Internal motion of the complex oscillators near main resonance

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Abstract An analytical study of the two degrees of freedom nonlinear dynamical system is presented. The internal motion of the system is separated and described by one fourth order differential equation. An approximate approach allows reducing the problem to the Duffing equation with adequate initial conditions. A novel idea for an effective study of nonlinear dynamical systems consisting in a concept of the socalled limiting phase trajectories is applied. Both qualitative and quantitative complex analyses have been performed. Important nonlinear dynamical transition type phenomena are detected and discussed. In particular, nonsteady forced system vibrations are investigated analytically. [doi:10.1063/2.1204302]

Keywords nonlinear dynamics, multiple scale method, complex oscillator, internal motion

The energy exchange problem occurring in the nonlinear dynamical systems is widely discussed in the literature.[1] This phenomenon is observable in the coupled oscillators as well. The dynamics of such a system is investigated here. The coupled oscillators play an important role in many fields, for example in mechanics, electronics, medicine, etc.[2]

Dynamics of coupled periodically driven oscillators is very complicated. Certain simplification of the equations of motion of the two degrees of freedom system reducing it to the Duffing equation is shown in Ref. 3.

The equations of motion of two linear oscillators connected together by a nonlinear equipment maybe formulated as follows in the most general form

$$\begin{aligned} &L_1(q_1) + L_n(q_2 - q_1) = F_1, \\ &L_2(q_2) - L_n(q_2 - q_1) = F_2, \end{aligned} \tag{1}$$

where q_1 and q_2 are general coordinates, L_1 and L_2 are linear differential operators, L_n is nonlinear differential operator, F_1 and F_2 are the external loads.[1]

Introducing the new denotations $\xi = q_1$, $\zeta = q_2 - q_1$, the system (1) reads

$$\begin{aligned} &L_1(\xi) + L_n(\zeta) = F_1, \\ &L_2(\zeta) + L_2(\xi) - L_n(\zeta) = F_2. \end{aligned} \tag{2}$$

After some algebraic transformations and taking advantage of linearity of L_1 and L_2, the equation describing the internal motion may take a form

$$\begin{aligned} &(L_1 + L_2)(-L_n(\zeta)) + L_1(L_2(\zeta)) = \\ &\quad L_1(F_2) - L_2(F_1). \end{aligned} \tag{3}$$

In consequence, only one equation allows to investigate the significant dynamical task of the system of two coupled oscillators of two degrees of freedom.

Two coupled oscillators are analyzed in the paper. One of them, the mass of which is much bigger than that of the other, is driven by an external periodic load. We make the simplifying assumption that, while the motion of the smaller mass is nonlinear, the motion of the main mass may be considered as linear. This is an interesting case from a practical point of view.[2–4] The small mass can be considered as an energetic sink from the main object.

We assume that the system vibrates in the neighborhood of the static equilibrium position. Introducing a variable describing the internal motion $y(\tau) = z(\tau) - \phi(\tau)$, where $z(\tau)$ and $\phi(\tau)$ are non dimensional coordinates depending on the non dimensional time, the equations of motion take the form

$$\begin{aligned} &\mu_1\ddot{\phi}(\tau) + \gamma_1\dot{\phi}(\tau) + \alpha_1\phi(\tau) - y(\tau) - \eta_e y^3(\tau) - \\ &\quad \gamma_e\dot{y}(\tau) = f\cos(p_0\tau), \end{aligned} \tag{4}$$

$$y(\tau) + \eta_e y^3(\tau) + \gamma_e\dot{y}(\tau) + \ddot{y}(\tau) + \ddot{\phi}(\tau) = 0, \tag{5}$$

where μ_1, α_1, γ_1, η_e, γ_e, f, p_0 are non dimensional parameters defined as functions of the original dimensional ones.[1]

The structure of the Eqs. (4) and (5) is similar to Eq. (2), where the respective operators are

$$\begin{aligned} &L_1(\chi) = \mu_1\frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2} + \gamma_1\frac{\mathrm{d}\chi}{\mathrm{d}\tau} + \alpha_1\chi, \\ &L_2(\chi) = \frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2}, \\ &L_n(\chi) = -\gamma_e\frac{\mathrm{d}^2\chi}{\mathrm{d}\tau} - \chi - \eta_e\chi^3, \end{aligned} \tag{6}$$

while

$$F_1 = f\cos(p_0\tau), \qquad F_2 = 0. \tag{7}$$

Hence, according to Eq. (3), the equation for internal motion reads

$$\left[(\mu_1 + 1)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \gamma_1\frac{\mathrm{d}}{\mathrm{d}\tau} + \alpha_1\right] \cdot$$

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$$\left(\frac{\mu_1}{\mu_1+1}\ddot{y}+\gamma_{\mathrm{e}}\dot{y}+y+\eta_{\mathrm{e}}y^3\right)+\frac{1}{\mu_1+1}\left(\gamma_1\frac{\mathrm{d}}{\mathrm{d}\,\tau}+\alpha_1\right)\ddot{y}=p_0^2 f\cos(p_0\tau). \tag{8}$$

The second term on the left hand side of Eq. (8) is omitted in further analysis as being small comparing to others. In this way the approximate equation of the internal motion takes the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\,\tau^2}+\gamma_n\frac{\mathrm{d}}{\mathrm{d}\,\tau}+\alpha_n^2\right)g\left(\tau\right)=\frac{f\,p_0^2}{(\mu_1+1)}\cos(p_0\tau), \tag{9}$$

where

$$g\left(\tau\right)=\frac{\mu_1}{\mu_1+1}\ddot{y}\left(\tau\right)+y\left(\tau\right)+\eta_{\mathrm{e}}y^3\left(\tau\right)+\gamma_{\mathrm{e}}\dot{y}\left(\tau\right), \tag{10}$$

and $\gamma_n=\gamma_1/(\mu_1+1)$, $\alpha_n^2=\alpha_1/(\mu_1+1)$.

Equation (9) has the following analytical solution

$$g\left(\tau\right)=\frac{f\,p_0^2}{(\mu_1+1)\varDelta}\Big[\left(\alpha_n^2-p_0^2\right)\cos\left(p_0\tau\right)+\gamma_n p_0\sin\left(p_0\tau\right)\Big]+\mathrm{e}^{-\gamma_n\tau/2}\Big[d_1\cos\left(\beta\tau\right)+d_2\sin\left(\beta\tau\right)\Big], \tag{11}$$

where $\beta=\sqrt{\alpha_n^2-(\gamma_n/2)^2}$, $\varDelta=\left(\alpha_n^2-p_0^2\right)^2+p_0^2\gamma_n^2$, d_1 and d_2 are integration constants.

Substituting Eq. (11) into Eq. (10) and neglecting the decaying term in Eq. (11), the Duffing type equation is obtained

$$\ddot{y}+\gamma_{\mathrm{e}}\,\dot{y}+y+\eta_{\mathrm{e}}y^3=P\cos\left(p_0\tau+\varPhi\right), \tag{12}$$

where

$$\tan\varPhi=\frac{p_0\gamma_n}{\alpha_n^2-p_0^2},\quad P=-\frac{f\,p_0^2}{(\mu_1+1)\sqrt{\varDelta}}.$$

Further analysis relates to Eq. (12). Assuming that the damping is weak and that nonlinearity and amplitude of the external load are of the order of small parameter $\varepsilon\ll 1$, the equation of internal motion reads

$$\ddot{y}+2\varepsilon\gamma\dot{y}+y+8\varepsilon\eta y^3=2\varepsilon F\cos\left(p_0\tau+\varPhi\right), \tag{13}$$

where $2\varepsilon\gamma=\gamma_{\mathrm{e}}$, $8\varepsilon\eta=\eta_{\mathrm{e}}$, $2\varepsilon F=P$.

Let us introduce the variable $v\left(\tau\right)=\dot{y}\left(\tau\right)$ and suppose that the initial conditions correspond to the rest, that is $y(0)=0$, $\dot{y}\left(0\right)=0$. Equation (13) takes the form

$$\begin{aligned}&v-\dot{y}=0,\\&\dot{v}+2\varepsilon\gamma\dot{y}+y+8\varepsilon\eta y^3=2\varepsilon F\cos\left(p_0\tau+\varPhi\right).\end{aligned} \tag{14}$$

Afterwards, the new complex variables $\varPsi=(v+\mathrm{i}\,y)\,\mathrm{e}^{-\mathrm{i}\,\tau}$ and $\bar{\varPsi}=(v-\mathrm{i}\,y)\,\mathrm{e}^{\mathrm{i}\,\tau}$ are introduced.[4] Then the system (14) can be replaced by the following equation

$$\frac{\mathrm{d}\,\varPsi}{\mathrm{d}\,\tau}+\gamma\varepsilon\left(\varPsi+\bar{\varPsi}\mathrm{e}^{-2\mathrm{i}\,\tau}\right)+\mathrm{i}\,\eta\varepsilon\Big(\varPsi^3\mathrm{e}^{2\mathrm{i}\,\tau}-\bar{\varPsi}^3\mathrm{e}^{-4\mathrm{i}\,\tau}-3\left|\varPsi\right|^2\varPsi+3\left|\varPsi\right|^2\bar{\varPsi}\mathrm{e}^{-2\mathrm{i}\,\tau}\Big)=2\varepsilon\mathrm{e}^{-\mathrm{i}\,\tau}F\sin\left(p_0\tau+\varPhi\right), \tag{15}$$

with the initial condition $\varPsi\left(0\right)=0$. The conjugate equation to Eq. (15) can be obtained in similar way.

The case of the main resonance $p_0\approx 1$ is then considered. The multiple time scale method is used to obtain the analytical solution of Eq. (15).[5] Let us introduce the co-called detuning parameter σ as a measure of the distance of the strict resonance. Then the substitution $p_0=1+\sigma=1+\varepsilon\tilde{\sigma}$ into Eq. (15) is made. The assumed form of the solution is

$$\varPsi\left(\tau\right)=\varPsi_0\left(\tau_0,\tau_1\right)+\varepsilon\,\varPsi_1\left(\tau_0,\tau_1\right), \tag{16}$$

where time scales are $\tau_0=\tau$ and $\tau_1=\varepsilon\tau$.

After substituting the expansion (16) into Eq. (15) and eliminating secular terms, the solvability condition gives the equation for the main asymptotic approximation

$$\frac{\partial\varPsi_0}{\partial\tau_1}+\gamma\varPsi_0-3\mathrm{i}\,\eta\left|\varPsi_0\right|^2\varPsi_0=F\mathrm{e}^{\mathrm{i}\,(\tilde{\sigma}\tau_1+\varPhi)}. \tag{17}$$

Let $\varPsi_0\left(\tau_1\right)=a\left(\tau_1\right)\mathrm{e}^{\mathrm{i}\,\delta(\tau_1)}$, where $a\left(\tau_1\right)$ and $\delta\left(\tau_1\right)$ are real functions. Then using this substitution in Eq. (17), separating its real and imaginary parts and returning to the original notations as in Eq. (12), we obtain

$$\begin{aligned}&\frac{\mathrm{d}\,a}{\mathrm{d}\,\tau}+\frac{1}{2}\gamma_{\mathrm{e}}a=\frac{P}{2}\cos\theta,\\&-a\frac{\mathrm{d}\,\theta}{\mathrm{d}\,\tau}+a\sigma-\frac{3}{8}\eta_{\mathrm{e}}a^3=\frac{P}{2}\sin\theta,\end{aligned} \tag{18}$$

where $\theta=\sigma\tau-\delta+\varPhi$ is the modified phase. Thanks to application of θ, the system (18) is autonomous.

The steady state solution of Eq. (18) gives $\theta=\pm\pi/2+n\pi$, where $n\in C$ and

$$2\sigma a-\frac{3}{4}\eta_{\mathrm{e}}a^3=\pm P. \tag{19}$$

If the interval is limited to $-\pi<\theta<\pi$, the value $\theta=-\pi/2$ corresponds to $-P$ in Eq. (19), while $\theta=\pi/2$ corresponds to $+P$. When the nonlinear parameter $\eta_{\mathrm{e}}<\eta_{\mathrm{c1}}=128\sigma^3/(81P^2)$ then for $\theta=-\pi/2$, Eq. (19) has one positive root corresponding to the resonance center a_{r}, while for $\theta=\pi/2$, there are two positive amplitudes: quasi-linear center a_{q} and the saddle point a_{s}. The above discussion is illustrated in Fig. 1.

We note that in the absence of damping, the system (18) has the first integral

$$H=-a\frac{P}{2}\sin\theta+\sigma\frac{a^2}{2}-\frac{3}{32}\eta_{\mathrm{e}}a^4=\mathrm{const}, \tag{20}$$

which allows to draw the trajectories of motion in the plane $(a,\,\theta)$. The positions of the steady state points

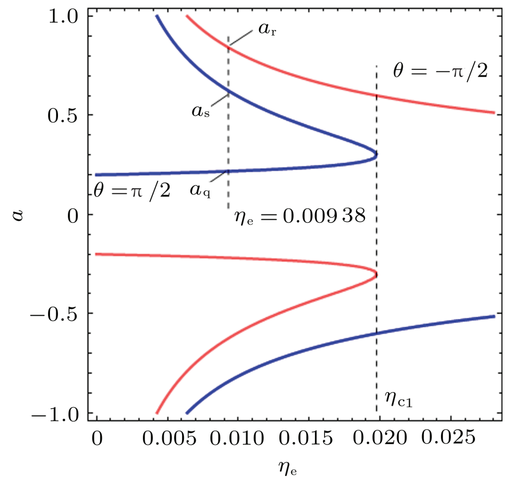

Fig. 1. Real roots of Eq. (19); positive branch–steady state amplitudes with respect to $\eta_{\rm e}$, for $\sigma = 0.002$ and $P = 0.000\,8$.

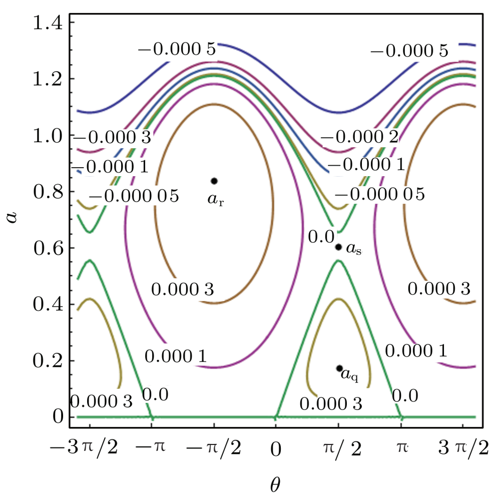

Fig. 2. Trajectories of motion for various values of H in Eq. (18).

of Eq. (18) and trajectories of motion are illustrated in Fig. 2.

An interesting case of the motion will appear for $H = 0$ when maximal energy exchange between the system and external loading occurs. For that case the maximal amplitudes can be derived and the another critical value of the nonlinearity parameter $\eta_{\rm e} = \eta_{\rm c} = 64\sigma^3/(81P^2) = \eta_{\rm c1}/2$ is detected. It separates qualitatively distinct types of vibrations. It can be easily seen that the extremes at the phase trajectories occur for $\theta = \pm\pi/2 + n\pi$. The maximal amplitudes in the non-stationary motion are drawn in Fig. 3.

The points 4 and 2 in Fig. 3 correspond to the maximal amplitudes for quasi-linear and strongly nonlinear cases, respectively. The points 1 and 3 indicate the solutions lying on the open curve at the phase trajectory and do not describe amplitude of vibrations (see also Fig. 4). The limiting phase trajectories for various non-

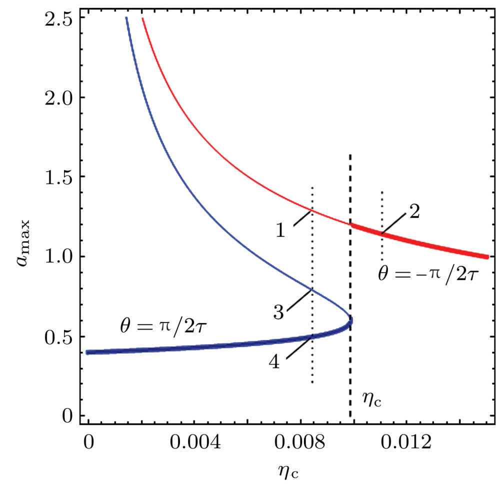

Fig. 3. The real roots of Eq. (20) for $H = 0$; the thick lines are maximal amplitudes in the non-stationary vibrations.

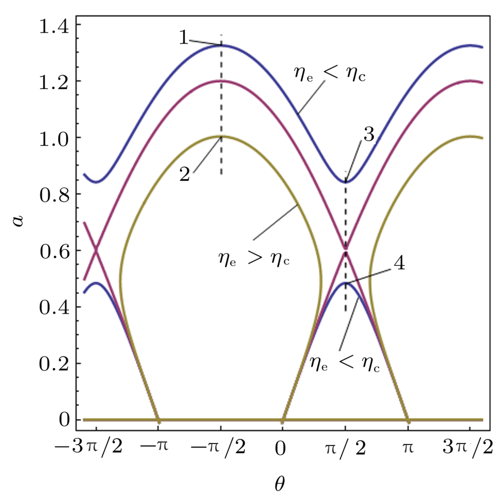

Fig. 4. Phase trajectories for $H = 0$; 1, 2, 3, 4 are real roots of Eq. (20).

linearity $\eta_{\rm e}$ are drawn in Fig. 4.

The graphs in Figs. 1–4 are made for $\sigma = 0.002$ and $P = 0.000\,8$. Then $\eta_{\rm c1} = 0.019\,753\,1$.

The non-stationary dynamical process has been investigated analytically. The way of testing the behavior of some kind of coupled nonlinear oscillators has been shown. $T_{\rm e}$ investigated system consists of two bodies. One of them has much greater mass than the second. The object of smaller mass can be considered as an energetic sink, and used as a passive damper. The main external resonance has been tested. The internal motion of two degrees of freedom system has been described by the effective equation, which takes the form of Duffing equation, after some simplifying assumptions.

Two critical values of the parameter $\eta_{\rm e}$, responsible for nonlinearity, have been noticed for which major qualitative and quantitative changes appear in the motion. Dependence of maximal amplitude versus η in the non-stationary process is also derived and graphically presented.

The complex description and multiple scale analysis allow to obtain qualitatively important information about the system dynamics. Especially the limiting phase trajectories allow to understand and describe the phase and temporal behavior of the internal motion of a coupled forced oscillator in its nonsteady vibrations.

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