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CHAOTIC VIBRATIONS OF TWO-LAYERED BEAMS AND PLATES WITH GEOMETRIC, PHYSICAL AND DESIGN NONLINEARITIES

J. AWREJCEWICZ[*,‡], A. V. KRYSKO[†,§], V. V. BOCHKAREV[†,¶], T. V. BABENKOVA[†,**], I. V. PAPKOVA[†,††] and J. MROZOWSKI[*]

[*]*Department of Automation and Biomechanics, Technical University of Lodz, Poland*

[†]*Department of Mathematics and Modeling, Saratov State Technical University, Russian Federation*

[‡]*awrejcew@p.lodz.pl*

[§]*anton.krysko@gmail.com*

[¶]*ra4ccl@yandex.ru*

[**]*BabenkovaT@yandex.ru*

[††]*ikravzova@mail.ru*

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In this paper, the theory of nonlinear interaction of two-layered beams and plates taking into account design, geometric and physical nonlinearities is developed. The theory is mainly developed relying on the first approximation of the Euler–Bernoulli hypothesis. Winkler type relation between clamping and contact pressure is applied allowing the contact pressure to be removed from the quantities being sought. Strongly nonlinear partial differential equations are solved using the finite difference method regarding space and time coordinates. On each time step the iteration procedure, which improves the contact area between the beams is applied and also the method of changeable stiffness parameters is used. A computational example regarding dynamic interaction of two beams depending on a gap between the beams is given. Each beam is subjected to transversal sign-changeable load, and the upper beam is hinged, whereas the bottom beam is clamped. It has been shown that for some fixed system parameters and with an increase of the external load amplitude, synchronization between two beams occurs with the upper beam vibration frequency. Qualitative analysis of the interaction of two noncoupled beams is also extended to the study of noncoupled plates. Charts of beam vibration types versus control parameters $\{q_0, \omega_p\}$, i.e. the frequency and amplitude of excitation are constructed. Similar and previously described competitions have been reported in the case of two-layered plates.

Keywords: Nonlinearity; beam; plate; chaos; synchronization.

1. Introduction

The fundamental idea of the proposed approach is associated with the analysis of chaotic vibrations of contact problems of beams taking into account design, geometric and physical nonlinearities. The designed nonlinearity concerns the activation and removal of one-sided constraints. Physical nonlinearity is associated with nonlinear relations between strains and stresses, whereas geometric nonlinearity is connected with nonlinear relations between deformations and displacements in the form proposed by von Kármán for each of the beams. We deal with the following system of equations

$$\ddot{U} + \varepsilon\dot{U} + L(U) = F - Mq_k\psi \tag{1}$$

with boundary conditions $GU(A) = H(A)$, initial conditions $U(t = 0) = \Phi_1$ and $\dot{U}(t = 0) = \Phi_2$, and a stamp of no penetration condition $f(\mathbf{R} + \mathbf{U}) \geq 0$,

$A \in \omega$, where ω is the contact zone, L is the matrix differential operator, U and $\mathbf{U}$ are the vector function and the vector displacements of a point of surface Ω, respectively; F is the vector function of the externally distributed load, $f(\omega) = 1, \psi(A \notin \omega) = 0$; M denotes a column with the element responsible for the projection of the equilibrium equation on the normal of surface Ω equal to 1, and the rest of the elements are equal to 0; $f_k = 0$ represents the stamp surface (outside stamp $f > 0$). The mentioned non-penetration condition is given in the following linear form

$$f(\mathbf{R}) + \mathbf{U}\nabla f(\mathbf{R}) \geq 0, \quad A \in \omega. \tag{2}$$

In what follows we are going to find a solution to problems (1) and (2). If space ω is not defined, then owing to the Lukash definition [Lukash *et al.*, 1981], a nonlinear design problem is obtained. Observe that in many engineering cases the contact reaction may consist of the transversal force and torque concentrated on boundary ω, represented by a vector tangent to the boundary ω.

On the other hand, in the case of static problems another approach is often applied. Namely, the problem is reduced to that of integral Fredholm first order equation of the form

$$\int_\omega G(A,B)q_k(B)d\omega = g(A), \quad A \in \omega, \tag{3}$$

where $G(A,B)$ is the Green function (defining deflection at points A and B), a force concentrated normal to the beam surface acts; $g(A)$ is the sum composed of an expression which describes part of the stamp surface in contact and a function defining the stamp (rigid body) displacement.

If Eq. (3) is obtained within the Euler–Bernoulli hypothesis, then it has only a general solution: the contact reaction represents concentrated forces on the contact zone boundaries (if the contact zone is not given) and a torque for the known (given) space ω [Galina, 1976a; Popov & Tolkachev, 1980].

The contact problem of beams theory can be improved via the regularization procedures, which mainly deal with transition from Eq. (3) to the Fredholm second type equation of the form

$$Kq_k(A) + \int_\omega G(A,B)q_k(B)d\omega = f(A). \tag{4}$$

Methods of mathematical regularization require the use of the regular operator definition [Popov & Tolkachev, 1980; Tikhonov & Arsenin, 1979]. It is evident that the mathematically approved method of regular parameter choice is that of a boundary layer minimum proposed by Tikhonov [Tikhonov & Arsenin, 1979; Lavrentev *et al.*, 1980]. This method allows us to use only minimum *a priori* information, but it also requires a solution to the complimentary problem of the functional minimum search.

There are also approaches devoted to regularization of the integral equations of the studied contact problems. They are focused on an improvement of physical modeling and are referred to as the physical regularization [Grigoluk & Tolmachev, 1975]. In [Grigoluk & Tolmachev, 1975] an additional elastic layer is introduced to the contact zone, governing real properties of micro-geometry of the contacting surfaces and having the following properties: (i) deformation of the contact layer is local and does not depend on the movement of contacting bodies; (ii) a mutual influence of the contact deformations is cancelled, i.e. the displacement of an arbitrary part of the contact layer is not influenced by displacements of other parts of the layer; (iii) deformation of the contact layer is proportional to the interaction of contact forces. Formulation of contact conditions in the integral form leads to a second order integral Fredholm type equation. Coefficient K is also treated as the regularization parameter which qualitatively characterizes properties of the contacting surfaces and is defined experimentally [Aleksandrov & Romalis, 1986; Demkin, 1970; Levina & Reshetov, 1971].

One of the methods of K determination in Eq. (4) relies on the identification of transversal clamping with displacement of the half-space boundary subjected to contact pressure [Konveristov & Spirina, 1979]. In references [Aleksandrov, 1962; Aleksandrov & Mkhitarian, 1983; Bogatyrenko *et al.*, 1982] the formula which governs normal layer of low thickness displacement is stiffly coupled with nondeformable foundation [Vorovich *et al.*, 1974], and it is shown that when the layer thickness decreases then its mechanical properties approach the Winkler foundation properties [Vorovich *et al.*, 1974]. In [Kantor, 1983, 1990; Kantor & Bogatyrenko, 1986] the approach devoted to solutions of the contact problems of nonlinear shells theory is proposed. It consists of removing the contact pressure q_k from the unknown functions with the help of the Winkler type coupling. The mentioned approach is equivalent to that of formula (4), and it allows us to neglect the tedious task of constructing the Green function, and hence solutions can be found directly through equilibrium equations (1). On the other hand, a study of chaotic vibrations of the contact problems of nonlinear mechanics of thin-walled structures is rarely presented (see for example [Awrejcewicz & Krysko, 2001, 2003a, 2003b, 2003c; Awrejcewicz *et al.*, 2002; Awrejcewicz *et al.*, 2004; Krysko *et al.*, 2003]). However, modeling and analysis of chaotic vibrations of two-layered beams with the mentioned three types of nonlinearity belongs to pioneering research.

2. Coupling between Contact Pressure and the Transversal Clamping of a Thin Beam

We begin to study this problem with the solution to the problem of a layer of thickness $2l_1$. Let in zone of length $2l$, the layer be loaded by pressure q in time instant Δt symmetrically regarding axes x and z (Fig. 1). We consider the plane case, and the pressure is approximated by the following series

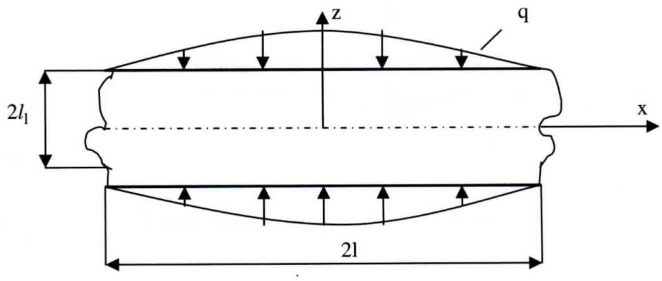


Fig. 1. Loading of the layer of length $2l$.

$$q(x) = \sum_{m=0}^{\infty} A_m \cos\frac{m\pi x}{l}. \tag{5}$$

Using a solution for stresses [Vorovich *et al.*, 1974], one gets

$$\sigma_x = 2\sum A_m F_m^-(z)\cos\beta x, \quad \sigma_z = 2\sum A_m F_m^+(z)\cos\beta x,$$
$$F_m^{\pm} = \frac{(\beta t\cosh\beta l_1 \pm \sinh\beta l_1)\cosh\beta z - \beta z\sinh\beta z\sinh\beta l_1}{\sinh 2\beta l_1 + 2\beta l_1}, \tag{6}$$

where $\beta = m\pi/l$. We take $q(x, \mathrm{V}t) = -\sigma_z(x, l_1, \mathrm{V}t)$, and in the case of plain stress state we have

$$\varepsilon_z = \frac{\partial u_z(x, z, \mathrm{V}t)}{\partial z} = \frac{1}{E}[(1-\nu^2)\sigma_z - \nu(1+\nu)\sigma_z].$$

Integrating the given formula with respect to z and taking into account that $\int_0^{l_1}\sigma_x dz = 0$, and with the external load along x being not activated, one gets

$$u_z(x, l_1, \mathrm{V}t) = \frac{1-\nu^2}{E}\int_0^{l_1}\sigma_z dz = 2(1-\nu^2)\sum_m A_m \times \frac{\cosh 2\beta l_1 - 1}{\beta(\sinh 2\beta l + 2\beta l_1)}\cos\beta z. \tag{7}$$

Developing the numerator and denominator of the fraction into the power series regarding $\beta l_1 \equiv c$ one obtains

$$u_z(x, l_1, \mathrm{V}t) = \frac{1-\nu^2}{E}l_1\sum_m A_m \frac{1 + \frac{c^2}{3} + \frac{2c^4}{45} + K}{1 + \frac{c^2}{3} + \frac{3c^4}{45} + K}\cos\beta x$$
$$= \frac{1-\nu^2}{E}l_1\sum_m A_m\left(1 - \frac{c^4}{45} + K\right)\cos\beta x$$
$$= \frac{1-\nu^2}{E}l_1\left[q(\bar{x}, \mathrm{V}t) - \frac{1}{45}(\bar{l}_1)^4 q_x^{IV}(\bar{x}, \mathrm{V}t) + L\right],$$
$$\bar{x} = \frac{x}{l}, \quad \bar{l}_1 = \frac{l_1}{l}. \tag{8}$$

Denoting $2l_1 = h$, one gets the formula which couples transversal clamping V with function $q(\bar{x}, \mathrm{V}t)$:

$$\frac{1}{1-\nu^2}\frac{E}{h}\mathrm{V} = q(\bar{x}, \mathrm{V}t) - \frac{1}{720}\left(\frac{h}{l}\right)^4 q_{\bar{x}}^{IV}(\bar{x}, \mathrm{V}t) + L. \tag{9}$$

Let us study the clamping of the layer by two symmetrical smooth stamps without corners (Fig. 2). The solution of the contact problem, where $U_z(x, l_1)$ is the function given on interval $-l \le x \le l$, satisfies equation [Galina, 1976b]

$$\int_{-\infty}^{\infty} q(\xi)K\left(\frac{x-l_1}{l_1}\right)d\xi = \pi\bar{S}_0 u_z(x, l_1), \tag{10}$$

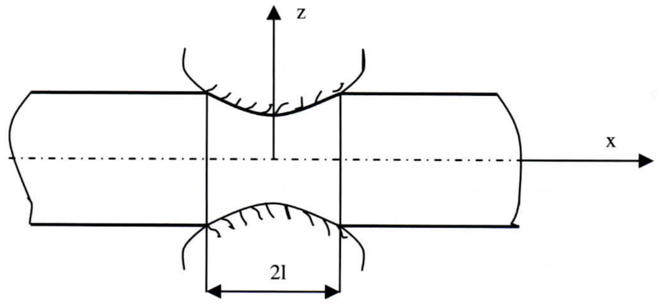


Fig. 2. Clamping by two symmetrical smooth stamps.

if in outside interval $|x| \leq l$ for $|z| = l_1$ the following conditions hold $\sigma_z = \tau = 0$, where

$$K(p) = \int_0^\infty \frac{ch2u - 1}{sh2u + 2u} \frac{\cos up}{u} du \equiv \int_0^\infty \frac{L(u)}{u} \cos up\, du,$$

$$\overline{S}_0 = \frac{E}{2(1-\nu^2)}.$$

A solution to Eq. (10) in the form

$$q(x, \mathrm{V}t) = \frac{\overline{S}_0}{\pi l_1^2} \int_{-\infty}^{\infty} u_z(h, l_1) K^*\left(\frac{x - l_1}{l_1}\right) d\xi,$$

$$K^*(p) = \int_0^\infty \frac{u \cos pu}{L(u)} du$$

is given in [Korn & Korn, 1968], where with the help of the series

$$\frac{u}{L(u)} = \frac{1}{A} + \sum_{i=1}^{\infty} B_i u^{(2i)}(p)$$

we get

$$K^*(p) = \frac{\pi}{A}\delta_0(p) + \pi \sum_{i=1}^{\infty} (-1)^i B_i \delta^{(2i)}(p),$$

and $\delta_0^{(2i)}$ denotes the second order derivative. Let us find an explicit solution using the formulas [Korn & Korn, 1968]:

$$\delta_0(p) = \delta_0\left(\frac{x - \xi}{l_1}\right) = l_1 \delta_0(x - \xi),$$

$$\delta_0(x - \xi) = -\delta_0(\xi - x),$$

$$\int_{-l_1}^{l} f(\xi)\delta_0(x - \xi) d\xi = f(x),$$

$$\int_{-l_1}^{l} f(\xi)\delta_0^{(2i)}(x - \xi) d\xi = f^{(2i)}(x), \quad |x \leq l|.$$

Owing to the evenness of function $K^*(p)$, we get $K^*((\xi - x)/l_1) = K^*((x - \xi)/l_1)$. Substituting the series for $u/L(u)$ into the solution of Eq. (10), integrating it in the interval $(-l, l)$ and taking into account relations for the general functions, one gets

$$q(x, \mathrm{V}t) = \frac{1}{2(1-\nu^2)} \frac{E}{l_1}[u_z(x, l_1, \mathrm{V}t) - B_1 l_1^2 u_z + B_2 l_1^4 u_z^{IV} - L].$$

Development of $u/L(u)$ into a series regarding u gives

$$\frac{u}{L(u)} = \frac{u(\sinh u + 2u)}{\cosh u - 1} = 2\left(1 + \frac{1}{45}u^4 + L\right).$$

Therefore, owing to the introduced formulas one gets $1/A = 2$, $B_1 = 0$, $B_2 = 2/45$, and finally

$$q(\overline{x}, \mathrm{V}t) = \frac{1}{1-\nu^2} \frac{E}{h}\left[\mathrm{V} + \frac{1}{720}\left(\frac{h}{l}\right)^4 \mathrm{V}^{IV} + L\right]. \tag{11}$$

One may check that relations (9) and (11) are mutually invertible. It means that a lack of deflection and extension deformations of the middle beam surfaces leads to the coupling between clamping and contact pressure to differ from that of the Winkler type by the fourth and higher derivative multiplied by small coefficients. Even for $h/l = 1$, which is the characteristic contact zone dimension, the first of the mentioned coefficients is equal to 1/720, whereas usually for contacting beams the following estimation holds $h \ll l$. This observation allows us to conclude that the Winkler model (11) possesses sufficiently high accuracy. A similar conclusion was obtained by the authors in [Aleksandrov & Romalis, 1986], who carried out the asymptotic analysis of an exact solution for a layer, as well as by Kantor [1990], who solved a wide class of problems for axially symmetric shells.

In the case of tangential deformations occurring during deflection and extension of the contacting beams, the coefficient by V is changed, and small additional terms appear. From the formula for clamping [Pelekh & Sukhorolskiy, 1980] for plates, one gets

$$q_k = \frac{3(1-\nu)}{4(1+\nu)(1-2\nu)} \frac{E}{h}[\mathrm{V} + O(h^2 \mathrm{V}_0 w)],$$

whereas for beams one obtains

$$q_k = \frac{3}{4}\frac{E}{h}[\mathrm{V} + O(h^2 \mathrm{V}_0 w)].$$

The clamping formula yields also [Grigoluk & Tolmachev, 1975]:

$$q_k = \frac{16}{3}\frac{E}{h}[\mathrm{V} + O(h^2 \mathrm{V}_0 w)].$$

Analogous formulas are obtained in reference [Bloch, 1975], whereas similar relations are compared in [Bloch, 1977]. They differ only by the value of coefficient E/h, which in all cases is close to one. Taking into account the estimations and application of the Winkler type relation between clamping and contact pressure during solutions to the contact problems of the shells with the Kirchhoff theory, we express further a contact pressure between beam and stamp by the difference $\mathrm{V} = w - a$, where w denotes the normal displacement of the beam middle surface (see [Grigoluk & Tolmachev, 1975; Galina, 1976a; Pelekh & Sukhorolskiy, 1980; Aleksandrov & Mkhitarian, 1983]), and

$$q_k = K\frac{E}{h}(w - a), \quad a > 0. \tag{12}$$

In the solution to the contact problem between two beams, one gets

$$q_k = K\frac{E}{h}(w_1 - a - w_2). \tag{13}$$

Beams numeration is introduced with respect to the positive direction of a normal to beam surface. If the contact takes place then $q_k > 0$. If between beams there is a thin washer, it may be included into considerations by changing K. Formula (13) is written for the case of the contact between beams with the same values of E and h. If this is not the case, then we take

$$w_i^z\left(\frac{h}{2}\right) - w_i = \frac{1 - v^2}{E_1}\frac{h_i}{2}q_k, \quad i = 1, 2.$$

Let us transform the given formulas for condition $w_2^z(-h/2) = w_2^z(h/2)$. One obtains

$$q_k = 2\left(1 + \frac{E_1 h_2}{E_2 h_1}\right) K\frac{E_1}{h_1}(w_1 - a - w_2). \tag{14}$$

In real problems, the fraction on the right-hand side is of order one, therefore further for the sake of simplification we apply (13), considering E and h as the characteristic quantities.

3. Fundamental Relations of the Nonlinear Beams Theory

The validity of theoretical investigations of beams with hybrid type contacts of layers has already been discussed. We consider now two types of problems. The first one deals with the analysis of stress state and the dynamics of layered beams with the occurrence of separated zones including nonideal ones of layers appearing due to technical defects and peculiarities in the exploitation of a construction. This problem is discussed in numerous works, for example in [Pelekh & Sukhorolskiy, 1980]. The second type of problems occur during the computation of beams composed of equidistant layers coupled with each other on the beam ends and interacting in on one-sided manner. The structures which include these beams as elements are widely applied in technology, and they are characterized by a large number of layers. Sometimes external layers differ from internal ones by thickness and mechanical properties, and gaps between them are allowed. The layers may slide with and without friction. The occurrence of stick-slip zones is rather less probable, since the pressure between layers is small. We deal with the mentioned beams. Contact conditions between zones may depend on the coordinates and may include various beam imperfections. Conditions of the layers welding (a break in the vertical direction and a rotation in a tangent direction) are not considered. The behavior of the layers obeys the theory of deformation and plasticity and geometric nonlinear theory of averaged von Kármán deflection. As it has been previously described, the function of contact pressure is excluded from the number of unknowns.

Next, we study a two-layered package composed of two beams of thickness h_l $(l = 1, 2)$. The following notation is applied: h_l — beam thickness; h_{0l} — beam thickness in its center; $w_l(x, t)$ — beam deflection; $u_l(x, t)$ — deflections in the averaged surface; E_l — material elasticity modulus; b_l — beam width; b_{0l} — beam width in its center; t — time; ε_l — damping coefficient; a — beam length; ρ_l — material density; G_{0l} — shear modulus; v_l — Poisson's coefficient; e_{il} and σ_{il} — deformation and stress intensity, respectively; e_{sl} and σ_{sl} deformation and stress flow intensity, respectively; K_l — volume modulus of elasticity, ε_{0l} — volume modulus of deformation; K — the Winkler coefficient. Let beams occupy a space $\Omega_l = \{(x, z) \,|\, 0 \le x \le a, \alpha_l \le z \le \beta_l\}, (l = 1, 2)$ in R^2, where $\alpha_1 = -h_1/2, \alpha_2 = (h_1/2 + \delta)$, $\beta_1 = h_1/2, \beta_2 = (h_1/2 + \delta + h_2)$ and the introduced notation holds for the first and second beams, respectively. The origin of coordinates is shifted to the left-hand side of the upper beam on its middle

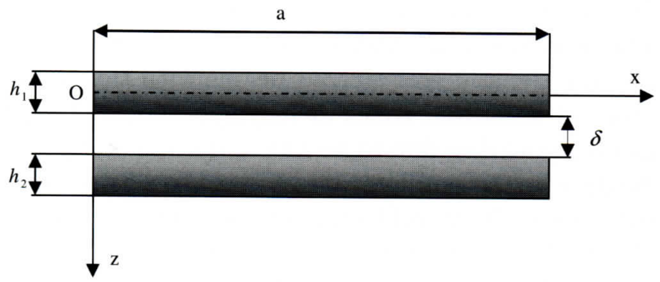


Fig. 3. Two beams with a gap.

surface, and the distance between beams is δ. Axis oz is directed downwards, and the package length is a (Fig. 3). Although the beams are made from an isotropic material, a nonhomogeneous material is applied such that the extension E_l and shear modules G_l, volume deformation K_l, transversal deformation coefficient γ_l, and flow zone coefficient σ_{sl} are functions of x and z.

It is further assumed that the physical parameters of material E_l, G_l, K_l, ν_l, i.e. Young modulus, shear modulus, volume deformation, and Poisson's coefficient are one-valued functions of a point and associated with its deformable state. The deformable point state is characterized by volume deformation type ε_{0l} and deformation intensity e_{il}. As it has been already mentioned, in solutions of contact problems of the theory of beams satisfying the Euler–Bernoulli hypothesis, the Winkler coupling between clamping and contact pressure is applied between beams of the same thickness $h_1 = h_2 = h$ and material properties $E_1 = E_2 = E$ (see [Aleksandrov & Romalis, 1986]).

Function $q_k(w)$, defined by formula (13), is linear in relation to transversal beam measure in the contact zone. The application of nonlinear coupling q_k with deflection w does not complicate the construction of the further developed method of solution of contact problems, i.e. in the equilibrium equation q_k is substituted by explicit expression $q_k(w)$. Furthermore, this substitution holds also in equations of nonlinear beams theory. We assume also during the derivation of the beam motion governing equations that relation $E = E(x, z, \varepsilon_0, e_i)$ is given, and the so-called Hencky's theory of elastic-plastic deformations is applied [Iliushin, 1948; Birger, 1951; Budiansky, 1959; Rabotnov, 1966].

The following nondimensional parameters are introduced:

$$x = \bar{x}a, \quad z = \bar{z}h_{0l}, \quad h_l = \bar{h}_l h_{0l}, \quad w_l = \bar{w}_l h_{0l},$$

$$E_l = \bar{E}_l G_{0l}, \quad b_l = \bar{b}_l, \quad p_l = \bar{p}_l h_{0l}, \quad K = \bar{K}\frac{h_0^4 b_0}{a^4},$$

$$t = \bar{t}\frac{a^2}{h_{0l}}\sqrt{\frac{\rho_l}{G_{0l}b_{0l}}}, \quad \varepsilon_l = \bar{\varepsilon}_l \frac{a^2}{h_{0l}^2}\sqrt{\frac{\rho_l}{G_{0l}b_{0l}}}.$$

Taking into account the Euler–Bernoulli hypothesis, the nonlinear stress-strain relation, and the theory of small elastic-plastic deformations, one gets the following nondimensional equations governing beam dynamics (bars over nondimensional quantities are omitted):

$$b_l h_l \frac{\partial^2 u_l}{\partial t^2} = \frac{\partial}{\partial x}\left[E_{0l}\left(u_l' + \frac{1}{2}(w_l')^2\right) - E_{1l}w_l''\right], \tag{15}$$

$$\begin{aligned} b_l h_l \frac{\partial^2 w_l}{\partial t^2} + \varepsilon_l \frac{\partial w_l}{\partial t} &= q_l^* + \frac{\partial^2}{\partial x^2}\left[E_{1l}\left(u_l' + \frac{1}{2}(w_l')^2\right) - E_{2l}w_l''\right] \\ &\quad + \frac{\partial}{\partial x}\left\{w_l'\left[E_{0l}\left(u_l' + \frac{1}{2}(w_l')^2\right) - E_{1l}w_l''\right]\right\}, \\ &\qquad l = 0, 1, 2, \end{aligned} \tag{16}$$

where

$$E_{il} = b\int_{\alpha_l}^{\beta_l} E_l z^i dz, \quad q_l^* = q_l + q_{kl}, \quad i = 0, 1, 2. \tag{17}$$

Force q_l^*, acting on the beam, is defined by the sum of external periodic load q_l and contact forces q_{kl}. Observe that the analysis of beams interaction includes a study of jumps (lack of beam contact). Contact stresses are defined similarly to formula (13) in the following way:

$$q_{kl} = (-1)^l K\frac{E_1}{h_1}\left(w_1 - \delta - w_2\frac{h_{02}}{h_{01}}\right)\psi, \quad l = 1, 2, \tag{18}$$

where K is the proportionality coefficient between contact pressure and clamping. Function ψ defines the dimensions of the contact zone and is defined as

$$\psi = \frac{\left[1 + \operatorname{sgn}\left(w_1 - \delta - w_2\frac{h_{02}}{h_{01}}\right)\right]}{2}. \tag{19}$$

Note that the occurrence of multiplier ψ in the equations of motion leads to the problem being transferred to that of design nonlinearity. By design nonlinearity we mean such nonlinearity, when the computation scheme is changed during the deformation process. Although boundary conditions can be arbitrary, we study two variants, namely:

hinged nonmovable support at the ends

$$\begin{aligned}\frac{\partial^2 w_l(0,t)}{\partial x^2} &= w_l(0,t) = \frac{\partial^2 w_l(1,t)}{\partial x^2} \\ &= w_l(1,t) = 0, \\ u(0) &= u(1) = 0,\end{aligned} \tag{20}$$

and clamping at the ends

$$\begin{aligned}\frac{\partial w_l(0,t)}{\partial x} &= w_l(0,t) = \frac{\partial w_l(1,t)}{\partial x} \\ &= w_l(1,t) = 0; \\ u(0) &= u(1) = 0.\end{aligned} \tag{21}$$

The initial conditions follow

$$\begin{aligned}\dot{w}(x,0) = F_l(x), \quad & w_l(x,0) = f_l(x), \\ u_l(x,0) = 0, \quad & \dot{u}_l(x,0) = 0,\end{aligned} \tag{22}$$

where F_l and f_l are the functions governing velocity and deflection distributions at an initial time instant. In order to include material beams physical nonlinearity, the plastic deformation theory and the method of changeable elasticity parameters are used [Birger, 1951]. Therefore, the elasticity modulus and Poisson's coefficient are coupled with shear and deformation modules by the relation

$$E_l = \frac{9K_l G_l}{3K_l + G_l}. \tag{23}$$

Modulus K_l is assumed constant and equal to $1.94G_{0l}$. In the theory of plastic deformations, the shear modulus is defined as follows

$$G_l = \frac{1}{3}\frac{\sigma_{il}(e_{il})}{e_{il}}. \tag{24}$$

Diagrams of beam material deformation can be arbitrary, but in this work, we take an ideal elastic-plastic material:

$$\begin{aligned}\sigma_{il} &= 3G_{0l}e_{sl} \quad \text{for } e_{il} < e_{sl}, \\ \sigma_{il} &= \sigma_{sl} \quad \text{for } e_{il} \geq e_{sl}.\end{aligned} \tag{25}$$

The deformation intensity is defined by the formula

$$e_{il} = \frac{\sqrt{2}}{3}\left[(e_{xx} - e_{yy})^2 + (e_{yy} - e_{zz})^2 + (e_{xx} - e_{zz})^2 + \frac{3}{2}e_{xy}^2\right]^{\frac{1}{2}}. \tag{26}$$

Component e_{zz} can be found from the plane stress-strain condition $\sigma_{zz} = 0$, and hence

$$e_{zz} = -\frac{\nu}{1-\nu}(e_{xx} + e_{yy}).$$

If one neglects components e_{zz}, e_{yy} for the beam, then

$$e_{il} = \left|\frac{2}{3}e_{xx}\right| = \frac{2}{3}\left|\frac{\partial u_l}{\partial x} + \frac{1}{2}\left(\frac{\partial w_l}{\partial x}\right)^2 - z\frac{\partial^2 w_l}{\partial x^2}\right|. \tag{27}$$

Finally, external load variations along beam axis and in time can be taken in an arbitrary way.

4. Method of Solution

Integration of Eqs. (15) and (16) with initial and boundary conditions is carried out numerically. For this purpose space $D = \{(x,t)\,|\,0 \leq x \leq 1, 0 \leq t \leq T\}$ is covered by rectangular grids $x_i = ih_x$, $t_j = jh_t$ $(i = 0,1,2,\ldots,n; j = 0,1,2,\ldots)$, where $x_i = x_{i+1} - x_i = h_x = 1/n_x$ (n_x integer) and $h_t = t_{j+1} - t_j$, $h_z = 1.0/n_z$. In grid x_i, t_j differential equations (15) and (16) are approximately substituted by the corresponding finite difference relations. In order to increase the accuracy, the symmetric formulas for derivatives are applied. After some transformations, one gets

$$\begin{aligned}w_{li,j+1} &= \frac{1}{\left(1 + \dfrac{\varepsilon_l h_t}{2b_l h_l}\right)} \times \left[2w_{li,j} + \left(\frac{\varepsilon_l h_t}{2h_l} - 1\right)w_{li,j-1} + \frac{h_t^2}{b_l h_l}A_{li,j}\right], \\ u_{ij+1} &= \frac{h_t^2}{bh}\left[\frac{\partial E_{0l}}{\partial x}\left(u' + \frac{1}{2}(w')^2\right) + E_{0l}(u'' + w'w'') - \frac{\partial E_{1l}}{\partial x}w'' - E_{1l}w'''\right]_{ij} + 2u_{ij} - u_{ij-1},\end{aligned} \tag{28}$$

where

$$A_{li,j} = \frac{\partial^2}{\partial x^2}\left[E_{1l}\left(u_l' + \frac{1}{2}(w_l')^2\right) - E_{2l}w_l''\right] - \frac{\partial}{\partial x}\left[w_l' E_{0l}\left(u_l' + \frac{1}{2}(w_l')^2\right) - E_{1l}w''\right]_{i,j}.$$

Applying the grid method, boundary conditions (20) and (21) regarding a layer with number j are as follows:

(i) hinge

$$w_{l-1,j} - 2w_{l0,j} + w_{l1,j} = 0, \quad w_{l0,j} = 0,$$
$$w_{ln-1,j} - 2w_{ln,j} + w_{ln+1,j} = 0, \tag{29}$$
$$w_{ln,j} = 0, \quad u_{l0,j} = u_{ln,j} = 0,$$

(ii) clamping

$$w_{l-1,j} - w_{l1,j} = 0, \quad w_{l0,j} = 0;$$
$$w_{l-n-1,j} - w_{ln+1,j} = 0, \quad w_{ln,j} = 0, \tag{30}$$
$$u_{l0j} = u_{lnj} = 0.$$

The following boundary conditions are applied

$$\frac{w_{li,j+1} - w_{li,j}}{h_t} = F_{li}, \quad w_{li} = f_{li}, \quad u_{li} = u_{l0i}, \tag{31}$$

where $i = 1, 2, \ldots, n$.

In this case, the governing equations obey a three-layered scheme. In the beginning we compute $w_l(x,t)$ and $u_l(x,t)$ on layer $(j+1)$ using the values of $w_l(x,t)$ and $u_l(x,t)$ at two previous layers jth and $(j-1)$th, respectively. In order to begin computations, the values of $w_l(x,t)$ and $u_l(x,t)$ on a fictitious layer are introduced. In order to apply the method of changeable stiffness [Birger, 1951] the beam is partitioned along its thickness to n_z layers. Furthermore, at each time step for node x_j and layer by layer one finds deformation intensity (27), and using formulas (23)–(25) the elasticity modulus is estimated, whereas integrals are computed using the Simpson method (17). We apply a periodic transversal of the form $q_{l0}\cos(\omega_l t)$.

The described algorithm of solution of the beam vibration equations are studied from the point of view of convergence along spatial and time grids regarding a stationary problem. For this purpose, using the set up method [Feodosev, 1963], we find deflection in centrally hinged supported beam uniquely loaded along its length (the load does not depend on time and $\varepsilon = \varepsilon_{\rm cr}$ is the critical value for an efficient damping of vibrations). Analysis of the results shows that the beam deflection does not depend on time step within the interval $2\cdot 10^{-5} \le h_t \le 10^{-3}$. Change of partition number along the spatial coordinate from $n_x = 28$ to $n_x = 10$ gives the error of only 1%. For $h_t = 2\cdot 10^{-5}$ and $n_x = 28$ in the beam center, one achieves the deflection of 0.02346 versus 0.02343 obtained in the analytical way, i.e. by solving a static problem [Bernstein, 1961]. For the values of h_t and n_x, i.e. the same as in the static case, the dynamic problem is solved. Namely, the hinge supported beam subjected to transverse load has been studied, where $q_{10} = 1.5$ and $\omega_1 = 0.5$. Analysis of the computational results showed that the number of partitions along the beam length from $n_x = 30$ to $n_x = 10$ gives a computation error of the fundamental beam frequency of only 1.6%. Taking into account the obtained data, the number of partitions along axis Ox has been taken as $n_x = 30$, whereas time step $t = 2\cdot 10^{-5}$. Change of the number of layers along thickness is from 12 to 20 and gives the error of 0.2%, and hence n_z is equal to 12. Computations are carried out for $0 \le t \le 120$. We take pure aluminum as the beam material ($e_{is} = 0.98\cdot 10^{-3}$). The intensity of deformation flow $e_s = 2.45$, which corresponds to the ratio of beam length to its thickness, is equal to 50. Excitation frequency acting on the upper (bottom) beam is $\omega_1 = 0.5$ ($\omega_2 = 1$). Observe that we have solved further not an elastic–plastic problem, but rather a nonlinear elastic–plastic problem, where relief occurs on the same curve as loading.

5. Examples of Beam Computation

5.1. *Geometric and design nonlinear system composed of two uncoupled beams*

The problem is solved in a physically linear frame, i.e. we take $E = \text{const}$. The upper beam is hinged, whereas the bottom beam is clamped on both sides. We take $\omega_1 = 0.5, \omega_2 = 1.0$, and the ratios $q_{01}/q_{02} = 1/4$, $\delta = 0.1$. Analysis of the computational results is carried out using the methods of nonlinear dynamics and qualitative theory of differential equations. Signals $w_i(0.5,t)$, FFT (Frequency Fast Fourier Transforms), Poincaré sections, and the dependencies $w_i(x,t), q_k(x,t)$ are studied.

In order to illustrate dynamic processes we take $w_i(0.5,t)$, FFT for each beam ($i = 1, 2$), and functions $w_i(x,t), q_k(x,t)$, where $0 \le x \le 1.5 \le t \le 15$. Already for relatively small vibration amplitudes (Fig. 4) one may observe in the frequency spectrum of the second beam that the frequency of the first beam is less than ω_2. Beam contact is realized in a central point, which is well illustrated by graphs of contacting pressure, and the occurred vibrations are almost harmonic ones.

Additionally, in both beams frequency power spectra a component of independent frequency $\omega = 0.225$ occurs. An increase of load intensity $q_1 = 1.0$,

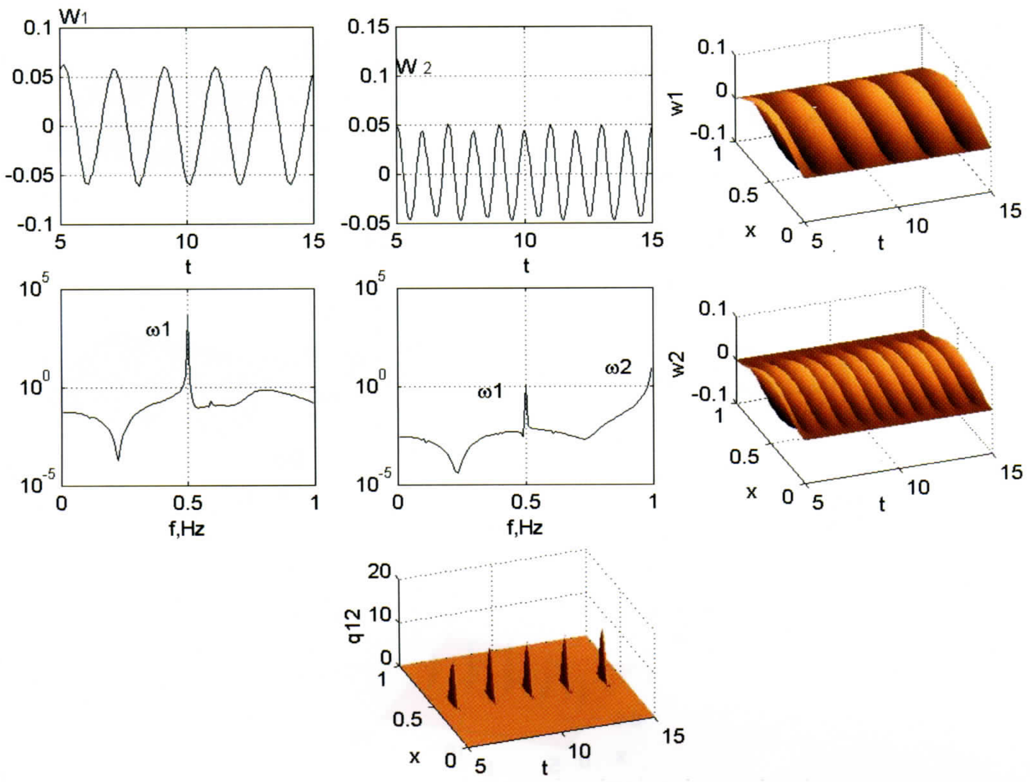


Fig. 4. Time and frequency characteristics of two beams.

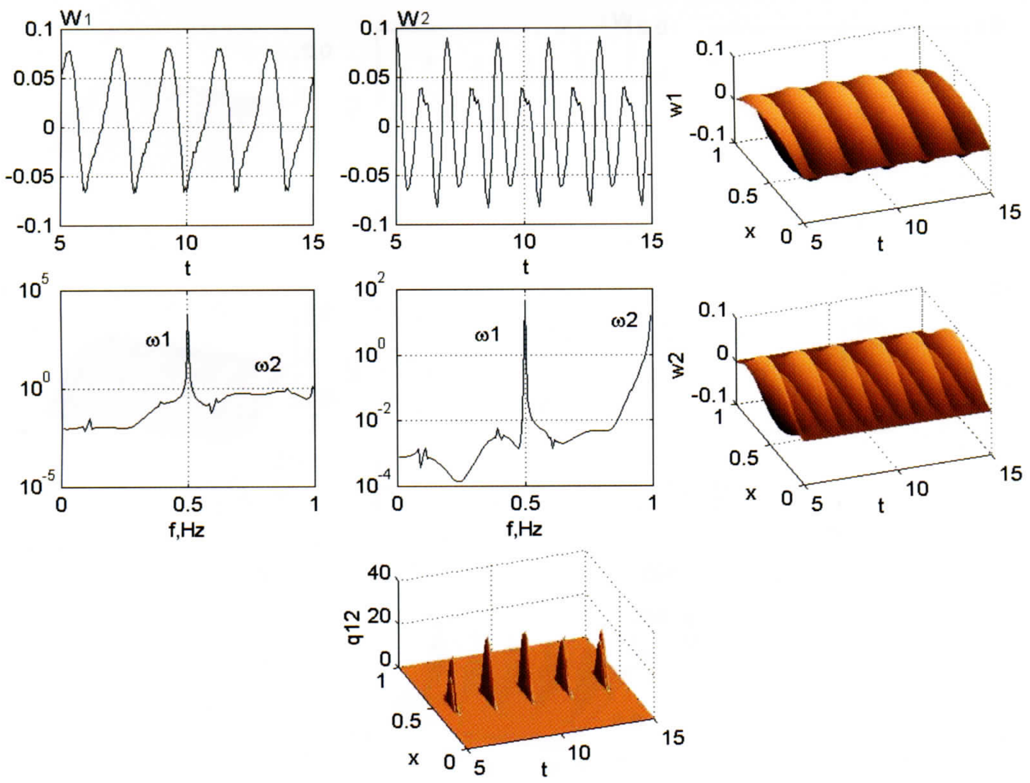


Fig. 5. Time and frequency characteristics of two beams ($q_{10} = 1.0, q_{20} = -4.0$).

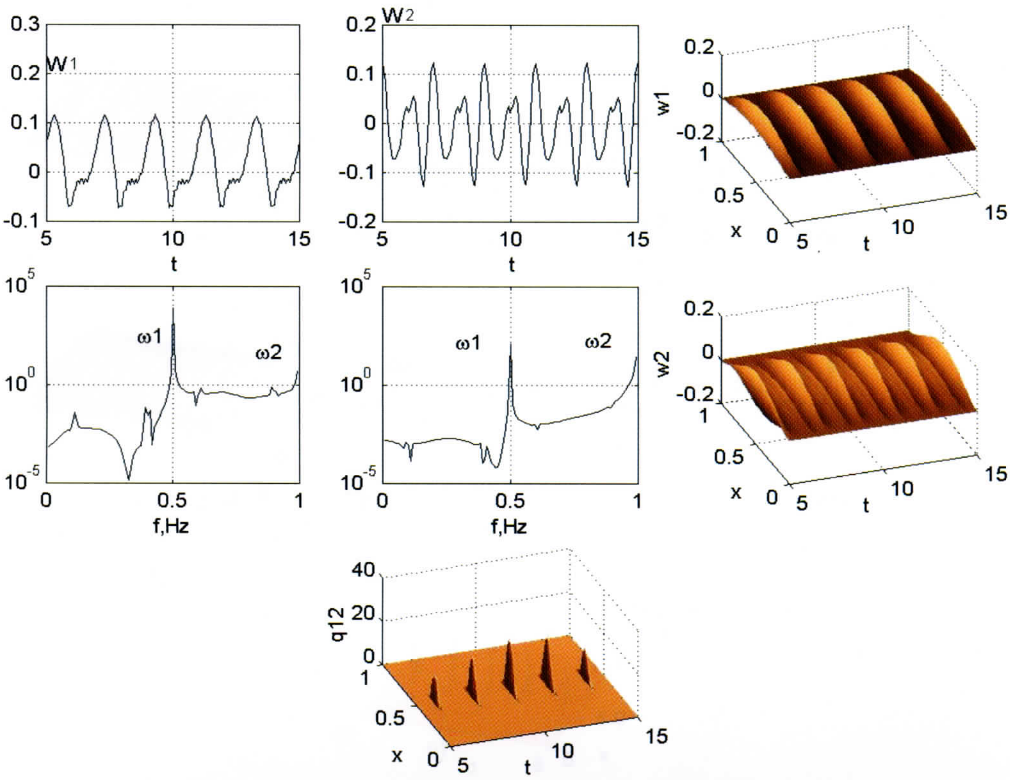


Fig. 6. Time and frequency characteristics of two beams ($q_{10} = 1.2, q_{20} = -4.8$).

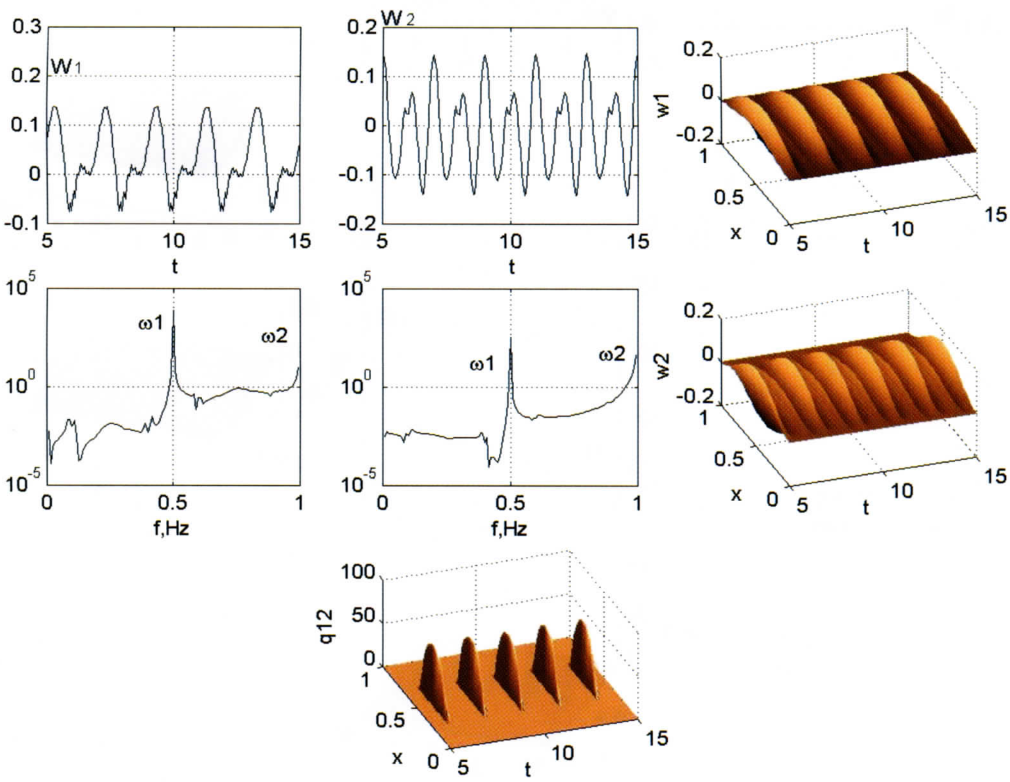


Fig. 7. Time and frequency characteristics of two beams ($q_{10} = 1.6, q_{20} = -6.4$).

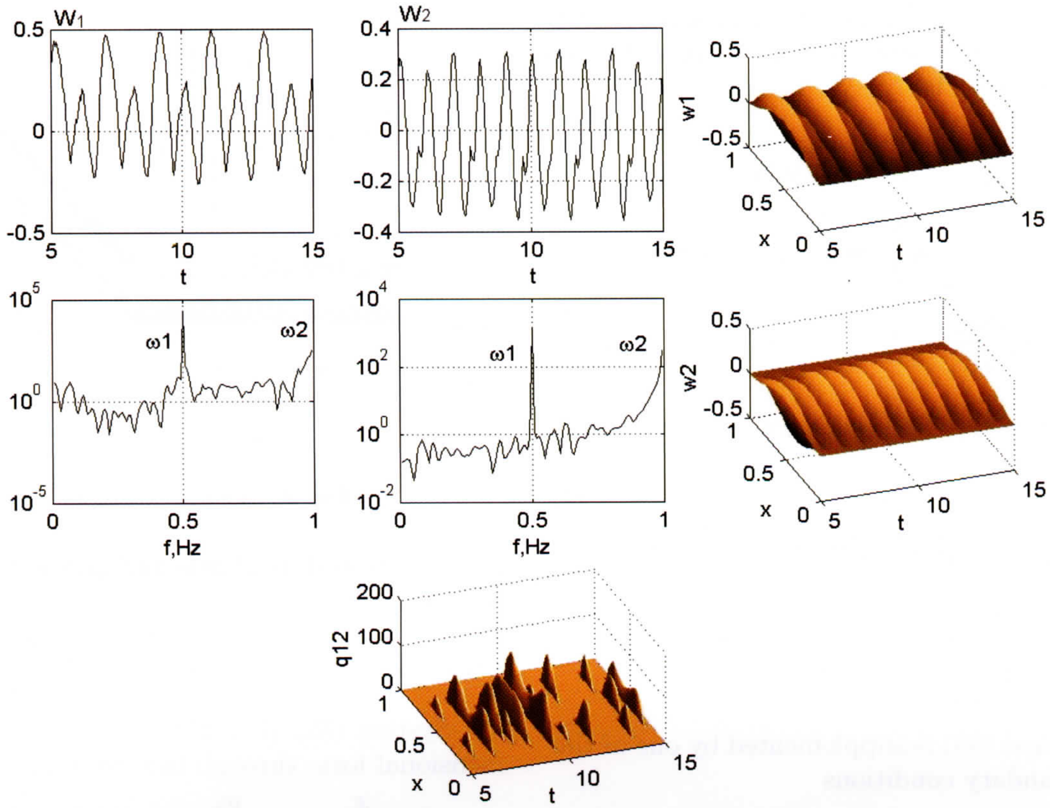


Fig. 8. Time and frequency characteristics of two beams ($q_{10} = 3.5, q_{20} = -14.0$).

$q_2 = -4.0$ on both beams (Fig. 5) does not change qualitatively beams interaction, but the frequency component ω_1 in the second beam frequency power spectrum increases and exceeds the excitation frequency component ω_2. Synchronization behavior is observed in the second beam vibrations and the amplitude associated with the first beam frequency increases. The synchronization behavior is well observed in the dependencies $w_1(x, t)$ and $w_2(x, t)$. New vibration components accompanied by frequencies $\omega = 0.1; 0.4; 0.6$ occur. The vibration form is modified, although the mutual contact is still realized in the vicinity of the beams center. An increase of the excitation amplitude regarding first $q_{10} = 1.2$ and second $q_{20} = -4.8$ beams causes that in the first beam frequency spectrum a component of the second beam frequency occurs, however with a small amplitude. The frequency spectrum becomes more complex and new frequency components appear ($\omega = 0.275; 0.9$). The contact pressure intensity is changed slightly in time (Fig. 6). An increase of the excitation amplitude on first (up to $q_{10} = 1.6$) and second ($q_{20} = -6.4$) beams causes the synchronization to be more evident, which is reported in Fig. 7.

In the frequency spectra new and independent frequencies appear, but their amplitudes are rather small, however the contact zone area and pressure increase substantially. Further increase of the excitation amplitude $q_{10} = 3.5, q_{20} = -14.0$ (Fig. 8) causes the occurrence of a chaotic dynamics manifested by a broad band frequency spectrum. Almost full synchronization of two beams occurs. Contact pressure has a stochastic character and maximum vibrations appear in the neighborhood of the beam quarter.

It should be emphasized that the analogous behavior appears also for other gap parameters.

6. Chaotic Vibrations of Two Uncoupled Plates

Let us derive a system of differential equations of two-layered uncoupled plates, when each of the layers satisfies the kinematic Kirchhoff hypothesis. The relative position of plates in space with the given coordinates O_{xyz} is defined in the following way: a middle surface of the first plate lies in $z = 0$, whereas of the second one in the $z = -1/2(\delta_1 - \delta_2) - h_k$ plane, where δ_i is the thickness of "i"th plate, and h_k denotes the distance between two plates in a nondeformable state. The system of governing equations is

$$\begin{cases} h\dfrac{\gamma}{g}\dfrac{\partial^2 w_1}{\partial t^2} + \varepsilon_1 \dfrac{\partial w_1}{\partial t} + A(w_1(x,y,t)) \\ \quad = q_1(x,y,t) - K\dfrac{E}{h}(w_1 - w_2 - h_k)\Psi(x,y,t), \\ h\dfrac{\gamma}{g}\dfrac{\partial^2 w_2}{\partial t^2} + \varepsilon_2 \dfrac{\partial w_2}{\partial t} + A(w_2(x,y,t)) \\ \quad = q_2(x,y,t) - K\dfrac{E}{h}(w_1 - w_2 - h_k)\Psi(x,y,t), \end{cases} \tag{32}$$

with the following attached initial conditions

$$w_i(t,x,y)|_{t=0} = f_i(x,y), \quad \left.\frac{\partial w_i}{\partial t}\right|_{t=0} = F_i(x,y), \quad i = 1,2. \tag{33}$$

System (32) and (33) is supplemented by one of the following boundary conditions

$$w_i|_{\partial\Omega_i} = \left.\frac{\partial w_i}{\partial n_i}\right|_{\partial\Omega_i} = 0, \tag{34}$$

$$w_1|_{\partial\Omega_1} = \left.\frac{\partial w_1}{\partial n_1}\right|_{\partial\Omega_1} = 0, \quad w_2|_{\partial\Omega_2} = \left.\frac{\partial^2 w_2}{\partial n_2^2}\right|_{\partial\Omega_2} = 0, \tag{35}$$

$$w_i|_{\partial\Omega_i} = \left.\frac{\partial^2 w_i}{\partial n_i^2}\right|_{\partial\Omega_i} = 0, \tag{36}$$

$$w_1|_{\partial\Omega_1} = \left.\frac{\partial^2 w_1}{\partial n_1^2}\right|_{\partial\Omega_1} = 0, \quad w_2|_{\partial\Omega_2} = \left.\frac{\partial w_2}{\partial n_2}\right|_{\partial\Omega_2} = 0, \tag{37}$$

where $q_1(t,x,y) = q_0 \sin(\omega_p t)$ is the function of external load acting on the first plate, $q_2 = 0$.

Let the plates occupy in R^2 the space $\Omega_i = \{|(x,y)|, 0 \le x \le a, 0 \le y \le b\}, i = 1,2$, and $\partial\Omega_i$ be the associated space boundary in R^2, K is the known constant, and $\Psi(x,y,t)$ is the contact space Ω^* indicator:

$$\Psi(x,y,z) = \frac{1 + \operatorname{sgn}(w_1(x,y,t) - w_2(x,y,t) - h_1)}{2}.$$

Although the differential operator $A(w_i(x,y,t))$ is in general a nonlinear one, but in this work each plate is treated as plastic and geometrically linear, and therefore

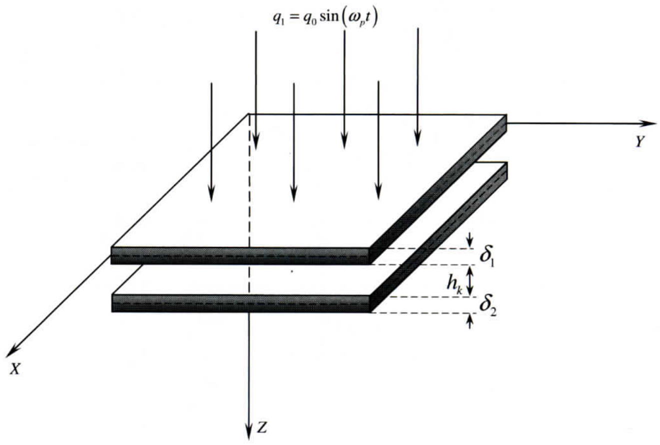


Fig. 9. Two plates with a gap.

$$A(w_i(x,y,t)) = \nabla^4 w_i = \frac{\partial^4 w_i}{\partial x^4} + 2\frac{\partial^4 w_i}{\partial x^2 \partial y^2} + \frac{\partial^4 w_i}{\partial y^4}.$$

System (32)–(37) is transformed into a nondimensional form through the following parameters

$$\bar{x} = \frac{x}{a}, \quad \bar{w} = \frac{w_i}{h}, \quad \bar{q}_1 = 2(1+\nu)\lambda_1^4 \frac{q_1}{E},$$

$$\lambda_1 = \frac{a}{h}, \quad \bar{y} = \frac{y}{b}, \quad \bar{h}_i = \frac{\delta_i}{h},$$

$$\bar{K} = 12(1-\nu^2)\lambda_1^4 K, \quad \bar{t} = \frac{h}{ab}\left(\frac{E}{1-\nu^2}\cdot\frac{g}{\gamma}\right)^{1/2} t.$$

Finally, system (32) is given in the following nondimensional form

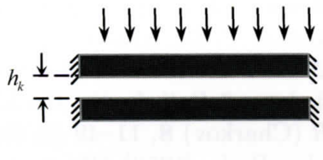
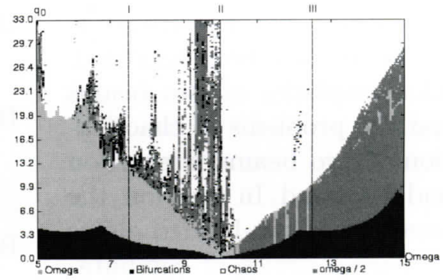
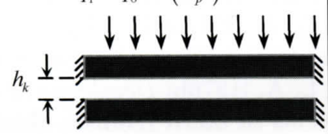
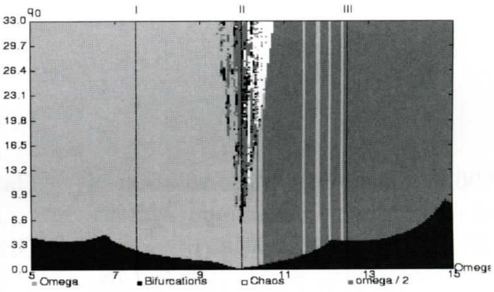
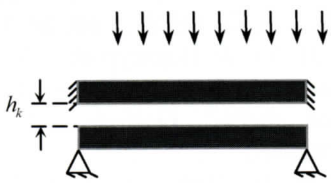
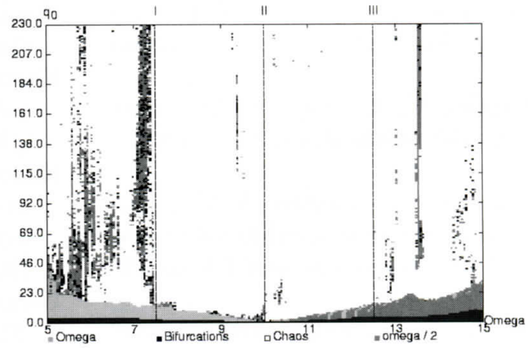
$$\begin{cases} \dfrac{\partial^2 w_1}{\partial t^2} + \varepsilon_1 \dfrac{\partial w_1}{\partial t} + \nabla^4 w_1 \\ \quad = q_1(x,y,t) - K\dfrac{E}{h}(w_1 - w_2 - h_k)\Psi(x,y,t), \\ \dfrac{\partial^2 w_2}{\partial t^2} + \varepsilon_2 \dfrac{\partial w_2}{\partial t} + \nabla^4 w_2 \\ \quad = q_2(x,y,t) - K\dfrac{E}{h}(w_1 - w_2 - h_k)\Psi(x,y,t), \end{cases} \tag{38}$$

where bars are already omitted. The obtained PDEs are reduced to the second order ODEs via the finite difference method with approximation $O(h^2)$. Further, the system is transformed to the first order ODEs and then solved via the fourth order Runge–Kutta method. Space and time steps are chosen via the Runge principle and $\Delta t = 0.001$, whereas grid step is 23×23 of space Ω.

Next, we study vibrations of two nonwelded plates of constant thickness ($\bar{h}_1 = \bar{h}_2$), made from an isotropic material with the Poisson's coefficient $\nu = 0.3$ subjected to uniformly distributed sign-changeable load. System behavior is investigated for two types of boundary conditions [clamping–clamping (34) and clamping-hinge (35)] and for various values of the gap between plates ($h_k = 0.01$ and $h_k = 0.1$). In order to investigate the two-layered beams behavior, charts of vibration character are constructed (Table 1). Three vertical lines are introduced: $\omega_p = \omega_0$ frequency of linear vibration, left and right lines correspond to frequencies $\omega_p - (\omega_p/2)$ and $\omega_p + (\omega_p/2)$, respectively. The introduced charts illustrate nonlinear dynamics of two-layered nonwelded beams.

The analysis of the system dynamics for various gap parameters and boundary conditions showed an increase of the gap between plates [Tables 1(a) and 1(b)] and decreases of the chaotic vibrations zones, whereas zones of periodic and quasi-periodic vibrations are increased. In the case of boundary

Table 1. Schemes of two-layered nonwelded plates and dynamics indicator charts.

(a) $q_1 = q_0 \sin(\omega_p t)$  $\varepsilon_1 = \varepsilon_2 = 0.1$ $h_k = 0.01$	
(b) $q_1 = q_0 \sin(\omega_p t)$  $\varepsilon_1 = \varepsilon_2 = 0.1$ $h_k = 0.1$	
(c) $q_1 = q_0 \sin(\omega_p t)$  $\varepsilon_1 = \varepsilon_2 = 0.1$ $h_k = 0.01$	

Notation

- Periodic vibrations
- Quasi-periodic vibrations
- Period doubling bifurcation
- Chaos

conditions [Tables 1(a) and 1(c)] one may observe that the boundary condition change of the second beam (from clamping to hinge) causes an essential increase of chaotic zones, and the periodic and quasi-periodic zones decrease.

7. Concluding Remarks

Theory of nonlinear interaction of two-layered beams have been introduced. Then a series of computational examples regarding regular, bifurcational and chaotic dynamics of the investigated objects have been reported. The general theory of the problem has been formulated, and a coupling between contact pressure and the transversal clamping of a thin beam has been modeled. Time and frequency characteristics of two beams have been reported, and the problems of chaotization and synchronization of two beams interaction have been illustrated and discussed. In addition, the theory of nonlinear interaction of two-layered plates have been introduced. Then a series of computational examples regarding regular, bifurcational and chaotic dynamics of the investigated objects have been reported.

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