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ANALYSIS OF CHAOTIC VIBRATIONS OF FLEXIBLE PLATES USING FAST FOURIER TRANSFORMS AND WAVELETS

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In this paper chaotic vibrations of flexible plates of infinite length are studied. The Kirchhoff–Love hypotheses are used to derive the nondimensional partial differential equations governing the plate dynamics. The finite difference method (FDM) and finite element method (FEM) are applied to validate the numerical results. The numerical analysis includes both standard (time histories, fast Fourier Transform, phase portraits, Poincaré sections, Lyapunov exponents) as well as wavelet-based approaches. The latter one includes the so called Gauss 1, Gauss 8, Mexican Hat and Morlet wavelets. In particular, various plate dynamical regimes including the periodic, quasi-periodic, sub-harmonic, chaotic vibrations as well as bifurcations of the plate are

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illustrated and studied. In addition, the convergence of numerical results obtained via different wavelets is analyzed.

Keywords: Chaos; bifurcation; plates; fast Fourier transform; wavelets.

1. Introduction

Recent achievements obtained in 80−90th years of the previous century devoted to numerical analysis of PDEs allowed direct use of the earlier developed theories regarding mechanics of deformable objects giving possibility for getting reliable results concerning nonlinear behavior of complex mechanical structures. Elastic panels plates/shells are widely used as elements of thin walled structures applied in air and rocket industries. It is clear that investigation of those elements subjected to an action of dynamical loading belongs to one of the fundamental problems regarding stability and durability of various structures. Another problem refers to the estimation of a construction dynamical regime versus parameters of external loading (excitation amplitude and frequency) taking into account dispersive properties of a surrounding medium, where the being analyzed construction is embedded.

On the other hand the recent developed theories of bifurcation and chaos allowed achieving new results being close to real nonlinear dynamical behavior of plates, panels and shells as well as their combinations and interaction in various mechanical structures.

In Ref. 1, both global bifurcations and chaotic dynamics of parametrically excited simply supported rectangular this plate have been studied analytically and numerically. Both fractal dimension and maximum Lyapunov exponent concepts are used in Ref. 2 to study chaotic dynamics a simply supported large deflection rectangular plate with thermo-mechanical coupling. Bifurcation points, period doubling phenomena under various lateral and bi-axial loads of thermo-mechanical coupling factors and aspect ratios have been detected and discussed. Periodic and chaotic dynamics and jumping phenomena of a simply supported cross-ply laminated rectangular thin plate subjected to the parametric excitation using von Kármán-type equation have been analyzed in Ref. 3. Asymptotic four-dimensional nonlinear averaged equations regarding the amplitude and phase of plate vibrations, and then relations between the steady-state nonlinear responses and the amplitude and frequency of the parametric excitations have been derived. Nonlinear and chaotic dynamics of a parametrically excited simply supported symmetric thin plate has been reduced to a two degree-of-freedom system, and then the multiple scales method is used to derive the averaged equations in Ref. 4. In particular, the occurrence of periodic, quasi-periodic and chaotic dynamics for a parametrically excited four-edges simply supported plate versus the parametric excitation magnitudes have been studied. Zhang *et al.*[5] have studied global bifurcations and multi-pulse chaotic dynamics of a simply supported laminated composite piezoelectric rectangular thin plate subjected to combined parametric and transverse excitations. The von

Kármán-type equation and the Reddy's third order shear deformation plate theory supported by the Galerkin procedure and the Melnikov method have been successfully applied to predict and study the multi-pulse plate chaotic dynamics. Zhang *et al.*[6] applied the extended Melnikov technique to study multi-pulse Shilnikov type chaos of a parametrically and externally excited simply supported rectangular thin plate using the von Kármán equation and Galerkin approach. Chaotic and bifurcational dynamics of various mechanical structural members (flexible plates, cylinder-like panels, rectangular spherical and cylindrical shells, axially symmetric plates, as well as spherical and conical shells) have been studied recently in Refs. 7−9. First an emphasis is put for reliability and validation of the numerically obtained results using the Finite Difference, Bubnov-Galerkin and Ritz methods.[7] Then, either known or novel scenarios of routes to chaos are detected, illustrated and discussed.[8] Finally, transitions chaos−hyper chaos as well as chaos−hyper chaos−hyper-hyper chaos are illustrated and studied.[9] It should be emphasized that the wavelet-type analysis has been also applied to study coherent structures[10] as well as the regenerative cutting processes.[11] In this paper, we extend our earlier approaches devoted to application of the wavelets to study nonlinear dynamics of continuous mechanical systems.[12,13]

2. Object of Investigation

In what follows we consider infinitely length flexible one-layered thin plates of a wideness a, height h and a curvature k_x. The plate is loaded through its whole surface by $q(x, t)$ acting in direction of a normal to the middle plate surface (see Fig. 1).

Developed and further applied mathematical model is based on the following hypotheses: (i) any plate cross-section being normal to the middle surface after deformation remains straight and normal to the middle surface, i.e. the cross-section height does not change; (ii) inertia effects of rotation of plate elements are not taken into account although inertial forces associated with a plate element movement along the normal are taken into account; (iii) external forces do not change their directions while the plate deformations take place.

Although the applied computational scheme is based on the Kirchhoff−Love hypotheses and is treated as the model of first approximation, it is sufficient for engineering oriented analysis as it has been pointed out in Ref. 14.

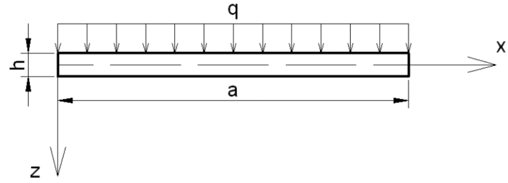


Fig. 1. The investigated plate.

Mathematical plate model is governed by two PDEs with regard to displacements of the form

$$\begin{aligned}
&Eh\left(\frac{\partial^2 u}{\partial x^2} - k_x\frac{\partial w}{\partial x} + L_3(w,w)\right) - \frac{\gamma}{g}h\frac{\partial^2 u}{\partial t^2} = 0,\\
&Eh\left\{-\frac{h}{12}\frac{\partial^4 w}{\partial x^4} + k_x\left[\frac{\partial u}{\partial x} - k_x w - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2}\right]\right.\\
&\qquad\left. + L_1(u,w) + L_2(w,w)\right\} + q - \frac{\gamma}{g}h\frac{\partial^2 w}{\partial t^2} - \varepsilon\frac{\gamma}{g}h\frac{\partial w}{\partial t} = 0,
\end{aligned} \tag{1}$$

$$L_1(u,w) = \frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}, \quad L_2(w,w) = \frac{3}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2},$$

$$L_3(w,w) = \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2},$$

where $w(x,t)$ denotes the element displacement in normal direction; $u(x,t)$ is the element displacement in longitudinal direction; ε denotes the dissipation coefficient; E is the Young modulus; h describes height of the transversal plate element cross-section; γ is the unit specific weight; g is the Earth acceleration; t denotes time, whereas $q = q_0 \sin\omega_p t$ is the external periodic load. After introducing the following nondimensional parameters,

$$\begin{aligned}
&\lambda = \frac{a}{h}, \quad \bar{w} = \frac{w}{h}, \quad \bar{u} = \frac{ua}{h^2}, \quad \bar{x} = \frac{x}{a}, \quad \bar{t} = \frac{t}{\tau},\\
&\tau = \frac{a}{c}, \quad c = \sqrt{\frac{Eg}{\gamma}}, \quad \bar{\varepsilon} = \frac{a}{c}, \quad \bar{q} = \frac{qa^4}{h^4E}, \quad \bar{k}_x = \frac{k_x a}{\lambda},
\end{aligned} \tag{2}$$

our PDEs take the following form:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} - k_x\frac{\partial w}{\partial x} + L_3(w,w) - \frac{\partial^2 u}{\partial t^2} = 0,\\
&\frac{1}{\lambda^2}\left\{-\frac{1}{12}\frac{\partial^4 w}{\partial x^4} + k_x\left[\frac{\partial u}{\partial x} - k_x w - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2}\right]\right.\\
&\qquad\left. + L_1(u,w) + L_2(w,w)\right\} + q - \frac{\partial^2 w}{\partial t^2} - \varepsilon\frac{\partial w}{\partial t} = 0.
\end{aligned} \tag{3}$$

In what follows one of the following boundary conditions will be used together with Eq. (3):

$$w(0,t) = w(a,t) = u(0,t) = u(a,t) = w'_x(0,t) = w'_x(a,t) = 0, \tag{4}$$

$$w(0,t) = w(a,t) = u(0,t) = u(a,t) = w''_{xx}(0,t) = w''_{xx}(a,t) = 0, \tag{5}$$

$$w(0,t) = w(a,t) = u(0,t) = u(a,t) = w'_x(0,t) = w''_x(a,t) = 0, \tag{6}$$

$$w(0,t) = w'_x(0,t) = u(0,t) = 0; \quad M_x(a,t) = N_x(a,t) = Q_x(a,t) = 0, \tag{7}$$

and the following initial conditions are taken:

$$w(x,0) = \dot{w}(x,0) = u(x,0) = \dot{u}(x,0) = 0. \tag{8}$$

In order to reduce our PDEs to a system of ODEs with respect to time we apply both finite difference approximations and the Taylor series in the neighborhood of a point x_i. Let us consider the following mesh space $G_N = \{0 \leq x_i \leq 1,\ x_i = i/N,\ i = 0, \ldots, N\}$.

The following difference operators with the approximation $O(c^2)$, where c denotes the step of spatial coordinate, are introduced: $\Lambda_x(\cdot_i) = \frac{(\cdot)_{i+1}-(\cdot)_{i-1}}{2c}$; $\Lambda_{x^2}(\cdot_i) = \frac{(\cdot)_{i+1}-2(\cdot)_i+(\cdot)_{i-1}}{c^2}$; $\Lambda_{x^4}(\cdot_i) = \frac{(\cdot)_{i+2}-(\cdot)_{i+1}+6(\cdot)_i-(\cdot)_{i-1}+(\cdot)_{i-2}}{c^4}$.

As a result of the above mentioned procedure we get the following system of second order ODEs:

$$\begin{aligned}
\ddot{u}_t &= \Lambda_{x^2}(u_i) - k_x\Lambda_x(w_i) + \Lambda_x(w_i)\Lambda_{x^2}(w_i), \\
\ddot{w}_t + \varepsilon\dot{w}_t &= \lambda^2\Big\{-\frac{1}{12}\Lambda_{x^4}(w_i) + k_x[\Lambda_x(u_i) - k_x w_i - w_i\Lambda_{x^2}(w_i)] \\
&\quad + \Lambda_{x^2}(u_i)\Lambda_x(w_i) + \Lambda_{x^2}(w_i)\Lambda_x(u_i) + \frac{3}{2}(\Lambda_x(w_i))^2\Lambda_{x^2}(w_i) + q\Big\}
\end{aligned} \tag{9}$$

Then system (9) is reduced to a set of first order ODEs and solved via the fourth order Runge−Kutta method. Boundary conditions (4)−(7) also undergo a similar like modification.

3. Numerical Analysis

It should be emphasized that the so far obtained system (9) cannot be solved analytically. In order to validate and verify the further obtained results through the method of finite differences (FDM), a comparison with results obtained via finite element method (FEM) in the form of Bubnov−Galerkin is carried out. The comparison results are reported in Table 1 including the following dynamical characteristics: a time history ($w(0.5,t)$), fast fourier transform (FFT) and a Morlet wavelet-spectrum in the 2D and 3D forms. Although one may observe signal differences but a qualitative character of the signal modulation is similar, what belongs to characteristic features while investigating a chaotic signal. Frequency spectra of the chaotic signal overlap qualitatively in vicinity of the fundamental frequencies, although a difference in a frequencies number occurs. On the other hand, wavelets exhibit the results convergence. Assuming the convergence of results obtained either by FEM or FDM, further FDM has been used because it requires less computational time in comparison to FEM.

In order to analyze the nonlinear dynamics of flexible long plates the following dynamical characteristics have been applied: a signal, a power spectrum, a phase portrait, a Poincaré section and the Lyapunov exponents for each set of parameters q_0, ω_p, allowing finally for a determination of various kinds of vibrations including

Table 1. Comparison between FDM and FEM.

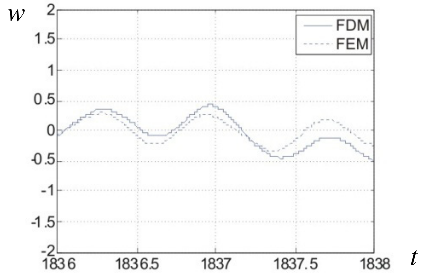
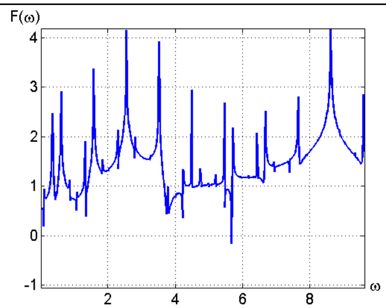
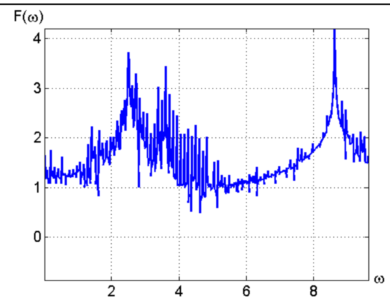
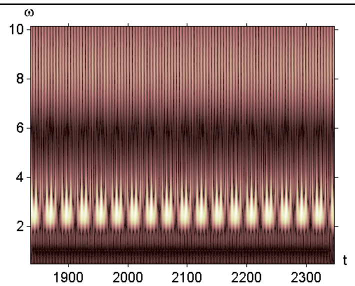
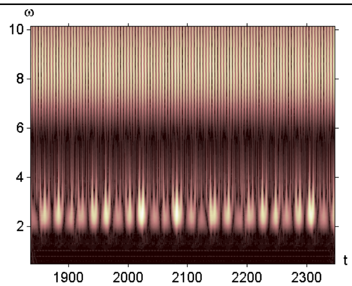
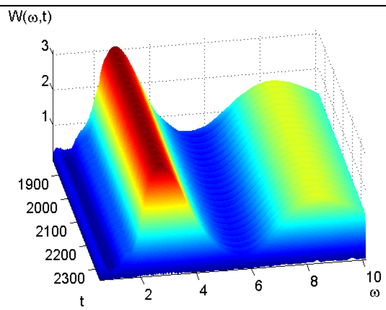
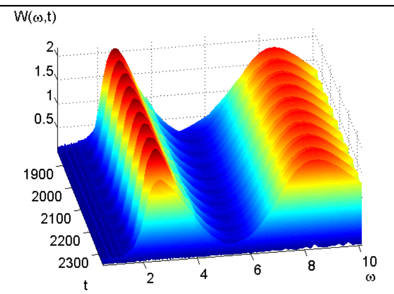
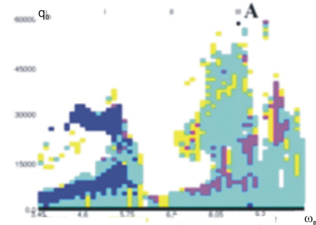
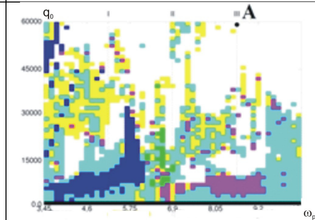
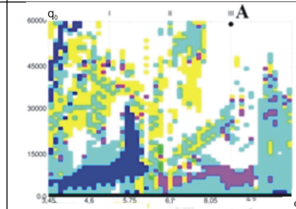
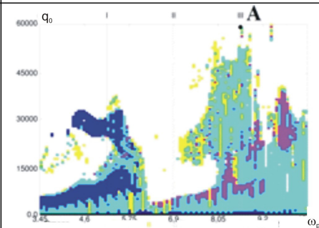
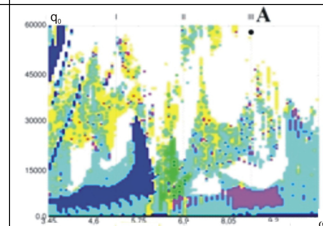
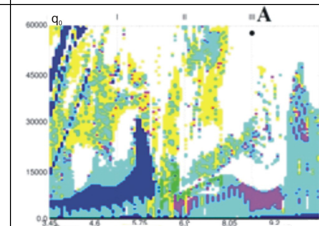
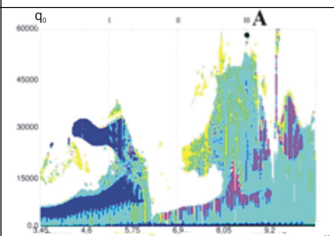
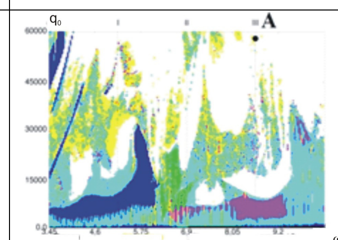
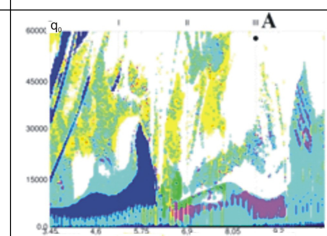
FDM	FEM
	
	
	
	

Table 2. Vibrational regimes versus number of nodes n.

	n=16	n=32	n=40
50×50			
100×100			
200×200			

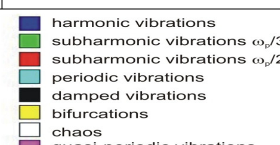
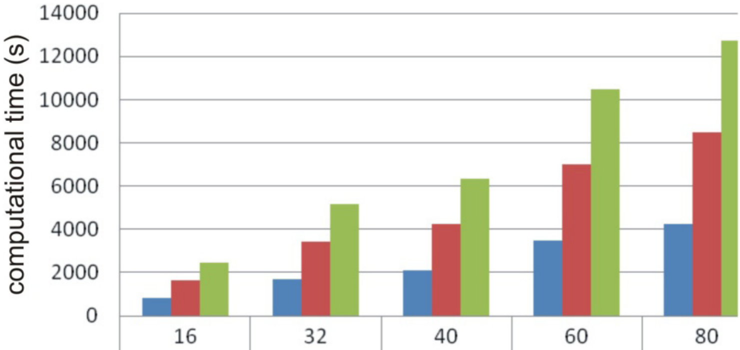


Table 3. Computational time versus number of spatial partitions n.



	16	32	40	60	80
100x100	826	1714	2120	3498	4240
200x200	1652	3428	4240	6996	8480
300x300	2478	5142	6360	10494	12720

periodic, quasi-periodic and chaotic ones. In result, various vibration regimes are presented in a graph form versus control parameters q_0, ω_p. Charts of vibration regimes exhibit various zones of nonlinear behavior allowing to either keep or avoid a required regime. However, to validate computational results one needs to define an optimal number of nodes (n) of a computational mesh, as well as a suitable number of partitions of intervals regarding both frequency (ω_p) and excitation amplitude (q_0). In Table 2 vibrational regimes versus $\{\omega_p, q_0\}$ are reported. Note that harmonic, periodic, subharmonic, quasi-periodic, chaotic vibrations as well as bifurcation dynamics are distinguished by different colours. However, here we do not study multiple solutions.

As it has been already mentioned, the construction of a vibration chart usually requires a long computational time. Owing to data reported in Table 3 one may conclude that the most optimal choice refers to the resolution of 200×200 and the computational time is proportional to the resolution number, i.e. the mesh nodes number n. Similar like observation holds also while increasing the resolution number.

Applied algorithm for charts constructions allows detection of periodic, quasi-periodic and chaotic regimes including bifurcation zones. However, results given by the chart do not allow estimating a convergence of the numerical process regarding the partition number (n) of the spatial coordinate. In order to verify results reliability the following parameters are fixed: $\varepsilon = 1$; $\lambda = \frac{a}{h} = 50$; $\omega = 8,625$; $q_0 = 59000$; $k_x = 0$, as well as boundary conditions (5) and initial conditions (8) are taken. For the given parameters the investigated system exhibits chaos, which results from the charts (see point A of Table 2).

Let us study the obtained signal regarding a number of partition of the spatial coordinate. For this purpose time histories frequency power spectra and Poincaré maps (Table 4) and wavelets (gaus1, gaus8, mexh, morl)— see Table 5 — are

Table 4. Time histories, frequency power spectra and Poincaré maps versus n.

n=16	n=32	n=60	n=80

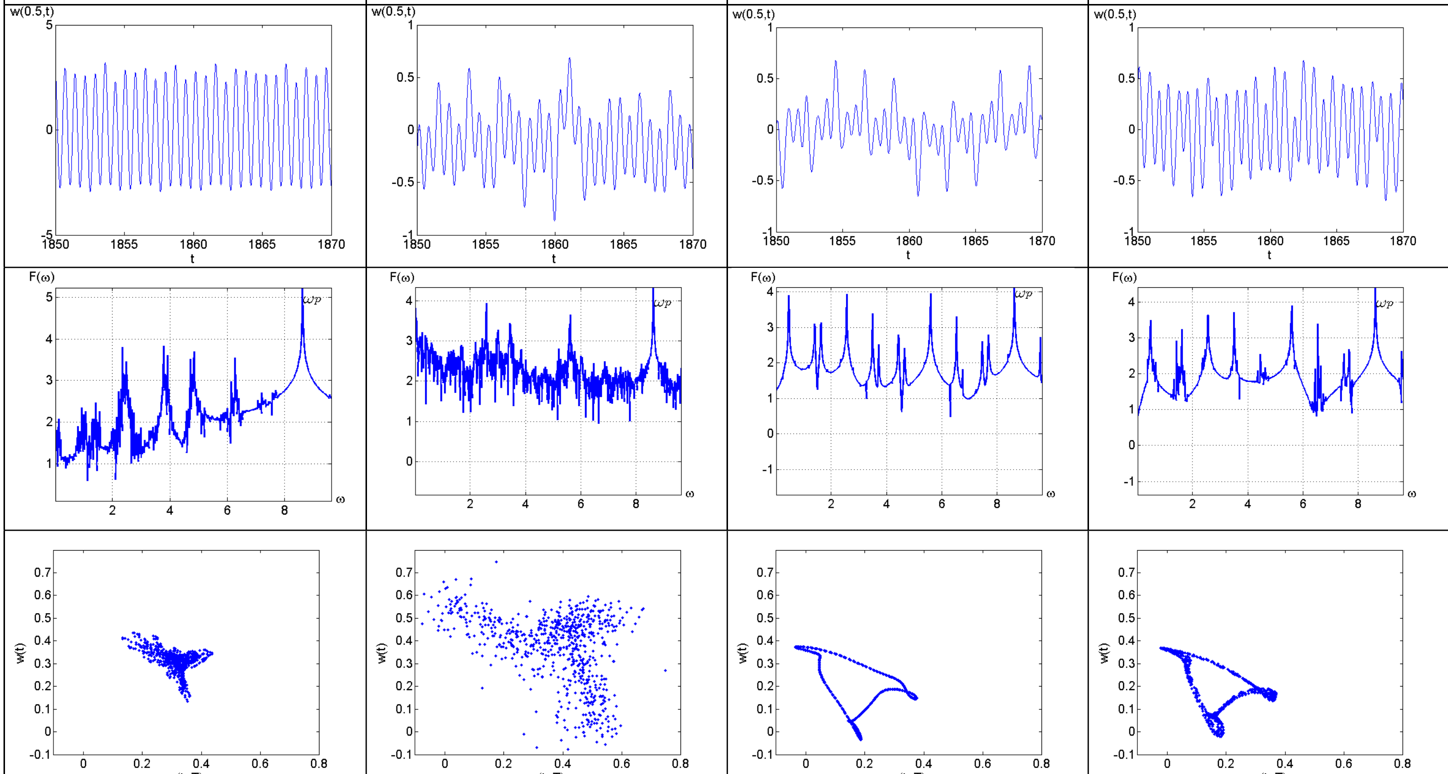
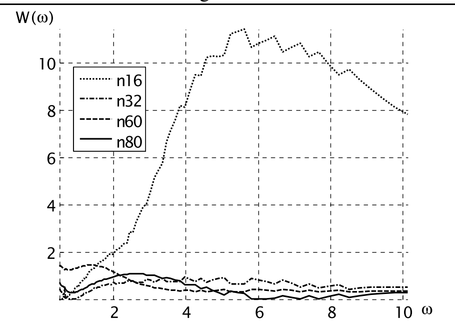
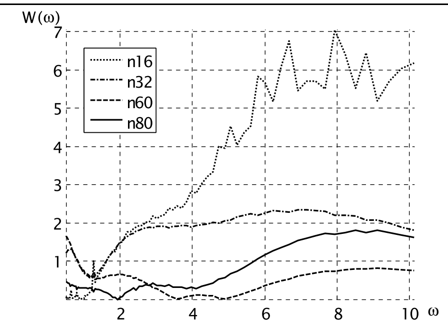
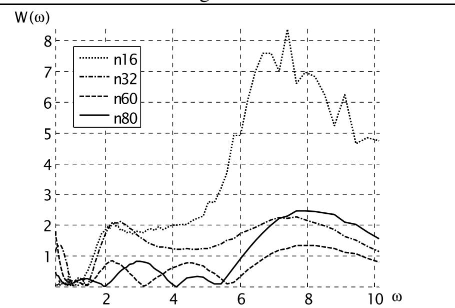
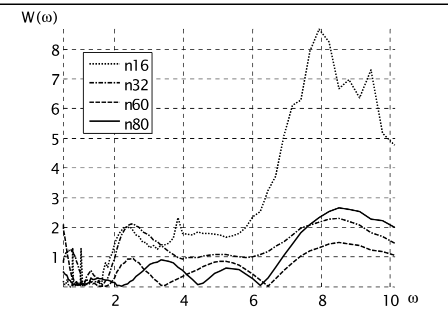


Table 5. Various wavelets for different n.

gaus1	mexh
	
gaus8	**morl**
	

constructed, where abbreviations of the applied wavelets correspond to Gauss 1, Gauss 8, Mexican Hat and Morlet ones, respectively.

A study of data of Table 4 shows that the time history $w(0.5, t)$ converges with respect to amplitude of the maximum deflection for $n \geq 32$. In the FFT power

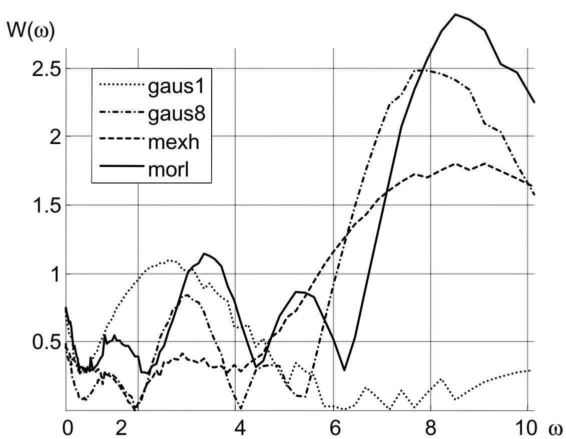


Fig. 2. Comparison of different wavelets.

spectrum for various partition of the spatial coordinate one may observe the same frequencies localization, and in particular the frequency of excitation ω_p, but again the convergence is not observed. In the Poincaré map for $n \geq 60$ a strange chaotic attractor appears, which is robust and preserves even for higher partition number. The wavelet transform becomes one of the most effective tools while investigated nonlinear dynamical systems. Data of Table 5 (wavelet-type analysis) yields the observation that for $n \geq 60$ for various Morlet wavelets the convergence is achieved regarding both energy $W(\omega)$ as well as positions distributions of frequencies ω. In the case of partition $n = 80$ the most convergent wavelets are the Morlet and Gauss of the Eighth (see Fig. 2).

The so far carried out analysis allows taking the following reliable choice: method of solution to differential equations—finite difference method (FDM) and the vibration charts resolution with the 200×200 number of partitions regarding the spatial coordinate ($n = 60$).

4. Concluding Remarks

In this paper the derived PDEs governing dynamics of flexible infinite plates are reduced to ODEs via either the FEM or FDMs. It has been illustrated that the results obtained via the FDM require less computational time in comparison to the FEM, and hence the FDM has been finally validated and used for further numerical analysis. In particular, we have shown how to choose an optimal number of nodes (n) of a computational mesh, as well as a suitable number of partitions (resolutions) of two control parameters (q_0, ω_p). It has been shown that depending on the choice of two mention parameters the investigated plate may exhibit the regular (periodic, harmonic, sub-harmonic, quasi-periodic), chaotic as well as bifurcation dynamics. It has been illustrated and discussed, among other, that the most suitable wavelets to study the chaotic and bifurcational dynamics of plates having the infinite length are the Morlet and Gauss 8 wavelets.

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References

1. W. Zhang, Global and chaotic dynamics for a parametrically excited thin plate, *J. Sound Vib.* **239**(5) (2001) 1013−1036.
2. Y.-L. Yeh, C.-K. Chen and H.-Y. Lai, Chaotic and bifurcation dynamics for simply supported rectangular plate of thermo-mechanical coupling in large deflection, *Chaos Solitons Fractals* **13**(7) (2002) 1493−1506.
3. M. Ye, Y. Sun, W. Zhang, X. Zhan and Q. Ding, Nonlinear oscillations and chaotic dynamics of an antisymmetric cross-ply laminated composite rectangular this plate under parametric excitation, *J. Sound Vib.* **287**(4−5) (2005) 723−758.

4. W. Zhang, S. Song and M. Ye, Further studies on nonlinear oscillations and chaos of a symmetric cross-ply laminated thin plate under parametric excitation, *Int. J. Bifur. Chaos* **16**(2) (2006) 325–347.
5. W. Zhang, J. H. Zhang, M. H. Yao and Z. G. Yao, Multi-pulse chaotic dynamics of non-autonomous nonlinear system for a laminated composite piezoelectric rectangular plate, *Acta Mech.* **211**(1–2) (2009) 23–47.
6. W. Zhang, J. H. Zhang and M. H. Yao, The extended Melnikov method for non-autonomous nonlinear dynamical systems and application to multi-pulse chaotic dynamics of a buckled thin plate, *Nonlinear Anal. Real World Appl.* **11**(3) (2010) 1441–1457.
7. J. Awrejcewicz, V. A. Krysko, I. V. Papkova and A. V. Krysko, Routes to chaos in continuous mechanical systems. Part 1. Mathematical models and solution methods, *Chaos Solitons Fractals* **45** (2012) 687–708.
8. J. Awrejcewicz, V. A. Krysko, I. V. Papkova and A. V. Krysko, Routes to chaos in continuous mechanical systems: Part 2, Modelling transitions from regular to chaotic dynamics, *Chaos Solitons Fractals* **45** (2012) 709–720.
9. J. Awrejcewicz, V. A. Krysko, I. V. Papkova and A. V. Krysko, Routes to chaos in continuous mechanical systems. Part 3. The Lyapunov exponent, hyper, hyper-hyper and spatial-temporal chaos, *Chaos Solitons Fractals* **45** (2012) 721–736.
10. W. J. Staszewski and K. Worden, Wavelet analysis of time-series: Coherent structures, chaos and noise, *Int. J. Bifur. Chaos* **9** (1999) 455–471.
11. G. Litak, A. K. Sen and A. Syta, Intermittent and chaotic vibrations in a regenerative cutting process, *Chaos Solitons Fractals* **41** (2009) 2115–2122.
12. J. Awrejcewicz, A. V. Krysko and V. Soldatov, On the wavelet transform application to a study of chaotic vibrations of the infinite length flexible panels driven longitudinally, *Int. J. Bifur. Chaos* **19**(10) (2009) 3347–3371.
13. J. Awrejcewicz, A. V. Krysko and V. Soldatov, Analysis of the nonlinear dynamics of the Timoshenko flexible beams using wavelets, *J. Comput. Nonlinear Dyn.* **7**(1) (2012) 011005-1–011005-14.
14. A. C. Volmir, *Nonlinear Dynamics Plates and Shells* (Nauka, Moscow, 1972).