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Sensitivity analysis in design of constructions made of functionally graded materials

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Abstract
This article is focused on analysis of influence of functionally graded material parameters in the problem of longitudinal rod deformations. This analysis is based on exact and asymptotic solutions. Accuracy rating of the proposed asymptotic method of calculating deformations in constructions made of functionally graded material is also given.

Keywords
Functionally graded material, longitudinal rod deformation, asymptotic approach, homogenization method

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Introduction

The functionally graded constructions and materials are represented by structures, whose characteristics are continuously changing along one or more directions due to a certain rule. Designing functionally graded structures (FGSs) with a simultaneous control of the gradient properties allows improving their operating characteristics. FGS can be divided into homogeneous (for example, plates and shells of variable thickness) and heterogeneous (non-homogeneous) structures (for instance, ribbed, perforated or corrugated plates and shells, as well as composite materials). Fundamental results, allowing to estimate a sensitivity of the main characteristics of the FGS according to the changes of design parameters, for homogeneous structures have been reported in Haug et al.[1] and Choi and Kim.[2] However, in practice, it is frequently required to increase the costs for manufacturing of such FGS. On the other hand, it is achievable to reduce costs and provide the functionally graded material (FGM) heterogeneous characteristics using their quasi-periodic properties.

Only recently, the above-mentioned FGMs have been applied in manufacturing of FGS (for instance, alloys composed of stiff grains and metallic ligament, which content is continuously changing within the material volume). Nevertheless, there is a lack of any theoretical basis concerning sensitivity estimation of the FGM constructions along with the changes of material characteristics. Publications on the subject mainly concern the 'direct' problem solution: evaluation of different physical fields in FGS. Main approach during analysis and design of FGMs is being based on the finite element method, but it is associated with many serious computational problems.[3–5] A different approach consists in extending micromechanical models originally developed for statistically uniform random composites to the case of statistically non-uniform (graded) random composites.[6–8] Non-linear analysis of plates and shells made of FGM is studied in Shen.[9] Analytical solution of the mechanical behaviour of FGMs based on the first- and third-order shear deformation theory is reported in Akbarzadeh et al.[10] and Khazaeinejad and Najafizadeh.[11] Galerkin's approach for buckling analysis of rectangular plates

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composed of FGMs is applied by Najafizadeh et al.[12] On the other hand, an analytical method based on the Mindlin plate theory is introduced by Mohammadi et al.[13] In papers Anthoine[14] and Nematollahi et al.,[15] the homogenization theory for periodic media is generalized to the case of quasi-periodic media. In addition, in that work, state of the art of computational methods devoted to FGMs is presented and discussed.

For solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with periodic coefficients, the asymptotic homogenization method has been developed, for example, being reported in Manevitch et al.[16] In Bolshakov et al.,[17] this method is used to calculate periodic properties exhibited by non-homogeneous composite materials with regard to micro-mechanical effects. Furthermore, in Andrianov et al.,[18,19] a modification of the homogenization method for calculating the quasi-periodic structures has been proposed. These modifications are applied to compute physical fields in the FGMs.

The aim of this article is to investigate the dependence between reaction of deformation of FGSs and changes in the characteristics of FGM. For this purpose, the exact solution is studied. We apply here the modified method of Andrianov et al.,[18,19] which yields analytical solutions. This allows analysing the mode of deformation of the FGSs, without resorting to the variational methods.[1,2]

In 'Problem formulation and exact solution' section, an equation of longitudinal rod deformation made of FGMs is given; analysis of influence of a function, defining gradient properties on the FGM parameters is carried out and bounds for this function are formulated. The exact solution of the problem is found. Then, the 'Asymptotic solution' form for small gradients is formulated, which allows to study a sensitivity of the mode of deformation to gradient changes. 'Computational examples and accuracy estimation of the asymptotic solution' section yields numerical examples of exact and asymptotic solutions, comparison of which allowed estimating the accuracy of the asymptotic solution and analysing the gradient influence on the deformable condition. Subsequent sections investigate the influence of heterogeneous parameters and gradient on strained condition based on the asymptotical solution, as well as a scheme of decrease in the strain amplitude with the help of the gradient concept is formulated.

Problem formulation and exact solution

Consider the longitudinal deformation $u(x)$ of a rod made of FGMs with fixed ends and subjected to action of the distributed longitudinal load $q(x)$. This deformation is governed by the following boundary value problem

$$\frac{d}{dx}\left\{a\left[\frac{f(x)}{\varepsilon}\right]\frac{du}{dx}\right\} = q(x) \tag{1}$$

$$u(0) = u(1) = 0 \tag{2}$$

where $a = a(x)$ is an elasticity coefficient with respect to the rod elongation–compression being a periodic function with regard to x and with the period $\varepsilon << 1$ and $f(x)$ a function characterizing the material gradient properties. In what follows, we further use the abbreviation gradient instead of a material gradient. This gradient is reached due to functional changes of grains concentration and their size. Thus, the approximate formula for variable step of non-homogeneous cells $T(x)$ can be described in the following way

$$\varepsilon = \Delta f(x) \approx f'(x)\Delta x; \;\; T = \Delta x \approx \frac{\varepsilon}{f'(x)} \tag{3}$$

Consequently, if $f'(x) > 1$, then the non-homogeneous step decreases, and if $0 < f'(x) < 1$, then it increases.

Figure 1 shows the changes in heterogeneity of cell boundaries $x_1, x_2, x_3, \ldots,$ for the arbitrarily given function $y = f(x)$, where the line 1 ($y = x$) corresponds to a regular structure.

According to Figure 1 and equation (3), the following constraints are set for function $f(x)$, where the number of non-homogeneous cells is permanent

$$f(0) = 0; \; f(1) = 1; \; f'(x) > 0 \tag{4}$$

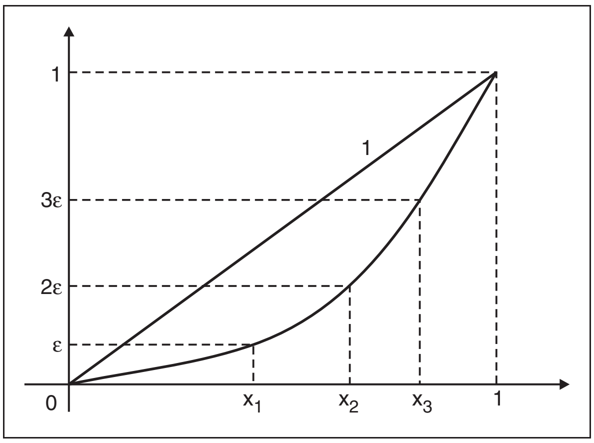

Figure 1. Nomogram of changes in heterogeneity of cell boundaries $x_1, x_2, x_3, \ldots,$.

As an example, we consider two typical gradient cases. The first case is when the step of non-homogeneity decreases or increases monotonically along the rod length. This gradient can be described as follows

$$f(x) = \alpha x^2 + \beta x + \gamma \quad (5)$$

Taking into account conditions (4), equation (5) reads

$$f(x) = \alpha x^2 + (1 - \alpha)x \quad (6)$$

where α is a parameter, characterizing a gradient magnitude and direction (for $\alpha > 0$ the non-homogeneity step decreases, whereas for $\alpha < 0$ it increases).

Condition (4) yields the constraint $|\alpha| < 1$. Furthermore, large value of $|\alpha|$ corresponds to the large gradient value (higher speed of the non-homogeneity step variation). It is significant that for arbitrary values of α formula (6) guarantees symmetry of intervals of either increase or decrease of the gradient step regarding the rod centre $x = 0.5$. In Figure 2, the function $f(x)$ (curve 1) as well as its derivative $f'(x)$ (curve 2), which were defined by equation (6) for $\alpha = 0.5$, are shown. Derivative $f'(x)$ makes possible to determine intervals where the non-homogeneity step is larger than the initial one (for zero gradient $\alpha = 0$). These are the intervals where curve 2 is located below or above the line $x = 1$ (intervals with the decreased step).

The second typical gradient case corresponds to increase (decrease) of the heterogeneous step to the rod centre and increase (decrease) of the step on the rod edges (Figure 3). This type of gradient can be described by a cubic function of the following form

$$f(x) = 2\alpha x^3 - 3\alpha x^2 + (\alpha - 1)x \quad (7)$$

where α defines the gradient magnitude and direction, while condition (4) yields the constraint $-1 < \alpha < 1$.

Formula (7) also provides symmetry of the intervals (either increase or decrease of the heterogeneity step regarding the rod centre). In Figure 3, the cubic type gradient $f(x)$ (curve 1) and its derivative $f'(x)$ (curve 2) have been defined by equation (7) for $\alpha = -0.5$.

The exact solution of equation (1) is achieved using double integration

$$u = \int \frac{C_1 + \int q(x)\mathrm{d}x}{a(f(x)/\varepsilon)}\mathrm{d}x + C_2 \quad (8)$$

where C_1 and C_2 are the arbitrary integration constants defined by boundary conditions (2).

It should be emphasized that, although the solution of equation (1) is obtained, integrals of this expression cannot be analytically found for majority of functions $a(x)$, $f(x)$, and $q(x)$ that appear in practice. Furthermore, there are problems in getting a result numerically due to the fast changes of the function $a(f(x)/\varepsilon)$. The asymptotic solution proposed in this article is devoid of the mentioned drawbacks.

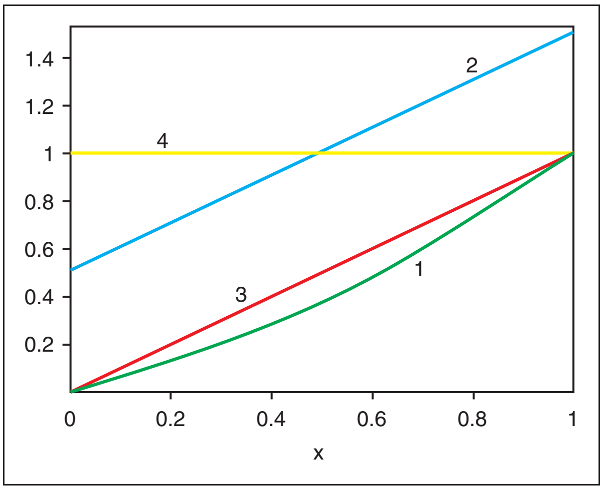

Figure 2. Function $f(x)$ that guarantees the monotonous decrease of the heterogeneity step (curve 1); line 2 – its gradient derivative; line 3 – function that guarantees constant heterogeneity step; line 3 – corresponds to zero gradient of $f(x)$.

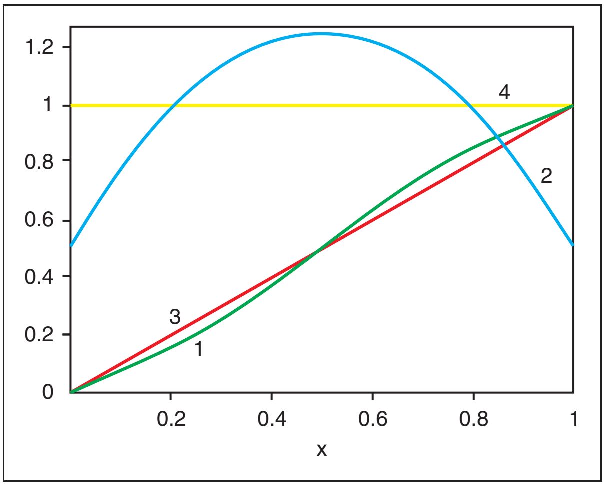

Figure 3. Function $f(x)$ guaranteeing decreased step of heterogeneity in the centre found from formula (7) ($\alpha = 0.5$ corresponds to curve 1, whereas its derivative $f'(x)$ corresponds to curve 2; lines 3 and 4 correspond to zero gradient of the FGM properties $\alpha = 0$).

It is significant that investigation of influence of gradient parameters $f(x)$ on the mode of deformation has a sense that only the following restriction holds for the considered variations of the function $f(x)$

$$c = \int_0^1 a\left(\frac{f(x)}{\varepsilon}\right)\mathrm{d}x = const \quad (9)$$

Observe that condition (9) together with (4) guarantee constant both qualitative and quantitative FGM properties with respect to the rod length for given and earlier defined variations of $f(x)$. Furthermore, the existing conditions guaranteeing the permanent number of heterogeneous cells (4) are not enough to satisfy the isoperimetric condition (9), since the step of heterogeneity is changing due to the non-linear rule (3). The investigations of function $f(x)$ for keeping the condition (9) in a general form belongs to rather difficult task; therefore, this condition is verified numerically only for some specific examples.

Asymptotic solution

An asymptotic solution to the boundary value problems (1) and (2) for a periodically non-homogeneous medium ($f(x) = x$) has been obtained in Bolshakov et al.,[17] with a help of homogenization procedure. In order to solve a similar problem, but for a quasi-periodically non-homogeneous medium, the modification of this method is applied.[19] In addition, we assume that the material gradient satisfies the following condition

$$f'(x) \sim 1 \tag{10}$$

This gradient is further referred to be 'small' one. The condition (10) guarantees small step changes of the non-homogeneities (3) for neighbouring cells. However, owing to large number of these cells, the gradient of all length will be relatively large (Figure 4).

Define two variables: $\eta = x$ and $\xi = f(x)/\varepsilon$, which will be considered as independent ones; hence, one gets

$$\frac{d}{dx} = \frac{\partial}{\partial \eta} + \frac{f'(\eta)}{\varepsilon}\frac{\partial}{\partial \xi} \tag{11}$$

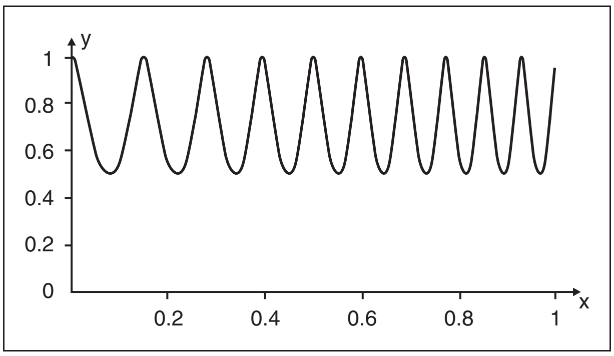

Figure 4. Elasticity coefficient (20) for small gradient: $f(x) = 0.4x^2 + 0.6x$; $z = 1$.

and instead of the initial ODE (1), one considers a PDE. Its solution can be searched in the following manner

$$u = u_0(\eta, \xi) + \varepsilon u_1(\eta, \xi) + \ldots \tag{12}$$

where u_0, u_1 are the periodic functions with respect to ξ and have the period 1.

Substituting equations (11) and (12) into equation (1), and comparing terms standing by the same powers of ε, the following recurrent system of equations is obtained

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_0}{\partial \xi}\right] = 0 \\ &f'(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_1}{\partial \xi}\right] + \frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_0}{\partial \eta}\right] = 0 \\ &f'^2(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_2}{\partial \xi}\right] + f'(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_1}{\partial \eta}\right] \\ &\quad + a(\xi)\frac{\partial}{\partial \eta}\left[f'(\eta)\frac{\partial u_1}{\partial \xi}\right] + a(\xi)\frac{\partial^2 u_0}{\partial \eta^2} = q(\eta) \end{aligned} \tag{13}$$

From the first equation of the system (13), it is evident that $u_0 = u_0(\eta)$. Then, the second equation of (13) gives

$$f'(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_1}{\partial \xi}\right] = -\frac{\partial a(\xi)}{\partial \xi}\frac{\partial u_0}{\partial \eta}$$

and hence

$$\frac{\partial u_1}{\partial \xi} = -\frac{1}{f'(\eta)}\frac{\partial u_0}{\partial \eta} + \frac{D_1(\eta)}{a(\xi)} \tag{14}$$

The constant of integration $D_1(\eta)$ is defined from a condition of periodicity of the function u_1 with respect to ξ

$$D_1(\eta) = \tilde{a}\,\frac{du_0}{d\eta}, \quad \tilde{a} = \left[\int_0^1 a^{-1} d\xi\right]^{-1}$$

Eliminating the derivative $\partial u_1/\partial \xi$ from the third equation of (13), one gets

$$f'^2(\eta)\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_2}{\partial \xi}\right) + f'(\eta)\left(a\frac{\partial u_1}{\partial \eta}\right) + \tilde{a}\frac{\partial^2 u_0}{\partial \eta^2} = q \tag{15}$$

The homogenization procedure is applied to equation (15) and then every term of this equation undergoes action of the operator $\int_0^1(\ldots)d\xi$. The first two terms are equal to zero due to their periodicity and finally we get

$$\tilde{a}\frac{\partial^2 u_0}{\partial \eta^2} = q(\eta) \tag{16}$$

The following boundary condition is attached

$$u_0 = 0 \text{ for } \eta = 0, 1 \tag{17}$$

Problem defined by equations (16) and (17) is called the homogenized problem. It describes the deformation of a homogeneous rod. The elastic properties of the material of this homogeneous rod are close to the effective properties of the initial material. In the case of a periodic non-homogeneous material, this property has been reported in Bolshakov et al.[17] This statement is verified for FGMs further in this article.

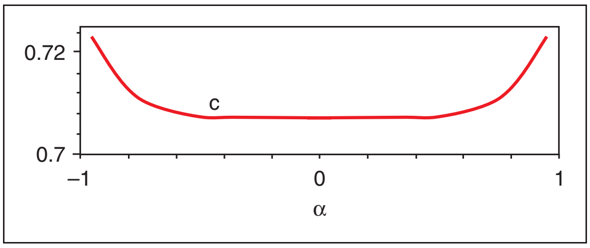

Figure 5. Influence of the quadratic gradient magnitude α on keeping condition (9), z = 1.

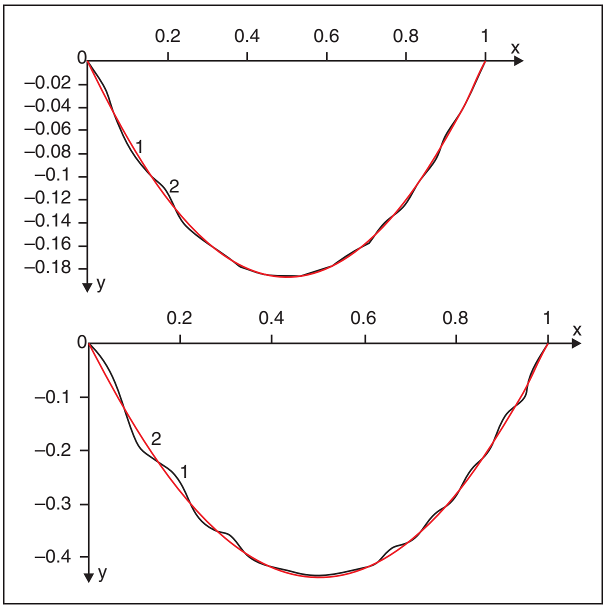

Figure 6. Comparison of the exact solution (curve 1) and homogenized solution u_0 (curve 2) for z = 1 and z = 5.

There is no reflection of material gradient in equation (16), so it coincides with homogenized equation presented in Bolshakov et al.[17] Gradient property is considered during defining the first correction term to the homogenized solution

$$u_1 = f'(\eta^{-1}) \int \left(a^{-1}\tilde{a} - 1\right) \mathrm{d}\xi \frac{\mathrm{d}u_0}{\mathrm{d}\eta} + D_2(\eta) \tag{18}$$

where D_2 is defined via boundary conditions. Further, equation (15) allows to find u_2 and so on.

Thus, asymptotic solutions (12), (16) and (18) show that the gradient change of the forms (3), (4) and (10) modifies the deformable condition of the rod only by small amount in comparison with deformation (16) of the equivalent homogeneous rod (order ε). The corrector (18) is an oscillating function around the homogenized solution (16) (Figures 5 to 7). Therefore, the functional flexibility[1,2]

$$I_1 = \int_0^1 uq\mathrm{d}x \tag{19}$$

cannot be applied as a functional of the construction reaction to the gradient changes.

Computational examples and accuracy estimation of the asymptotic solution

As an example, we take the formula for elasticity coefficient allowing getting the exact solution (8) in an analytical form

$$a = \frac{1}{1 + z\sin^2(2\pi n f(x))} \tag{20}$$

where $n = 1/\varepsilon$ is the number of non-homogeneity cells; parameter z the 'magnitude' of heterogeneity (ratio of characteristic magnitudes; in this case, it refers to elasticity coefficients, grains and the material ligament).

It should be mentioned that material gradient could also be introduced with a help of variability of this parameter. The same FGMs for $z = z(x)$ and $f(x) = x$ are studied in Andrianov et al.[19]

Further, we apply $n = 5$ and get 10 oscillations of the coefficient a. Note that for real constructions made of FGM, there are much more such oscillations, which increase the accuracy of asymptotic method, but they complicate numerical solution. Figure 4 shows the elasticity coefficient for the quadratic gradient (6) for $\alpha = 0.4$ and $z = 1$.

The influence of gradient magnitude α on keeping the isoperimetric condition (9) for the elasticity coefficient (20) and quadratic gradient (6) and $z = 1$ is shown in Figure 5.

Figure 5 allows to estimate how the gradient magnitude α influences the validity of the isoperimetric condition (9).

For large values of heterogeneity, the sensitivity of condition (9) increases. Namely, for $|\alpha| = 0.95$ by 2.2%, for $|\alpha| = 0.8$; 0.6 by 0.72% and 0.15%, respectively. For example, for $z = 5$ the corresponding increase will be 6.7; 2.01 and 0.39%.

It is significant to take into account changing intervals for gradient α in sensitivity analysis. Though, violation of the conditions (9) may introduce the computation error (the dependence in Figure 5 has been obtained through numerical integration using the package Maple).

The accuracy analysis of proposed asymptotic method comparing exact solution (8) and asymptotic solutions (16) and (18) for $q = 1$ have been carried out. Figure 6 shows the comparison of the exact solution (8) and u_0 solution of homogenized equation (16) for $z = 1$ and $z = 5$, $f(x) = 0.4x^2 + 0.6x$.

From Figure 6, it is clear that computation error of homogenized solution u_0 is essentially dependent on the magnitude of heterogeneity (the more z the more computation error will be). Therefore, if small heterogeneity occurs in the asymptotic method, it is necessary to take into account the additional correction terms $u_1, u_2, \ldots,$. At the same time, when the first correction u_1 has been applied, the corresponding solution has been found. This solution almost coincides with the exact solution (so much, that it was obliged to depict the solution with points), see Figure 7. The solution accuracy $u_0 + u_1/n$ has a low dependence on a heterogeneity magnitude, which is demonstrated in Figure 7.

Constraint on the gradient magnitude (10) has an efficient influence on the asymptotic method accuracy. Consider a quadratic gradient (6) for $\alpha = 1$

$$f(x) = x^2, \; z = 1. \tag{21}$$

In this case, the condition (10) near a left edge is not kept. Figure 8 shows the elasticity coefficient (20) for the mentioned parameters.

Comparison of the exact solution (8), homogenized solution u_0 and improved homogenized solution $u_0 + u_1/n$ for $f(x) = x^2$, $z = 1$ is shown in Figure 9.

From Figure 9, it is clear that as soon as $f'(x)$ approaches 1 (and it is possible near the right edge), the accuracy of the asymptotic method grows.

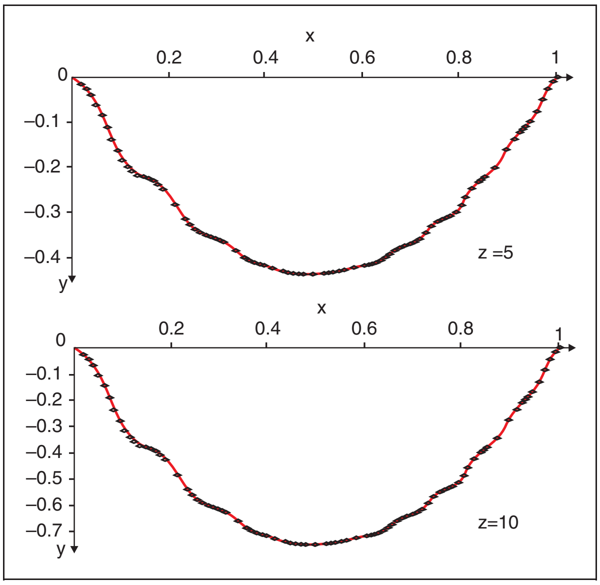

Figure 7. Comparison of the exact solution (depicted by points) and homogenized solution $u_0 + u_1/n$ for $z = 5$ and $z = 10$.

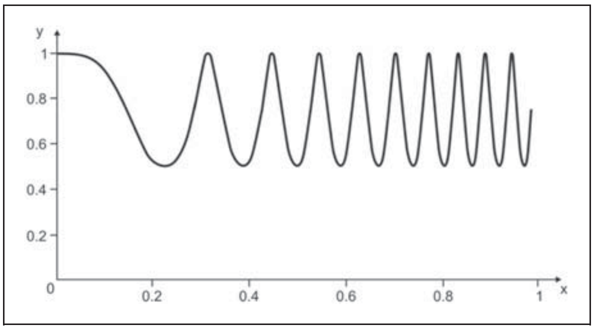

Figure 8. Elasticity coefficient (20) for large gradient $f(x) = x^2$; $z = 1$.

Heterogeneity influence

In what follows, we analyse a sensitivity of the deformable condition on a heterogeneity magnitude. For reaching this purpose, we compare the solutions of problems (1), (2) and (20) ($q = 1$) for zero gradient $f(x) = x$ with deformation of homogeneous rod, which inflexibility is uniformly distributed on the length ($a = c$) for heterogeneity $z = 1$ and $z = 5$ (Figure 10).

From Figure 10, it becomes clear that growth of heterogeneity causes the growth of deformation. This heightened sensibility of heterogeneous structures to external load is widely used in measuring instruments, where they are applied as bulging elements (for example, springs, convoluted diaphragms, accordion boots).

Consider the heterogeneous rod with elasticity coefficient (20) as a spring: one of its ends is fixed, while to the other end an external load p is applied and the distributed load is absent. Deformation of the rod

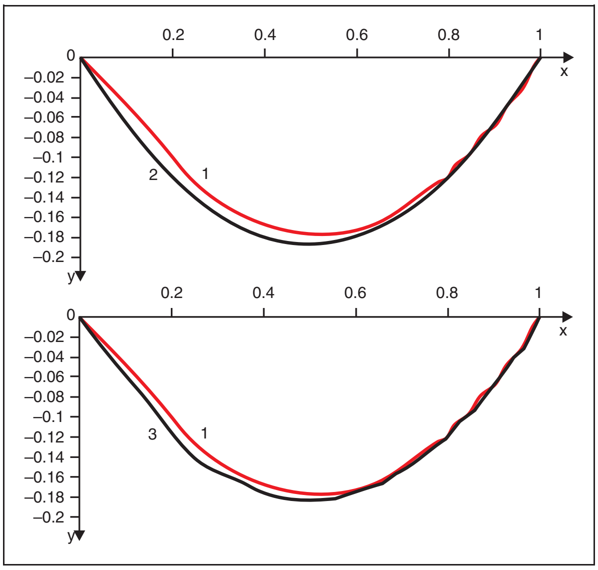

Figure 9. Comparison of the exact solution (curve 1), homogenized solution u_0 (curve 2) and improved homogenized solution $u_0 + u_1/n$ (curve 3) for $f(x) = x^2$, $z = 1$.

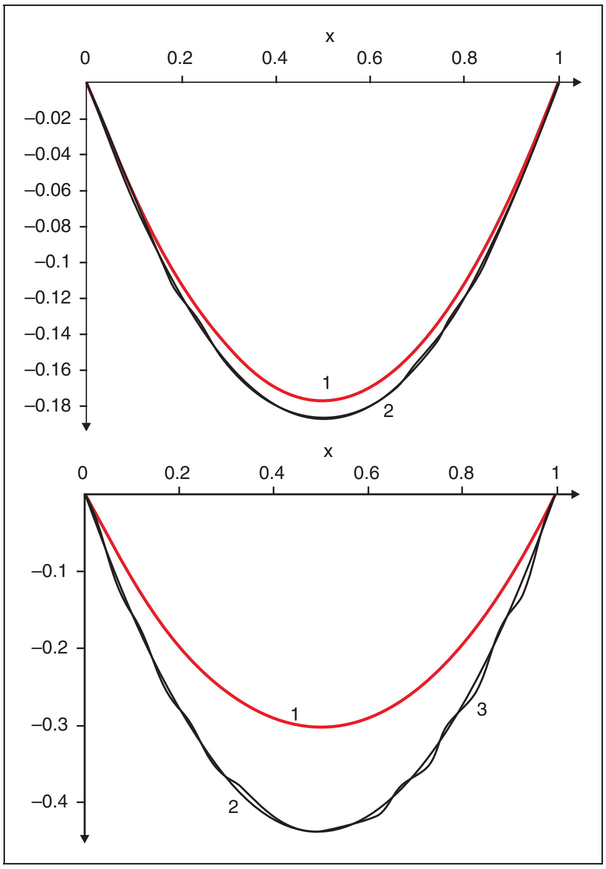

Figure 10. Comparison of homogeneous (curve 1) and heterogeneous (curve 2) rod deformation (curve 3 corresponds to the homogenized solution u_0).

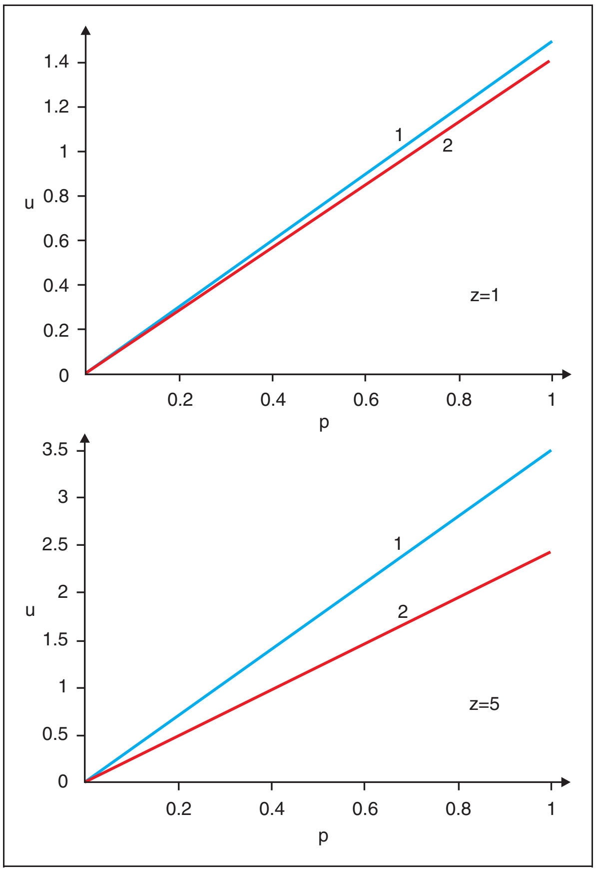

Figure 11. Elastic characteristic of heterogeneous (curve 1) and homogeneous rod (curve 2).

(Figure 11) is described by ODE (1) for $q(x) = 0$ and the following boundary conditions

$$u(0) = 0; \quad \frac{du}{dx} = p, \text{ for } x = 1 \tag{22}$$

The sensitivity of the mentioned spring is defined by an angle of characteristic curve that expresses the dependence between displacement $u(1)$ and load magnitude p. Figure 12 shows such dependence for $z = 1$ and $z = 5$ and zero gradient ($f(x) = x$) (lines 1); lines 2 show the elastic characteristic of homogeneous rod with uniformly distributed inflexibility ($a = c$).

From Figure 12, we assume that the sensitivity of the studied spring grows along with heterogeneity growth. Furthermore, owing to our computations, the gradient does not change the elastic characteristic. Figure 13 demonstrates comparison of the rod deformation (equations (1), (20) and (22) for zero gradient – curve 1) and with the gradient $f(x) = 0.3x^2 + 0.7x$ – curve 2. Figure 12 represents displacements of the rod end $u(1)$ being equal for both rods.

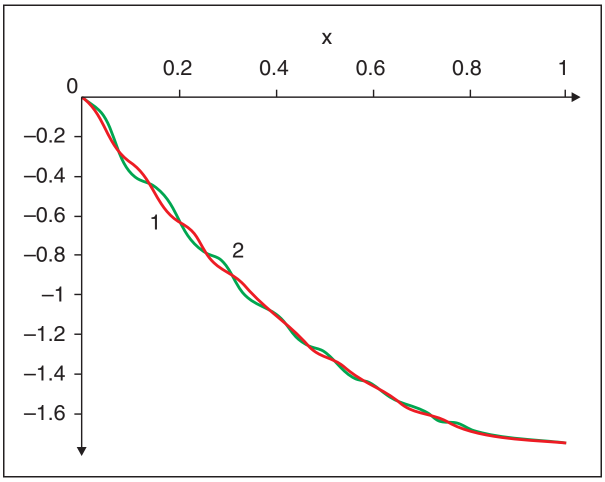

Figure 12. Comparison of rod deformation with zero gradient (curve 1) and with gradient $f(x) = 0.3x^2 + 0.7x$ (curve 2), $z = 5$.

Strained condition

Strains in studied problems (1), (2) and (20) ($q = 1$) appear via derivative du/dx, so an investigation of the parameters FGM influence on a magnitude of that derivative has been carried out. Figure 14 compares the derivative du/dx found using the exact solution (8) with the derivative du_0/dx found through the homogenized solution (16). The comparison shows that using the obtained homogenized solution is not enough accurate for defining strains even approximately.

From Figure 14, we could also conclude that heterogeneity causes oscillating concentrations of strains, which grow along with growing of heterogeneity. Large computational error that appears during the calculations of the strain using σ only the homogenized solution can be explained by the correcting term u_1 in the asymptotic decomposition of strain σ in zero-order approximation. In what follows, substitute the asymptotic expression (12) into formal expression for the strain $\sigma = du/dx$ and in the formula for derivative (11)

$$\sigma = \left(\frac{\partial}{\partial \eta} + \frac{f'(\eta)}{\varepsilon}\frac{\partial}{\partial \xi}\right)(u_0(\xi,\eta) + \varepsilon u_1(\xi,\eta) + \varepsilon^2 u_2(\xi,\eta) + \cdots) = \frac{\partial u_0}{\partial \eta} + f'(\eta)\frac{\partial u_1}{\partial \xi} + \varepsilon\left(\frac{\partial u_1}{\partial \eta} + f'(\eta)\frac{\partial u_2}{\partial \xi}\right) + \cdots$$

Thus, the formal expression for zero strain approximation is as follows

$$\sigma = \frac{\partial u_0}{\partial \eta} + f'(\eta)\frac{\partial u_1}{\partial \xi} \tag{23}$$

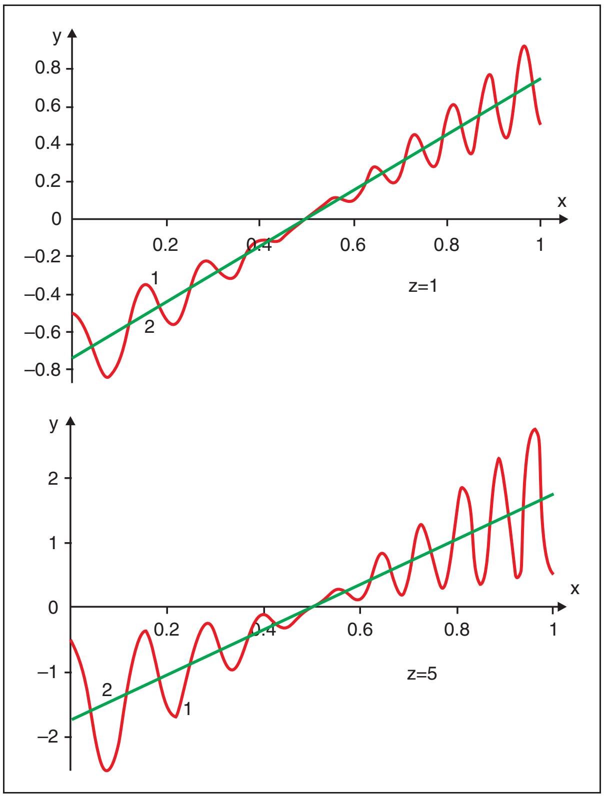

Figure 13. Comparison of the derivative du/dx (curve 1) and with derivative du_0/dx (curve 2) for $f(x) = 0.4x^2 + 0.6x$.

Due to equation (18), the formula (23) can be written in the following way

$$\sigma_0 = a^{-1}\tilde{a}\frac{\partial u_0}{\partial \eta} \tag{24}$$

Using formula (24) for defining strains, we can obtain a solution for any amount of non-homogeneities which overlaps with the exact solution (Figure 14).

The gradient function $f(x)$ has been already included in zero-order term of asymptotic solution (24). This fact indicates larger sensitivity of the strain field in comparison with the deformation field. While the gradient is under control, it is possible to improve the strained condition. According to this, Figure 15 shows changes in the strained condition governed by problems (1), (2) and (20) ($q = 1$) caused by the quadratic gradient (6).

The comparison presented in Figure 15 indicates that quadratic gradient decreases the maximum compressive stress, but increases the maximum tensile stress. Such a 'redistribution' of stresses could be helpful in some cases. Analysing Figures 13 and 15 and using gradient, the rule of decreasing of maximum strain concentration values could be formulated: it is necessary to increase the step of heterogeneity on the intervals of heightened homogenized strains du_0/dx,

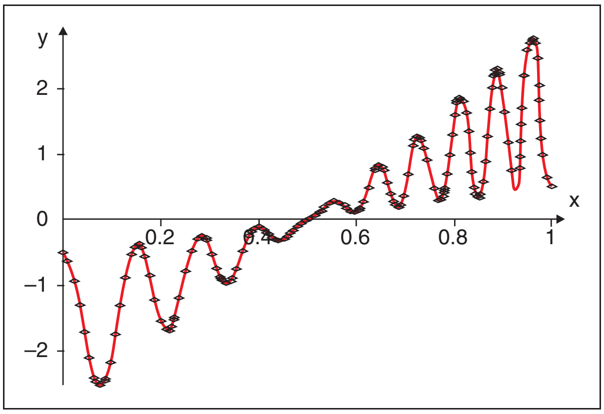

Figure 14. Comparison of strain found using the exact solution (depicted with points) and strain found using formula (22) (curve 2) for $z = 5$; $f(x) = 0.4x^2 + 0.6x$.

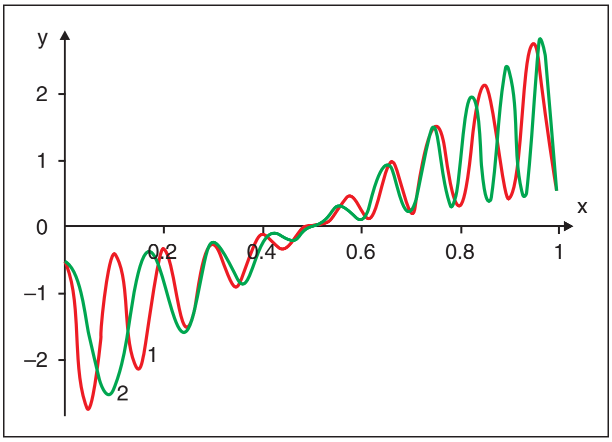

Figure 15. Comparison of strains with zero gradient (curve 1) and with gradient $f(x) = 0.4x^2 + 0.6x$ (curve 2), $z = 5$.

and vice versa on the intervals of reduced homogenized strains. Hence, for the mentioned problem, there is a need to increase the step of heterogeneity near the rod edges and to reduce it in the rod centre. Such a distribution of heterogeneous step could be guaranteed by the cubic gradient (7) (as an example). In Figure 16, the graph of elasticity coefficient (20) in cubic gradient is presented.

Note that it is impossible to obtain an exact analytical solution for even simple enough expressions for $f(x)$ in cubic gradient (7) for the studied problems (1), (2) and (20) ($q = 1$) with the help of ordinary mathematical packages. That is why in Figure 17 the comparison of strains found for zero gradients using the exact solution (curve 1), strains found for cubic gradient $\alpha = 0.5$ (curve 2) and $\alpha = 0.7$ (curve 3) using the asymptotic solution (24) are reported.

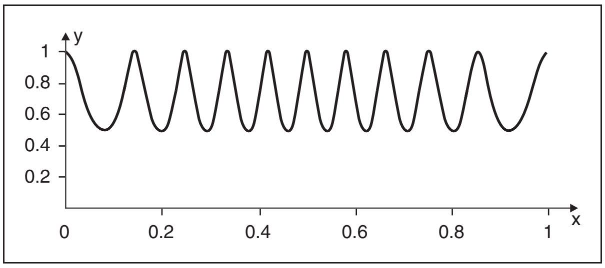

Figure 16. Elasticity coefficient in the cubic gradient (7), $\alpha = 0.5$, $z = 1$.

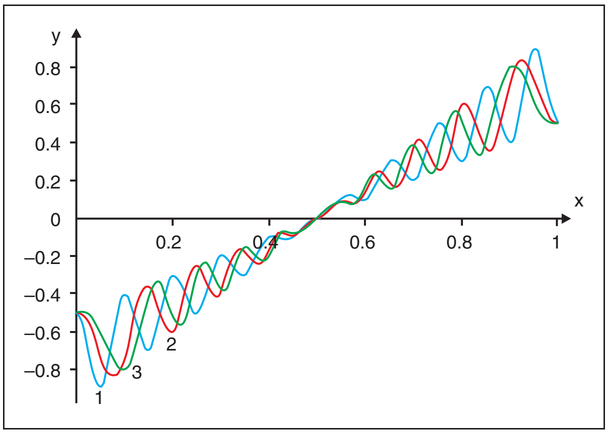

Figure 17. Comparison of strains for zero gradient (curve 1) and for cubic gradient $c = -0.5$ (curve 2) and $c = -0.7$ (curve 3), $z = 1$.

One may conclude from Figure 17 that it is possible to decrease the maximum strains by 10.8% for $c = 0.7$ and by 6.7% for $c = 0.5$ using the cubic gradient in comparison with zero material gradient.

Conclusions

Using the homogenized equation, the deformable conditions of the constructions made of FGMs can be found. Influence of gradient on deformable condition is small. The correcting terms to homogenized solution (caused by gradient) have an order ε being equal to a typical size of the heterogeneous cell. Strained condition sensitivity to gradient parameters is high. Controlling gradient allows reducing the maximum strain concentration values that are caused by heterogeneity of FGM.

There is a high effectiveness of applying proposed modification in homogenization method for calculating deformation in FGCs being based on comparison of exact and asymptotic solutions in the studied problem. Furthermore, the gradient magnitude has an essential influence on accuracy of the asymptotic method. The method gives good results only in the case of 'small' gradient, when the step of non-homogeneity changes a little along the non-homogeneous cell length (but throughout the length of the construction it could be quite large).

We assume that the described homogenization method will be also effective in investigations of different physical fields in FGMs for more complicated problems.

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