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PARAMETRIC AND EXTERNAL RESONANCES IN KINEMATICALLY AND EXTERNALLY EXCITED NONLINEAR SPRING PENDULUM

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A weakly nonlinear 2-DOF system, parametrically and externally excited, is studied. An intensive energy transfer between modes of vibrations is discovered. Multiple scales method is used for recognizing resonances occurring in the system. The amplitude response functions for some chosen cases of resonances are obtained and studied. Many of the engineering systems can be investigated in the presented way.

Keywords: Spring pendulum; parametric excitation; multiple scale method; resonance; asymptotic analysis; kinematic excitation.

1. Introduction

The paper concerns nonlinear dynamics of a parametric two degree-of-freedom system. Nonlinear dynamical systems are still of great interest and widely discussed in literature. Autoparameric excitation may occur in complex systems as a result of inertial coupling occurring in the equations of motion e.g. [Starosta & Awrejcewicz, 2009]. The phenomenon of energy transfer in such systems between modes of vibration is investigated by many authors [Sado, 1997; Vakakis & Gendelman, 2001; Vakakis *et al.*, 2003; Gendelman *et al.*, 2005]. The analytical study of intensive energy exchange in strongly nonlinear system of 1-DoF is presented in [Manevich & Musienko, 2009]. The dynamics of a highly nonlinear vibration absorber coupled to a harmonically excited two-degree-of-freedom system is investigated in [Starosvetsky & Gendelman, 2008].

The parametrical nonlinear systems are described in [Bajaj *et al.*, 1994]. The effect of parametric excitation on a three mass system was studied by [Tondl & Nabergoj, 2004].

In the last decades, asymptotic methods have been intensively developed and applied to solving nonlinear problems [Shivamoggi, 2002]. Some applications of these methods can be found in [Natsiavas, 1992; Awrejcewicz & Starosta, 2010]. Strong nonlinear systems may also be considered in this manner [Andrianov & Awrejcewicz, 2003].

Dynamical systems containing mathematical or physical pendulum play a significant role in technology. There are many papers investigating various kinds of pendulum [Tondl & Nabergoj, 2000; Zhu *et al.*, 2004; Starosta & Awrejcewicz, 2009]. The chaotic behavior of the spring pendulum is tested in [Lee & Park, 1997]. Because of the complicated motion of such systems, much attention is paid

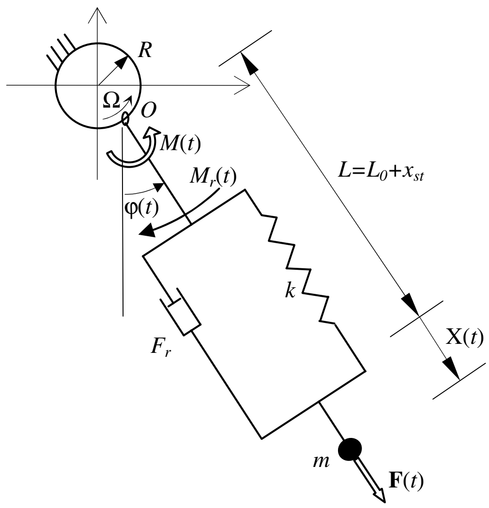

Fig. 1. Spring pendulum moving on circular path.

to the problems of control and stability [El-Serafi *et al.*, 2005]. In [Sieber & Krauskopf, 2005; Landry *et al.*, 2005] the control loop for the inverted pendulum is applied. The control system for the rapidly forced system of cart and pendulum is discussed in [Weibel *et al.*, 1997].

Our aim is also to obtain the dynamic analysis of the system with pendulum. In particular, the pendulum with changing length moving in a circular path is investigated (Fig. 1). The structure investigated in the paper can be recognized as a model of various engineering elements in machines or can simulate the motion of a floating body. In this paper, we focus on detecting the resonance conditions and the analysis of the chosen resonance case. The multiple scale method enables to recognize the parameters of the system that are dangerous due to the resonances and allows to illustrate frequency-amplitude response functions. Resonances of various kinds i.e. primary, parametric and combined are studied. All calculations were performed with the help of the computer algebra system *Mathematica*, in which several procedures were elaborated in order to automatize most of the operations.

2. Formulation of the Problem

We study the planar motion of a spring pendulum whose point of suspension moves with angular velocity Ω along a fixed circular path of radius R (Fig. 1). X and φ are the generalized co-ordinates. The moment $M(t) = M_0 \cos(t\Omega_2)$ and the linear viscous damping moment $M_r = B_2\dot{\varphi}$ act around the point O. The force $F(t) = F_0 \cos(t\Omega_1)$ and the linear viscous damping $F_r = B_1\dot{X}$ act on the mass m along the pendulum length (B_1 and B_2 are the viscous coefficients).

The kinetic energy of the system has the form

$$\begin{aligned} T = {} & \frac{m}{2}(R\Omega\cos(t\Omega) + \dot{X}(t)\sin(\varphi(t)) \\ & + \cos(\varphi(t))(L + X(t))\dot{\varphi}(t))^2 \\ & + \frac{m}{2}(R\Omega\sin(t\Omega) - \dot{X}(t)\cos(\varphi(t)) \\ & + \sin(\varphi(t))(L + X(t))\dot{\varphi}(t))^2, \end{aligned} \tag{1}$$

whereas the potential energy reads

$$\begin{aligned} V = {} & \frac{k}{2}\Big(X(t) + \frac{mg}{k}\Big)^2 - mg(R\cos(t\Omega) \\ & + \cos(\varphi(t))(L + X(t))), \end{aligned} \tag{2}$$

where $L = L_0 + (mg/k)$ denotes the length of the statically stretched pendulum at $\varphi = 0$ (L_0 is length of nonstretched spring), m is its mass, k denotes stiffness of the spring and g is the Earth's acceleration.

The governing equations of the system are as follows:

$$\begin{aligned} & \ddot{x}(t) + c_1\dot{x}(t) - (1 + x(t))(\dot{\varphi}(t))^2 + \omega_1^2 x(t) \\ & \quad + \omega_2^2(1 - \cos(\varphi(t))) - r\Omega^2\cos(t\Omega - \varphi(t)) \\ & = f_1\cos(t\Omega_1), \end{aligned} \tag{3}$$

$$\begin{aligned} & (1 + x(t))^2\ddot{\varphi}(t) + (c_2 + 2(1 + x(t))\dot{x}(t))\dot{\varphi}(t) \\ & \quad + \omega_2^2\sin(\varphi(t))(1 + x(t)) - r\Omega^2(1 + x(t)) \\ & \quad \times \sin(t\Omega - \varphi(t)) = f_2\cos(t\Omega_2), \end{aligned} \tag{4}$$

where $x = X/L$, $r = R/L$, $\omega_1^2 = k/m$, $\omega_2^2 = g/L$, $c_1 = B_1/m$, $c_2 = B_2/mL^2$, $f_1 = F_0/mL$, $f_2 = M_0/mL^2$.

Equations (3) and (4) should be supplemented by adequate initial conditions.

3. Solution Method

To solve the governing equations and to obtain the resonance conditions, the multiple scale method is applied. Trigonometric functions in Eqs. (3) and (4) can be approximated in the following way

$$\sin\varphi \cong \varphi - \frac{1}{3!}\varphi^3,$$

$$\cos\varphi \cong 1 - \frac{1}{2}\varphi^2,$$

$$\sin(t\Omega - \varphi) \cong \sin(t\Omega) - \cos(t\Omega) - \frac{1}{2}\varphi^2 \sin(t\Omega),$$

$$\cos(t\Omega - \varphi) \cong \cos(t\Omega) + \sin(t\Omega) - \frac{1}{2}\varphi^2 \cos(t\Omega),$$

since the motion in a small neighborhood of the static equilibrium position is considered. The amplitudes of vibrations are assumed to be of order of a small parameter ε, where $0 < \varepsilon \ll 1$, and hence $x = \varepsilon\tilde{x}, \varphi = \varepsilon\tilde{\varphi}$.

The generalized forces, damping coefficients and radius of the path are assumed in the form: $c_i = \varepsilon^2\tilde{c}_i, f_i = \varepsilon^3\tilde{f}_i, r = \varepsilon^2\tilde{r}, i = 1, 2$. The parameters $\tilde{f}_i, \tilde{c}_i, \tilde{r}$ are of the order of 1.

The functions $\tilde{x}$ and $\tilde{\varphi}$, are sought in the form

$$\begin{aligned}\tilde{x}(t;\varepsilon) &= \sum_{k=1}^{k=3} \varepsilon^k \tilde{x}_k(T_0, T_1, T_2) + O(\varepsilon^4),\\ \tilde{\varphi}(t;\varepsilon) &= \sum_{k=1}^{k=3} \varepsilon^k \tilde{\varphi}_k(T_0, T_1, T_2) + O(\varepsilon^4),\end{aligned} \tag{5}$$

where $T_0 = t, T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are various time scales.

The derivatives with respect to time t are calculated in terms of the new time scales as follows

$$\begin{aligned}\frac{d}{dt} &= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2},\\ \frac{d^2}{dt^2} &= \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1}\\ &\quad + \varepsilon^2\left(\frac{\partial^2}{\partial T_1^2} + 2\frac{\partial^2}{\partial T_0 \partial T_2}\right)O(\varepsilon^3).\end{aligned} \tag{6}$$

The definitions in Eqs. (5) and (6) transform the original equations to the set of following ordinary linear differential equations

(order ε^1)

$$\frac{\partial^2 \tilde{x}_1}{\partial T_0^2} + \omega_1^2 \tilde{x}_1 = 0, \quad \frac{\partial^2 \tilde{\varphi}_1}{\partial T_0^2} + \omega_2^2 \tilde{\varphi}_1 = 0; \tag{7}$$

(order ε^2)

$$\begin{aligned}\frac{\partial^2 \tilde{x}_2}{\partial T_0^2} + \omega_1^2 \tilde{x}_2 &= \tilde{r}\Omega^2 \cos(T_0\Omega) - \frac{1}{2}\omega_2^2 \tilde{\varphi}_1^2\\ &\quad - 2\frac{\partial^2 \tilde{x}_1}{\partial T_0 \partial T_1} + \left(\frac{\partial \tilde{\varphi}_1}{\partial T_0}\right)^2,\\ \frac{\partial^2 \tilde{\varphi}_2}{\partial T_0^2} + \omega_2^2 \tilde{\varphi}_2 &= \tilde{r}\Omega^2 \sin(T_0\Omega) - \omega_2^2 \tilde{x}_1 \tilde{\varphi}_1 - 2\frac{\partial \tilde{x}_1}{\partial T_0}\frac{\partial \tilde{\varphi}_1}{\partial T_0}\\ &\quad - 2\tilde{x}_1 \frac{\partial^2 \tilde{\varphi}_1}{\partial T_0^2} - 2\frac{\partial^2 \tilde{\varphi}_1}{\partial T_0 \partial T_1}.\end{aligned} \tag{8}$$

(order ε^3)

$$\begin{aligned}\frac{\partial^2 \tilde{x}_3}{\partial T_0^2} + \omega_1^2 \tilde{x}_3 &= \tilde{f}_1 \cos(T_0\Omega_1) + \tilde{r}\Omega^2 \tilde{\varphi}_1 \sin(\Omega T_0)\\ &\quad - \omega_2^2 \tilde{\varphi}_1 \tilde{\varphi}_2 - \frac{\partial \tilde{x}_1^2}{\partial T_1^2} - \tilde{c}_1 \frac{\partial \tilde{x}_1}{\partial T_0}\\ &\quad + 2\frac{\partial \tilde{\varphi}_1}{\partial T_1}\frac{\partial \tilde{\varphi}_1}{\partial T_0} + \tilde{x}_1\left(\frac{\partial \tilde{\varphi}_1}{\partial T_0}\right)^2\\ &\quad + 2\frac{\partial \tilde{\varphi}_1}{\partial T_0}\frac{\partial \tilde{\varphi}_2}{\partial T_0} - 2\frac{\partial^2 \tilde{x}_1}{\partial T_0 \partial T_2} - 2\frac{\partial^2 \tilde{x}_2}{\partial T_0 \partial T_1},\\ \frac{\partial^2 \tilde{\varphi}_3}{\partial T_0^2} + \omega_2^2 \tilde{\varphi}_3 &= \tilde{f}_2 \cos(T_0\Omega_2) + \tilde{r}\Omega^2 \tilde{x}_1 \sin(T_0\Omega)\\ &\quad - \tilde{r}\Omega^2 \tilde{\varphi}_1 \cos(T_0\Omega) - \omega_2^2 \tilde{x}_2 \tilde{\varphi}_1\\ &\quad - \omega_2^2 \tilde{x}_1 \tilde{\varphi}_2 - \frac{\partial^2 \tilde{\varphi}_1}{\partial T_1^2} - 2\frac{\partial \tilde{x}_1}{\partial T_0}\frac{\partial \tilde{\varphi}_1}{\partial T_1}\\ &\quad - \tilde{c}_2 \frac{\partial \tilde{\varphi}_1}{\partial T_0} - 2\frac{\partial \tilde{x}_1}{\partial T_1}\frac{\partial \tilde{\varphi}_1}{\partial T_0} - 2\tilde{x}_1 \frac{\partial \tilde{x}_1}{\partial T_0}\frac{\partial \tilde{\varphi}_1}{\partial T_0}\\ &\quad - 2\frac{\partial \tilde{x}_2}{\partial T_0}\frac{\partial \tilde{\varphi}_1}{\partial T_0} - 2\frac{\partial \tilde{x}_1}{\partial T_0}\frac{\partial \tilde{\varphi}_2}{\partial T_0} - 2\frac{\partial^2 \tilde{\varphi}_1}{\partial T_0 \partial T_2}\\ &\quad - 4\tilde{x}_1 \frac{\partial^2 \tilde{\varphi}_1}{\partial T_0 \partial T_1} - 2\frac{\partial^2 \tilde{\varphi}_2}{\partial T_0 \partial T_1} - \tilde{x}_1^2 \frac{\partial^2 \tilde{\varphi}_1}{\partial T_0^2}\\ &\quad - 2\tilde{x}_2 \frac{\partial^2 \tilde{\varphi}_1}{\partial T_0^2} - 2\tilde{x}_1 \frac{\partial^2 \tilde{\varphi}_2}{\partial T_0^2}.\end{aligned} \tag{9}$$

Solutions of Eqs. (7) are as follows:

$$\tilde{x}_1 = A_1 e^{iT_0\omega_1} + \overline{A}_1 e^{-iT_0\omega_1}, \tag{10}$$

$$\tilde{\varphi}_1 = A_2 e^{iT_0\omega_2} + \overline{A}_2 e^{-iT_0\omega_2}, \tag{11}$$

where A_1 and A_2 are unknown complex functions of slow time scales.

After eliminating secular terms, we obtain the following second and third order solutions:

$$\tilde{x}_2 = \frac{\omega_2^2 A_2 \overline{A}_2}{\omega_1^2} - \frac{e^{iT_0\Omega}\tilde{r}\Omega^2}{2(\Omega^2-\omega_1^2)} + \frac{3e^{i2T_0\omega_2}\omega_2^2 A_2^2}{2(4\omega_2^2-\omega_1^2)} + CC \tag{12}$$

$$\tilde{\varphi}_2 = \frac{ie^{iT_0\Omega}\tilde{r}\Omega^2}{2(\Omega^2-\omega_2^2)} - \frac{e^{iT_0(\omega_1+\omega_2)}\omega_2(2\omega_1+\omega_2)A_1A_2}{\omega_1(\omega_1+2\omega_2)} + \frac{e^{iT_0(\omega_1-\omega_2)}\omega_2(2\omega_1-\omega_2)A_1\overline{A}_2}{\omega_1(\omega_1-2\omega_2)} + CC, \tag{13}$$

where CC stands for the complex conjugates of the preceding terms.

The third order approximation is given by

$$\begin{aligned}\tilde{x}_3 = {} & \frac{ie^{iT_0(\Omega+\omega_2)}\tilde{r}\Omega^4 A_2}{2(\Omega^2-\omega_2^2)((\Omega+\omega_2)^2-\omega_1^2)} + \frac{ie^{iT_0(\Omega-\omega_2)}\tilde{r}\Omega^4 \overline{A}_2}{2(\Omega^2-\omega_2^2)((\Omega-\omega_2)^2-\omega_1^2)} \\ & + \frac{e^{iT_0(\omega_1+2\omega_2)}(\omega_1-\omega_2)\omega_2 A_1 A_2^2}{4\omega_1(2\omega_2+\omega_1)} + \frac{e^{iT_0(\omega_1-2\omega_2)}(\omega_1+\omega_2)\omega_2 A_1 \overline{A}_2^2}{4\omega_1(2\omega_2-\omega_1)} \\ & - \frac{e^{iT_0\Omega_1}\tilde{f}_1}{2(\Omega_1^2-\omega_1^2) + CC}\end{aligned} \tag{14}$$

$$\begin{aligned}\tilde{\varphi}_3 = {} & +\frac{ie^{iT_0(\Omega+\omega_1)}\tilde{r}\Omega^4 A_1}{2(\Omega^2-\omega_2^2)((\Omega+\omega_1)^2-\omega_2^2)} + \frac{ie^{iT_0(\Omega-\omega_1)}\tilde{r}\Omega^4 \overline{A}_1}{2(\Omega^2-\omega_2^2)((\Omega-\omega_1)^2-\omega_2^2)} \\ & + \frac{e^{iT_0(\Omega+\omega_2)}\tilde{r}\Omega(\Omega^2-\omega_1^2+\omega_2^2)A_2}{2(\Omega^2-\omega_1^2)(\Omega+2\omega_2)} + \frac{e^{iT_0(\Omega-\omega_2)}\tilde{r}\Omega(\Omega^2-\omega_1^2+\omega_2^2)\overline{A}_2}{2(\Omega^2-\omega_1^2)(\Omega-2\omega_2)} \\ & - \frac{e^{iT_0(2\omega_1+\omega_2)}\omega_2(2\omega_1^2+3\omega_1\omega_2+\omega_2^2)A_2A_1^2}{4\omega_1^2(2\omega_2+\omega_1)} + \frac{3e^{3iT_0\omega_2}\omega_2^2 A_2^3}{16(\omega_1^2-4\omega_2^2)} \\ & - \frac{e^{iT_0(2\omega_1-\omega_2)}\omega_2(2\omega_1^2-3\omega_1\omega_2+\omega_2^2)\overline{A}_2A_1^2}{4\omega_1^2(2\omega_2-\omega_1)} - \frac{e^{iT_0\Omega_2}\tilde{f}_2}{2(\Omega_2^2-\omega_2^2)} + CC\end{aligned} \tag{15}$$

The functions A_1 and A_2 can be calculated from secular terms and initial conditions due to Eqs. (3) and (4).

4. Parametric and External Resonances

From Eqs. (12)–(15), many various resonance cases can be detected from Eqs. (12)–(15). They are classified into resonances: primary external $\Omega_1 = \omega_1, \Omega_2 = \omega_2$, parametric $\Omega = \omega_1, \Omega = \omega_2$, internal: $\omega_1 = 2\omega_2$, combined: $\Omega = \pm(\omega_1-\omega_2), \Omega = \pm(\omega_1+\omega_2)$.

It should be noticed that the system behavior is very complex, especially when the natural frequencies satisfy certain resonance conditions. The parametric $\Omega \approx \omega_1$ and external $\omega_2 \approx \Omega_2$ resonances appearing simultaneously are discussed below. In order to study the resonances, we introduce detuning parameters σ_1 and σ_2 in the following way

$$\Omega = \omega_1 + \varepsilon\tilde{\sigma}_1, \quad \Omega_2 = \omega_2 + \varepsilon\tilde{\sigma}_2. \tag{16}$$

Introducing the resonance conditions [Eq. (16)] into Eqs. (12)–(15) and zeroing secular terms we can determine the solvability conditions for the tested case

$$-\frac{1}{2}e^{iT_1\tilde{\sigma}_1}\tilde{r}\Omega^2 + 2i\omega_1\frac{\partial A_1}{\partial T_1} = 0, \tag{17}$$

$$2i\omega_2\frac{\partial A_2}{\partial T_1} = 0, \tag{18}$$

$$i\tilde{c}_1\omega_1 A_1 + \frac{2A_1A_2\overline{A}_2\omega_2^2(7\omega_2^2-\omega_1^2)}{\omega_1^2-4\omega_2^2} + 2i\omega_1\frac{\partial A_1}{\partial T_2} + \frac{\partial^2 A_1}{\partial T_1^2} = 0, \tag{19}$$

$$-\frac{1}{2}e^{i\tilde{\sigma}_2T_1}\tilde{f}_2 + i\tilde{c}_2\omega_2A_2 - \frac{A_1A_2\overline{A}_2\omega_2^2(7\omega_2^2-2\omega_1^2)}{-\omega_1^2+4\omega_2^2} - \frac{A_2^2\overline{A}_2\omega_2^4(\omega_1^2+8\omega_2^2)}{-2\omega_1^4+8\omega_1^2\omega_2^2} + 2i\omega_2\frac{\partial A_2}{\partial T_2} + \frac{\partial^2 A_2}{\partial T_1^2} = 0 \tag{20}$$

Solvability conditions of Eqs. (17)–(20) yield the system of four partial equations from which we can get the unknown functions $A_1(T_1, T_2)$, $A_2(T_1, T_2)$ or, in particular, obtain the frequency responses for the mentioned resonance cases.

We introduce polar representation for the complex amplitudes in the following way

$$A_1(T_1,T_2)=\frac{\tilde{a}_1(T_1,T_2)}{2}e^{i\psi_1(T_1,T_2)},$$

$$A_2(T_1,T_2)=\frac{\tilde{a}_2(T_1,T_2)}{2}e^{i\psi_2(T_1,T_2)},$$

and the modified phase variables

$$\theta_1(T_1,T_2)=T_1\tilde{\sigma}_1-\psi_1(T_1,T_2),$$

$$\theta_2(T_1,T_2)=T_1\tilde{\sigma}_2-\psi_2(T_1,T_2).$$

Taking advantage of Eq. $(6)_1$ we get an autonomous modulation system of the form

$$\mathrm{i}\frac{da_1}{dt}+a_1\left(-\sigma_1+\frac{d\theta_1}{dt}\right)=-\frac{1}{2}\mathrm{i}a_1c_1+\frac{\omega_1^2-7\omega_2^2}{4\omega_1(\omega_1^2-4\omega_2^2)}\omega_2^2a_1a_2^2+\frac{r\Omega^2}{2\omega_1}\left(1-\frac{\sigma_1}{2\omega_1}\right)(\cos\theta_1+\mathrm{i}\sin\theta_1), \quad (21)$$

$$\mathrm{i}\frac{da_2}{dt}+a_2\left(-\sigma_2+\frac{d\theta_2}{dt}\right)=-\frac{1}{2}\mathrm{i}a_2c_2+\frac{\omega_1^2-7\omega_2^2}{4(\omega_1^2-4\omega_2^2)}\omega_2a_2a_1^2-\frac{a_2^3\omega_2^3}{16}\frac{\omega_1^2+8\omega_2^2}{\omega_1^2(\omega_1^2-4\omega_2^2)}+\frac{f_2}{2\omega_2}(\cos\theta_2+\mathrm{i}\sin\theta_2). \quad (22)$$

Steady state solution corresponds to zero values of the derivatives in Eqs. (21) and (22). Comparison of real and imaginary parts on both sides of the equations leads to the following set of algebraic equations

$$-\sigma_1a_1=\frac{\omega_1^2-7\omega_2^2}{4\omega_1(\omega_1^2-4\omega_2^2)}\omega_2^2a_1a_2^2+\frac{r\Omega^2}{2\omega_1}\left(1-\frac{\sigma_1}{2\omega_1}\right)\cos\theta_1, \quad (23)$$

$$0=2a_1c_1\omega_1^2+(\sigma_1-2\omega_1)r\Omega^2\sin\theta_1, \quad (24)$$

$$-\sigma_2a_2=\frac{\omega_1^2-7\omega_2^2}{4(\omega_1^2-4\omega_2^2)}\omega_2a_2a_1^2-\frac{a_2^3\omega_2^3}{16}\frac{\omega_1^2+8\omega_2^2}{\omega_1^2(\omega_1^2-4\omega_2^2)}+\frac{f_2}{2\omega_2}\cos\theta_2, \quad (25)$$

$$0=-\omega_2a_2c_2+f_2\sin\theta_2. \quad (26)$$

Eliminating from Eqs. (23)–(26) θ_1 and θ_2 we obtain frequency response functions:

(i) for parametric resonance

$$\left(-\sigma_1a_1+\frac{\omega_2^2(7\omega_2^2-\omega_1^2)a_1a_2^2}{4\omega_1(\omega_1^2-4\omega_2^2)}\right)^2+\frac{c_1^2}{4}a_1^2=\frac{R^2\Omega^4}{4\omega_1^2}; \quad (27)$$

(ii) for external resonance

$$\left(-\sigma_2a_2-\frac{\omega_2(\omega_1^2-7\omega_2^2)a_2a_1^2}{4(\omega_1^2-4\omega_2^2)}+\frac{\omega_2^3(\omega_1^2+8\omega_2^2)a_2^3}{16\omega_1^2(\omega_1^2-4\omega_2^2)}\right)^2+\frac{c_2^2}{4}a_2^2=\frac{f_2^2}{4\omega_2^2}, \quad (28)$$

where a_1 and a_2 are amplitudes of the longitudinal and swing vibrations, respectively.

5. Numerical Results

To illustrate the parametric resonance, we have chosen the following values of the system parameters: $\omega_1=10, \omega_2=2, r=0.1$.

Resonance curves in form amplitude a_1 versus detuning parameter σ_1 for some values of the damping coefficient c_1 are presented in Fig. 2. The influence of radius r on the resonance curves is given in Fig. 3.

In the case of external resonance, the following values of the system parameters are chosen: $\omega_1=15, \omega_2=6, r=0.1$.

Amplitudes a_2 versus detuning parameter σ_2 for chosen values of damping coefficient c_2 and amplitude f_2 are presented in Figs. 4 and 5.

It can be seen in Figs. 2 and 3 that the amplitude a_1 is a function monotonically increasing with r and decreasing with c_1. Similarly, the amplitude a_2 monotonically increases with the excitation f_2 and decreases with c_2. The curves are bent to the right, giving rise to the jump phenomenon.

The influence of ω_1 on frequency response for the external resonance is presented in Fig. 6. For $\omega_1>2\omega_2$, a soft spring effect is observed (curves are bent to the right). For $\omega_1<2\omega_2$ a hard spring effect is observed (curves are bent to the left). An interesting phenomenon here occurs namely at critical value of $\omega_1=\omega_{1c}$, the plot slope is minimal.

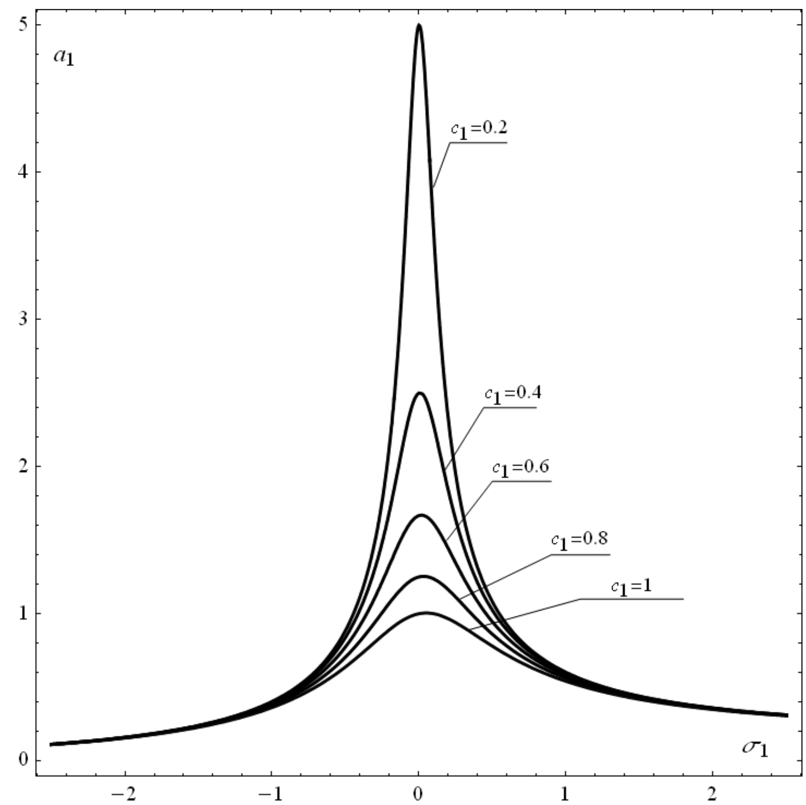

Fig. 2. Amplitude a_1 versus detuning parameter for different c_1 (for $a_2 = 0.06$).

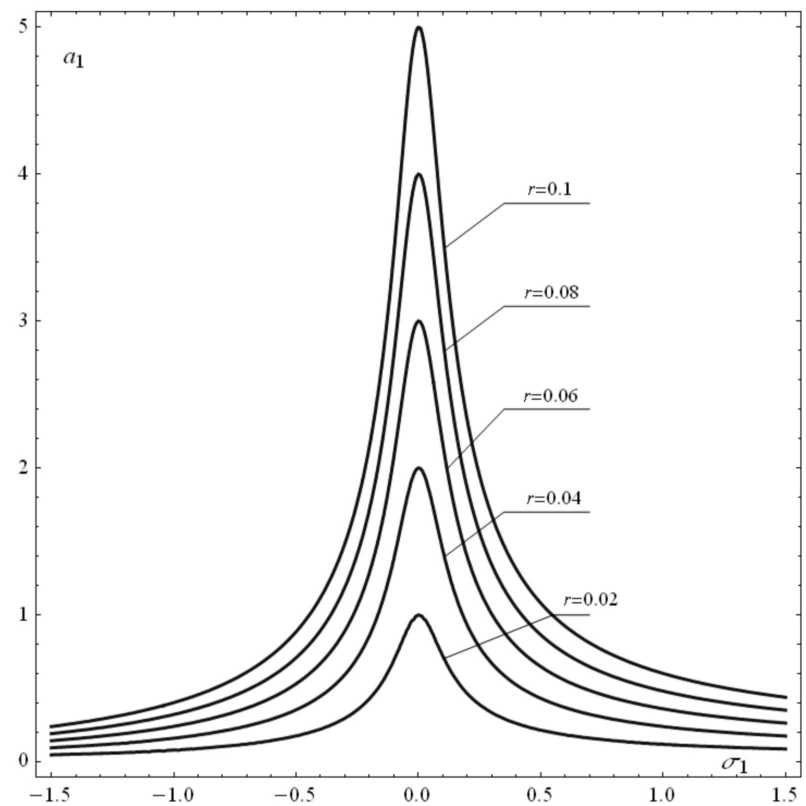

Fig. 3. Amplitude a_1 versus detuning parameter for different radius r (for $a_2 = 0.06$ and $c_1 = 0.2$).

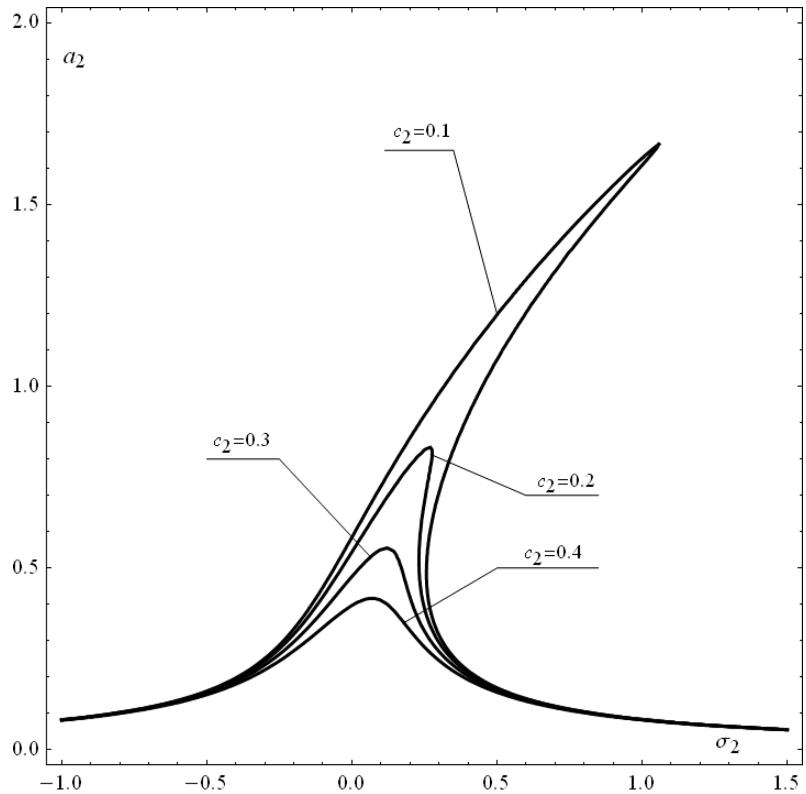

Fig. 4. Amplitude a_2 versus detuning parameter for different c_2 (for $a_1 = 0.08, f_2 = 1$).

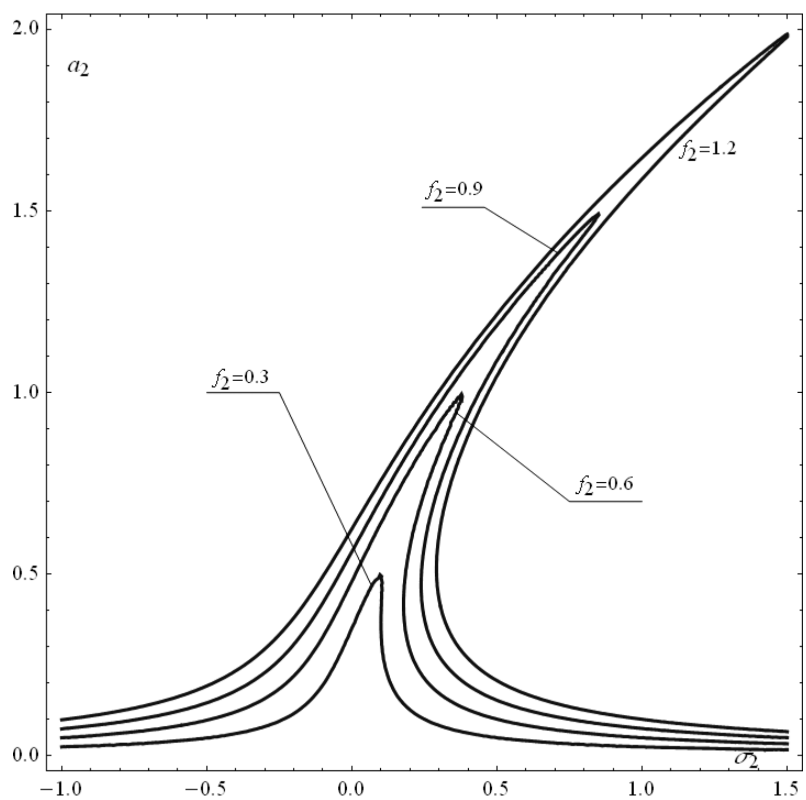

Fig. 5. Amplitude a_2 versus detuning parameter for different f_2 (for $a_1 = 0.08, c_2 = 0.1$).

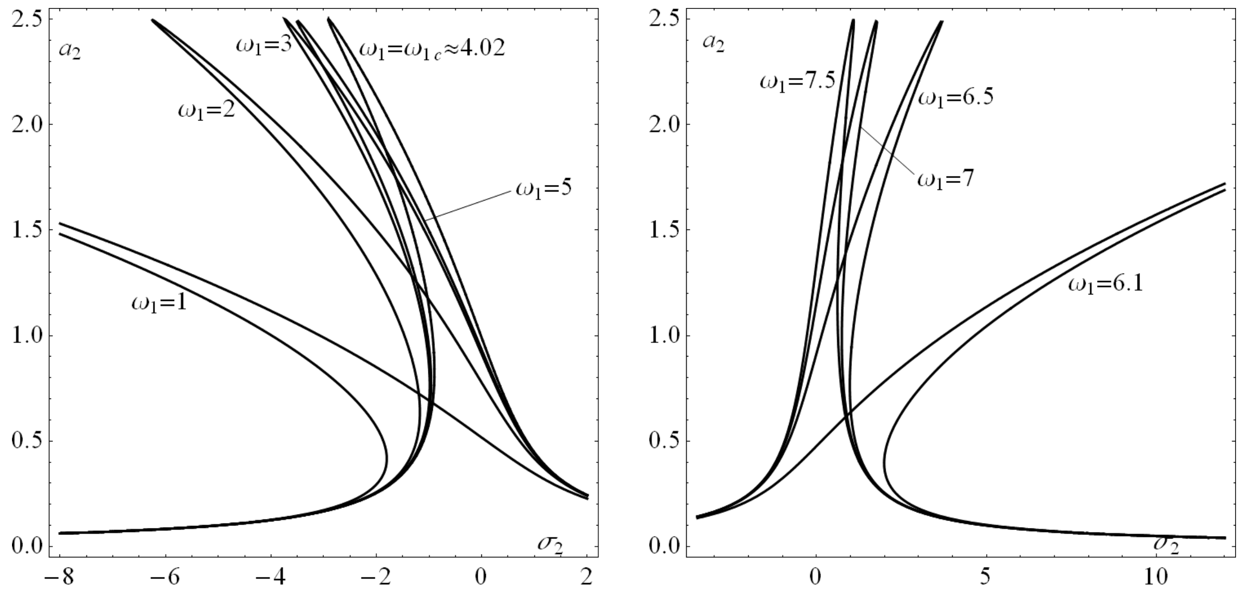

Fig. 6. Amplitude a_2 against detuning parameter (effects of natural frequency ω_1 variation).

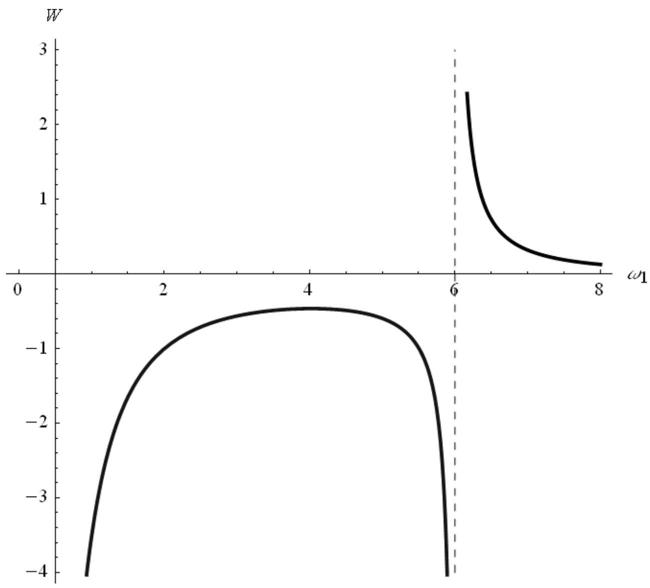

Fig. 7. The plot of the coefficient W, multiplied by a_2^3 in (16), versus ω_1.

The behavior of the system is associated with the change of sign of the coefficient

$$W = \frac{\omega_2^3(\omega_1^2 + 8\omega_2^2)}{16\omega_1^2(\omega_1^2 - 4\omega_2^2)}, \tag{29}$$

which is multiplied by a_2^3 in Eq. (28). The variation of coefficient at a_2^3 versus ω_1 is shown in Fig. 7.

6. Conclusions

The equations of motion for the studied system were effectively solved using the Multiple Scale Method adopting three time scales. The general solution, till third order, was achieved. We succeeded in presenting them in very concise form. The functions A_1 and A_2 can be calculated from the initial and solvability conditions.

The frequency response functions, regarding the stability of the system, are of special interest in the paper. All resonance cases for the third order expansion were detected.

Two chosen resonance cases, which occur simultaneously, have been discussed, namely: external ($\Omega_2 \approx \omega_2$) and parametric ($\Omega \approx \omega_1$). The frequency response functions are presented in graphical form. The nonlinear character of resonance for swings results from geometrical nonlinearity of the system. The shape of the resonance curve depends on the coefficient W [see Eq. (29)]. When W is negative, then we can observe the resonance suitable to the "soft" characteristics. Positive value of W makes the characteristics "hard".

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