic flexible circular cylindrical shell under uniform external pressures and of free oscillations of simply supported stringer shell demonstrate the efficiency and accuracy of the proposed method.

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1. Introduction

Approximate analytic integration of nonlinear differential equations of solid mechanics and, in particular, the theory of flexible elastic shells in most practical cases is based on the method of continuation of solution on the artificial parameter.[4] Two kinds of them are the most frequently used recently: the Adomian's decomposition method (ADM)[1,6] and homotopy analysis method (HAM).[8,9] Both methods have a significant impact on development of the theory of nonlinear ODEs analytical solution.

It is worth mentioning that ADM and HAM can be satisfactorily applied only with an effective method of summation. The most natural analytical continuation method is Padé approximants (PAs).[2,4,5] Recently, the method of PAs for single-variable functions (1D PAs) has been successfully extended to the approximation of two variable functions (2D PAs).[5,11] In the case of ODEs both methods can be combined on the basis of a new approach, the development of which this work is devoted to.

2. Modified Method of the Parameter Continuation

The modified method of the parameter continuation (MMPC) consists of perturbation technique of special form and the analytical continuation of obtained approximations by PAs. In the vicinity of regular point in the interval $\Omega : \xi \in [0, 1]$ any ODE or system of ODEs may be represented by a normal system of ODEs of the first order wirh respect to the unknown functions $u_i = u_i(\xi), i = \overline{1,n}$ with the BC on the bounds $\partial\Omega$:

$$Lu_i + R_i(\xi, u_1, \dots, u_n) + N_i(\xi, u_1, \dots, u_n) = g_i(\xi), \quad G_i(u_1, \dots, u_n)|_{\partial\Omega} = 0,$$
$$L = \frac{d}{d\xi}, \quad i = \overline{1,n}, \tag{1}$$

here L, R_i are the linear and N_i, G_j are the nonlinear differential operators. We assume also that point $\xi_0 = 0$ belongs to closure Ω, and R_i, N_i and G_j are the holomorphic functions for $\{u_i\}_{i=1}^n$.

MMPC coincides with the HAM for the case, when N_1 does not contain Lu_1, and with the ADM — when $g \equiv 0$, and thus generalizes them. The method does not imply the introduction of ≪trial≫ functions that satisfy the BC, they will be satisfied in successive approximations, and this gives us an opportunity to solve the BVP with complicated BCs.[2] To implement the MMPC, we introduce parameter ε as follows:

$$u_i = \sum_{j=0}^{\infty} u_{ij}^M \varepsilon^j, Lu_i = \varepsilon(g_i - R_i(u_1, \dots, u_n) - N_i(u_1, \dots, u_n)),$$
$$G_i(u_1|_{\partial\Omega}, \dots, u_n|_{\partial\Omega})|_{\partial\Omega} = 0, \quad i = \overline{1,n}. \tag{2}$$

Substituting power series into equations and splitting it with respect to the powers of ε, after summation of the coefficients with the same degrees of ξ for $\varepsilon = 1$ we get

$$\begin{aligned}
u_i &= \xi^0 (u_i|_{\partial\Omega} + \cdots) \\
&\quad + \xi^1 \left(g_{i0} - \sum_{r=1}^{n} \left(N_{ir}^0 u_r|_{\partial\Omega} + \frac{1}{2!} \sum_{p=1}^{n} N_{irp}^0 u_r|_{\partial\Omega} u_p|_{\partial\Omega} + \cdots \right) + \cdots \right) \\
&\quad + \xi^2 \left(\frac{g_{i1}}{2} - \frac{1}{2} \sum_{r=1}^{n} \left(N_{ir}^1 u_r|_{\partial\Omega} + \frac{1}{2!} \sum_{p=1}^{n} N_{irp}^1 u_r|_{\partial\Omega} u_p|_{\partial\Omega} + \cdots \right) \right. \\
&\quad - \sum_{r=1}^{n} \left(N_{ir}^0 \left(\frac{g_{r0}}{2} - \left(\frac{1}{2} \sum_{l=1}^{n} N_{rl}^0 u_l|_{\partial\Omega} + \frac{1}{2!} \sum_{q=1}^{n} N_{rlq}^0 u_l|_{\partial\Omega} u_q|_{\partial\Omega} + \cdots \right) \right. \right. \\
&\quad + \frac{1}{2!} \sum_{p=1}^{n} N_{irp}^0 \left(u_p|_{\partial\Omega} \left(\frac{g_{r0}}{2} - \frac{1}{2} \sum_{l=1}^{n} \left(N_{rl}^0 u_l|_{\partial\Omega} + \frac{1}{2!} \sum_{q=1}^{n} N_{rlq}^0 u_l|_{\partial\Omega} u_q|_{\partial\Omega} \right) \right) \right. \\
&\quad \left. + u_r|_{\partial\Omega} \left(\frac{g_{p0}}{2} - \frac{1}{2} \sum_{l=1}^{n} \left(N_{pl}^0 u_l|_{\partial\Omega} + \frac{1}{2!} \sum_{q=1}^{n} N_{plq}^0 u_l|_{\partial\Omega} u_q|_{\partial\Omega} \right) \right) \right) + \cdots \Bigg) \\
&\quad \left. + \cdots \right) + \cdots, \quad i = \overline{1,n}. \qquad (3)
\end{aligned}$$

Analysis of the obtained approximation suggests that, in contrast to the ADM and HAM, it gives the exact value of the coefficients in the power of the independent variable to the extent equal to the order of approximation (taking into account the expansion in power series of expressions in the equation). This guarantees stability of the computation with a limit-order approximation of the independent variable. MMPC approximation is simpler than ADM and HAM. The approximation thus obtained is converted to 1D PA with respect to ξ or 2D PA.

The proposed approach can be used to the nonlinear problems of plates and shells theory. The equations of static of geometrically nonlinear thin-walled structures can be reduced to the resolving equations, which contain the products and squares of the desired functions and their derivatives.[10]

3. Using the PAs

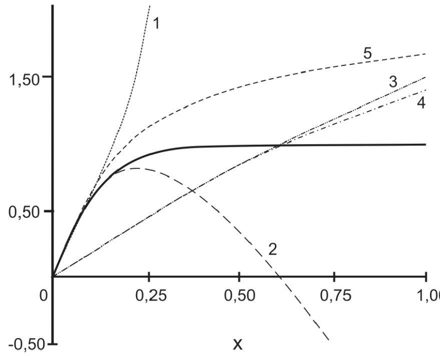
If the equations are solved with respect to the highest derivative, the coefficients of ADM and HAM approximants with the same degree of variable solutions ADM and HAM converge to each other as far as the order of approximation increases. It was shown in Ref. 2 that the solution of the ADM converges to the decomposition of exact solution in the Taylor series in the area of its holomorphy in the vicinity of zero. That is the reason that the same properties will have a solution of HAM in the case when the equation is in a normal form. This allows to use meromorphic continuation in the form of PAs.[5] For the ADM such a continuation procedure was proposed in

Ref. 2. Later, this approach was developed by a number of authors,[6] and was named modified Adomian's decomposition method and PAs (MADM-Padé). Thus, it is possible to use PAs to HAM with modifications, by decomposition of nonlinear terms in the series as for the independent variable, so for the desired function (MHAM-Padé).

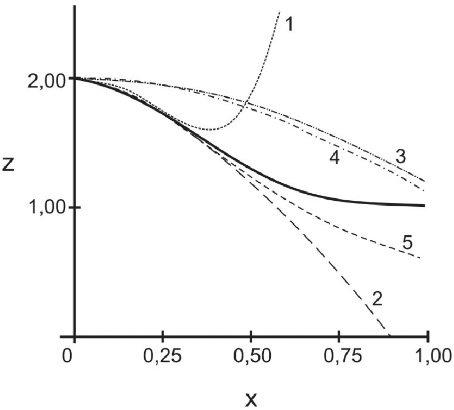
2D PAs in the form proposed Vavilov[11] is very promising for the use as an analytical continuation. This technique allows us to choose the coefficients of 2D Taylor series for construction of an unambiguous 2D PA with a given structure of the numerator and denominator, as well as ensures optimal PAs features in the sense of the Theorem of Montessus de Ballore-type. This means homogenous convergence of PA to the approximated function with an increase of the degree of the numerator and the denominator in all points of its meromorphy area. It should be noted that direct application of 2D PAs does not lead to the anticipated merging of 1D approximations. This is due to the initial requirements to the 2D approximation to ensure its transition to 1D in the case when the second variable is equal to zero.[11] At the same time as for the method of parameter continuation it is necessary to ensure such a transition when the parameter is equal to one.

4. Numerical Results for Examples

We consider three types of PAs — on the independent variable $z^{(x)}$, on the specified parameters $z^{(\varepsilon_1)}$, and 2D. Typical behavior of the approximations for the BVP, where natural small parameter ε is the factor at the highest derivative, is shown in Fig. 1(a) for $\varepsilon = 0.1$. The ADM approximation well describes the exact solution only for a distance which is comparable with the value of the natural small parameter ε. Despite the fact that the error of solutions of HAM is substantially lesser than the



(a) $\varepsilon z' + z = 1; z(0) = 0; x \geq 0$



(b) $\varepsilon z' + xz = x; z(0) = 2; x \geq 0$

Fig. 1. The exact solution (solid line) and approximate solutions ((i) three terms $z^{(x)}$ for ADM, (ii) $z^{(\varepsilon_1)}$ for ADM, (iii) three terms $z^{(x)}$ for HAM, (iv) $z^{(x)}$ for HAM, (v) 2D Padé for MMPC, ADM, HAM).

ADM, HAM does not accurately reflect the nature of solutions, namely the phenomenon of boundary layer in the vicinity of zero. At the same time, PAs for the ADM approximations for independent variable and PAs for the MMPC (1D and 2D) give satisfactory qualitative and quantitative results. Similar results give us an analysis of approximations whose coefficients are given depending on the variable for $\varepsilon = 0.2$ [Fig. 1(b)]. The graphs show that the solution is well described by the HAM approximation and MHAM-Padé ≪in average≫, and badly — in the boundary layer. The ADM approximation and MADM-Padé, on the contrary, is in good agreement with the behavior of solution in the vicinity of zero and in the bad one — on the stationary part. At the same time, 1D and 2D PAs, based on approximations of the MMPC, well described the solution in the whole interval.

5. Calculation of Nonlinear Deformation of Shells

The MMPC was applied to calculate the deformation and stability of a long flexible elastic circular cylindrical shell of radius R with half the central angle β_0 in the case of cylindrical bending under uniform external pressure with simple support of the longitudinal edges. The system of resolving equations in normal form is given in Ref. 7. Dependencies "measureless intensity of pressure P — deflection w/R" for the top cross-section of shell at different angles and dimensionless flexibility $C = 10^{-4}$ and "measureless intensity of the limit load P_b — size of half angle β_0" is shown in Fig. 2. For comparison, Fig. 2(b) also shows the dependence of the critical loads for inextensible shell obtained Timoshenko.[7] We see that dependences are in good agreement, while consideration of deformation of the longitudinal axis substantially affects the value of critical loads of construction.

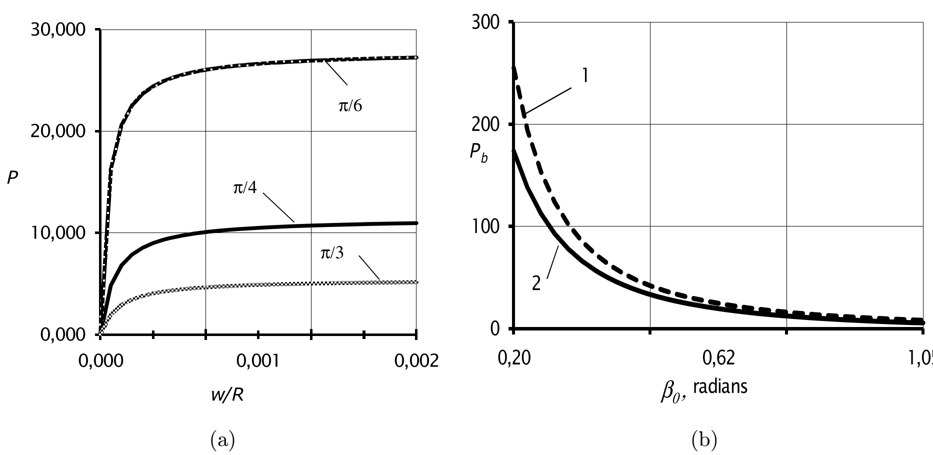


Fig. 2. Calculation of shell under uniformly distributed radial load. (a) the dependence of the measureless intensity of pressure P from the deflection w/R for different values of β_0 (the value of β_0 in radians is indicated by the curves), (b) the dependence of limit loads P_b from β_0 [(i) data[7], (ii) calculation].

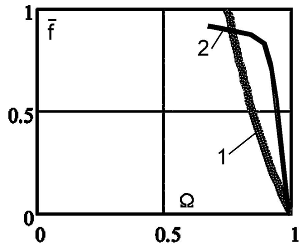


Fig. 3. The dependence of the oscillation frequency of stringer shell from the amplitude of the initial disturbance. [(i) according the proposed method, (ii) data,[3] all data are measureless, all the symbols are taken in accordance with Ref. 3].

The proposed method can be used in combination with the known asymptotic method. Consider the free vibrations of a flexible elastic circular cylindrical shell of radius R, thickness h and length L, backed by a set of uniformly stringers, with simple support at the ends. The calculation is based on mixed dynamical equations of the theory of shells after splitting them in powers of natural small parameters.[3,4] Governing equations can be reduced by the Bubnov–Galerkin method to the Cauchy problem with respect to $\xi = f_1/R$ on $t_1 = t\sqrt{B_1/\rho R^2}$ (all the symbols are taken in accordance with Ref. 3)

$$\ddot{\xi} + \alpha\xi\left[\left(\dot{\xi}\right)^2 + \xi\ddot{\xi}\right] + A_1\xi + A_2\xi^3 + A_3\xi^5 = 0, \quad t_1 = 0 : \xi = f, \quad \dot{\xi} = 0. \tag{4}$$

Application of the method to the problem (4) gives the approximation of second order for the artificial parameter for the frequency Ω of nonlinear oscillations. It is seen that the oscillations are non isochronous. This agrees well with previous results (Fig. 3), while is significantly reduced the volume of computations (in Ref. 3 to obtain similar results the approximation of fourth order is taken).

6. Conclusion

Proposed method of the parameter continuation (MMPC) enables simplification of the calculations both at the stage of constructing the model, and also within its continuation due to the precise values of the Taylor coefficients for the solution of the degree which is not exceeding the number of approximation. It was concluded that the application of 1D and 2D PAs is justified if it is applied to polynomials, which depend on the variable of integration. It has been shown that 2D PAs for the independent variable and for the artificial parameter used the scheme of Vavilov provides a satisfactory quality for the approximation behavior and minimizes its error. A study of numerical results for model examples confirms the advantage of MMPC approximations and shows that the application of PAs provides them with sufficient accuracy in the studied area. Calculations of nonlinear deformation and stability of elastic flexible circular cylindrical shell under uniform external pressures and of free oscillations of simply supported stringer shell demonstrate the efficiency and accuracy of the proposed method.

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