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Investigation of the stress-strain state of the laminated shallow shells by R-functions method combined with spline-approximation

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The bending behavior of the laminated shallow shells under static loading has been studied using the R-functions theory together with the spline-approximation. Formulation is based on the first order shear deformation theory. Due to usage of the R-functions theory the laminated shallow shells with complex shape and different types of the boundary conditions can be investigated. Application of the spline-approximation allows getting reliable and validated results for non concave domains and domains with holes. The proposed method is implemented in the appropriate software in framework of the mathematical package MAPLE. The analysis of influence of certain factors (curvature, packing of layers, geometrical parameters, boundary conditions) on a stress-strain state is carried out for shallow shells with cut-outs. The comparison of obtained results with those already known from literature and results obtained by using ANSYS are also presented.

1 Introduction

A wide range of applications of laminated shallow shells with an arbitrary shape in a modern industry causes necessity of an accurate investigation of the stress-strain state of such shells. The mathematical formulation of the problem is reduced to the solution of boundary value problems for systems of partial differential equations (PDEs). The complexity of the solution of such problems is caused by high orders of equilibrium equations, number of unknown functions, complex boundary conditions, suitable choice of fiber orientations, and other reasons.

The analysis of the literature had shown that in general case the investigation of a stress-strain state is carried out by numerical methods. The most commonly used methods are the finite element method (FEM) and finite difference method (FDM). The numerous variants of FEM were offered and approved [1–3]. According to this method a mesh of finite elements is generated over an examined domain to construct the approximating subspaces. For domains with complex forms the generation of FEM meshes leading to reliable results requires often a lot of time. Therefore, considerable efforts have been made to develop meshless methods.

R-functions method (RFM) is one of them. The efficiency of this method had been demonstrated on a wide class of mechanics problems including static and dynamic problems of plates and shallow shells [4–12]. The main idea of the RFM is the construction of the complete solution structures including indefinite components at any choice of which the boundary conditions will be satisfied exactly. As it had been shown by V. L. Rvachev [4], these indefinite components can be approximated by any complete system: power and trigonometric polynomials, splines, up-functions, polynomials of Chebyshev and Hermite, Legendre, and other systems. However an experience of RFM usage shows that the application of a polynomial approximation of the indefinite components leads to linear algebra problems with dense matrices. An increase in power of approximating polynomials and number of indefinite components causes bad conditioned matrices and further they generate unstable solutions regarding linear algebra problems. When multilayered shallow shells are studied, the number of unknown functions and hence the number of indefinite components in the solution structure can not be small, because the equilibrium equations contain three (classic theory) or five (refined theory) unknown functions. In addition, complex shapes or boundary conditions require increase of the polynomials power in order to control the convergence and validation of the obtained results. The usage of splines allows to avoid this problem and to get a better convergence of

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the results compared with a polynomial approximation. For the first time the combination of the R-functions theory and B-splines was used to investigate isotropic plate bending in monograph [4]. It should be emphasized that in the last decade the so-called Web-method, based on the joint of the FEM and web-splines got an active development. Main results related to some theoretical backgrounds of this approach and its applications were presented in monograph [13]. The R-functions theory was used in [13] and some tasks of the isotropic plates bending were solved by means of this method. In the present work we propose to use spline-approximation together with RFM to investigate a multilayered shallow shells bending.

2 Formulation

A laminated shallow shell with constant thickness, consisting of an arbitrary number of orthotropic layers with constant thicknesses is considered. It is assumed that the delamination between layers is absent. Due to shallowness of the shell the curvilinear coordinates commonly employed for shells can be directly replaced by the Cartesian coordinates x and y and the Lamé coefficients are put as $A = B = 1$. The investigation of bending of such a shell we will carry out using the first order refined theory, taking shear deformations in account. It is assumed that the transverse normal stress is negligible and normal to the reference surface of the shell before deformation remain straight, but not necessarily normal, after deformation (a relaxed Kirchhoff-Love's hypothesis).

Equilibrium equations of the shell subjected by the transverse load $q(x, y)$ only are presented in the following way [14]:

$$\begin{cases} N_{1,x} + N_{12,y} + k_1 Q_1 = 0, \\ N_{2,y} + N_{12,x} + k_2 Q_2 = 0, \\ Q_{1,x} + Q_{2,y} - k_1 N_1 - k_2 N_2 + q(x,y) = 0, \\ M_{1,x} + M_{12,y} - Q_1 = 0, \\ M_{2,y} + M_{12,x} - Q_2 = 0, \end{cases} \tag{1}$$

where system (1) is combined with corresponding boundary conditions. Internal forces and bending moments according to the first order refined theory are defined by the following relations:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} & K_{11} & K_{12} & K_{16} \\ C_{12} & C_{22} & C_{26} & K_{12} & K_{22} & K_{26} \\ C_{16} & C_{26} & C_{66} & K_{16} & K_{26} & K_{66} \\ K_{11} & K_{12} & K_{16} & D_{11} & D_{12} & D_{16} \\ K_{12} & K_{22} & K_{26} & D_{12} & D_{22} & D_{26} \\ K_{16} & K_{26} & K_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \\ \Psi_{1,x} \\ \Psi_{2,y} \\ \Psi_{1,y} + \Psi_{2,x} \end{bmatrix}.$$

Transverse forces are defined as:

$$Q_1 = k_4^2 C_{55} \varepsilon_{13} + k_5^2 C_{45} \varepsilon_{23}, \quad Q_2 = k_4^2 C_{45} \varepsilon_{13} + k_5^2 C_{44} \varepsilon_{23},$$

where k_4^2 and k_5^2 are shear correction factors.
Deformations ε_1, ε_2, ε_{12}, ε_{13}, ε_{23} are defined in the following way:

$$\varepsilon_1 = u_{,x} + k_1 w, \quad \varepsilon_2 = v_{,y} + k_2 w, \quad \varepsilon_{1,2} = u_{,y} + v_{,x},$$

$$\varepsilon_{13} = w_{,x} + \Psi_1 - k_1 u, \quad \varepsilon_{23} = w_{,y} + \Psi_2 - k_2 v,$$

where u and v are displacements of the midsurface, and Ψ_1, Ψ_2 are normal turning angles to the midsurface. Constants k_1, k_2 define midsurface curvatures of a shell. Coefficients C_{ij}, K_{ij}, D_{ij} are defined using characteristics of layers B_{ij}^l [14]:

$$C_{ij} = \sum_{l=1}^{S} \int_{h_{l-1}}^{h_l} B_{ij}^{(s)} dz, \quad (i,j) = \{1,2,6,4,5\},$$

$$K_{ij} = \sum_{l=1}^{S} \int_{h_{l-1}}^{h_l} B_{ij}^{(s)} z dz, \quad D_{ij} = \sum_{l=1}^{S} \int_{h_{l-1}}^{h_l} B_{ij}^{(s)} z^2 dz, \quad (i,j) = \{1,2,6\},$$

where $B_{ij}^{(s)}$ are stiffness coefficients of s-th layer, S denotes the number of layers.

3 The solution method

In order to study the bending of laminated shallow shells by RFM let us first formulate the variational statement of this problem. It is known that this formulation may be based on minimization of the Lagrange functional:

$$J = V - A. \tag{2}$$

Here V is a potential energy of deformation of a shell which is defined as

$$V = \frac{1}{2}\iint\limits_{\Omega}(N_1\varepsilon_1 + N_2\varepsilon_2 + N_{12}\varepsilon_{12} + M_1\Psi_{1,x} + M_2\Psi_{2,y} + M_{12}(\Psi_{1,y} + \Psi_{2,x}))\partial\Omega$$
$$+\frac{1}{2}\iint\limits_{\Omega}(Q_1\varepsilon_{13} + Q_2\varepsilon_{23})\partial\Omega$$

and A is a work of external forces. Taking only the transverse loading into account it can be presented as follows

$$A = \iint\limits_{\Omega} q(x,y)w\partial\Omega,$$

where $q(x,y)$ is an intensity of the external force and Ω is a domain, identical to the planform of a shell.

To minimize the functional (2) the Ritz method is used, where the basic functions are obtained by the RFM [4–5]. Therefore first it is needed to construct the solution structure of the boundary value problem. An analytical formula $U = B(P_1, P_2, \ldots, P_k, \omega, \omega_i)$ is called the solution structure, if it strictly satisfies all the given boundary conditions or at least the kinematical ones at any choice of indefinite functions $P_1, P_2, \ldots, P_k$ (components). Of course, this formula depends on the given boundary conditions and a shape of the given domain. For example, if the border is clamped then boundary conditions have the following form: $u = v = w = \Psi_1 = \Psi_2 = 0$. The solution structure in this case can be taken as:

$$u = \omega P_1, \quad v = \omega P_2, \quad w = \omega P_3, \quad \Psi_1 = \omega P_4, \quad \Psi_2 = \omega P_5, \tag{3}$$

where $\omega = 0$ is the equation of the clamped part of a border and $P_l, (l = \overline{1,5})$ are undefined components of the solution structure. It should be noted that the constructed solution structure may be used for domains with different shapes, and the function $\omega(x,y)$ plays the role of a parameter.

It should be noted that if boundary conditions are natural ones, for example, in case of the free edge $x = \mathrm{const}$, when boundary conditions have the following form: $N_1 = N_{12} = M_1 = M_{12} = Q_1$, then the simplest solution structure can be taken as: $u = P_1, v = P_2, w = P_3, \Psi_1 = P_4, \Psi_2 = P_5$. But if we have mixed boundary conditions, for example, combination of the clamped and free edge, then the corresponding solution structure may be taken in the form: $u = \omega_1 P_1, \quad v = \omega_1 P_2, \quad w = \omega_1 P_3, \quad \Psi_1 = \omega_1 P_4, \quad \Psi_2 = \omega_1 P_5$. Here $\omega_1 = 0$ is the equation only of a clamped part of the border.

Let us approximate the indefinite components $P_l, (l = \overline{1,5})$ of the solution structure by the Shoenberg spline functions of the 3-rd order [15]. To do that the investigated domain Ω is put into the rectangle $R = \{a \le x \le b, c \le y \le d\}$. The rectangle is then represented by a regular mesh with number of divisions N along Ox and M along Oy axes. The coordinates of the mesh nodes are defined as

$$x_i = a + i\frac{b-a}{N}, \quad y_j = c + j\frac{d-c}{M}, \quad (i = -1\ldots N+1, \quad j = -1\ldots M+1).$$

The indefinite components P_l are taken in the form

$$P_l = \sum_{i=-1}^{N+1}\sum_{j=-1}^{M+1} a_{ij}^{(l)}\Psi_{ij}(x,y), \tag{4}$$

where $\Psi_{ij}(x,y)$ is a two-dimensional B-spline obtained as a product of the single-dimensional ones with scaling and transition

$$\Psi_{ij}(x,y) = B_3\left(\frac{N(x-a)}{b-a} - i\right) B_3\left(\frac{M(y-c)}{d-c} - j\right). \tag{5}$$

The spline-function $B_3(t)$ in (5) is defined as

$$B_3(t) = \begin{cases} 0 & -\infty < t < -2 \\ 0.25(t+2)^3 & -2 < t \le -1 \\ -0,75t^3 - 1.5t^2 + 1 & -1 < t \le 0 \\ 0,75t^3 - 1.5t^2 + 1 & 0 < t \le 1 \\ -0.25t^3 + 1.5t^2 - 3t + 2 & 1 < t \le 2 \\ 0 & 2 < t < +\infty \end{cases}.$$

Basic functions are obtained by substituting (4) into a solution structure. For example for the clamped boundary (3) the basic functions have the following view:

$$u_{ij} = v_{ij} = w_{ij} = \psi_{1ij} = \psi_{2ij} = \omega \Psi_{ij}(x,y).$$

The undefined coefficients $a_{ij}^{(l)}$ in the decomposition (4) are obtained from the condition of the functional (2) minimum by the Ritz method. Thus the problem is reduced to the solution of an appropriate linear system of algebraic equations:

$$\frac{\partial J}{\partial a_k^{(l)}} = 0, \tag{6}$$

where $k \in [1; (M+3) \cdot (N+3)]$.

Obviously the obtained Ritz matrix will be sparse. The numerical realization of the proposed methods is carried out using analytical advantages of the mathematical package MAPLE and a number of developed programs written in C++, dealing with numerical methods of integration and solution of systems of linear algebraic equations. The numerical integration over an arbitrary domain is based on the Gauss quadrature. The Gauss formula is used for both B-splines in the interior of the domain and B-splines which are located near the boundary. The solution of systems of linear algebraic equations (6) is based on a bi-gradient iterative algorithm for sparse systems (BiCGStab) [16].

4 Numerical results

Example 1

As the first illustration of the RFM's application to the considered problems we will investigate a five-layers square plate (the length of a side equals to $2a$), with a symmetric structure. Suppose that the plate is subjected by a uniformly distributed load and is clamped over the border. Assume that all layers have the same thickness equal to $h/5$, provided that $h/(2a) = 0.1$. The mechanical properties of the layers are taken as follows: $E_1/E_2 = 25$; $G_{12}/E_2 = G_{13}/E_2 = 0.5$; $G_{23}/E_2 = 0.2$; $v_{12} = 0.25$. The lamina scheme is taken as the angle-ply $(\theta°, -\theta°, \theta°, -\theta°, \theta°)$, and θ will be put: 30°, 45°, 60°.

Let us investigate the convergence of the obtained results while increasing a mesh density. The obtained results we will compare with similar results, computed by the ANSYS package, based on FEM. First two types of layered elements utilizing the first-order refined theory are used: SHELL181 (linear element), SHELL281 (quadratic element).

The non-dimensional values of maximum deflections $\overline{w} = wh^3 E_2 10^3 / q_0 (2a)^4$ and stresses $(\overline{\sigma}_x, \overline{\sigma}_y) = (\sigma_x, \sigma_y)/(q_0(2a)^2)$ on external surfaces of shells are presented in Table 1 for different types of lamination scheme. As follows from the convergence of the presented results (see for example results for $\theta = 30°, 60°$) the linear element (SHELL181) gives poor predictions (compared with RFM and SHELL281) for stresses and needs higher levels of area decomposition; thus, for further investigations quadratic element (SHELL281) is used. For quadratic element the results obtained by FEM are slightly higher than appropriate results obtained by RFM. But the agreement of the results is enough good. The difference between them is from 1% up to 5.5%. The analysis of results presented in Table 1 shows that the maximum deflection is reached for the angle-ply lamination scheme with $\theta° = 45°$, whilst the minimum ones are obtained for angles $\theta° = 30°, 60°$.

Example 2

Five layered symmetrically laminated plates and shallow spherical shells with the cut-out (Fig. 1) subjected to a uniformly distributed load are investigated.

The geometrical parameters are: $a/b = 1; a_1/b_1 = 1; a_1/a = 0.3; h/(2a) = 0.1$. Mechanical parameters are taken the same as in Example 1. Two types of boundary conditions are considered: I-type (CC) – a shell clamped over all

Table 1 Maximum deflections $\overline{w}$ and stresses $\overline{\sigma}_x, \overline{\sigma}_y$ of the clamped five layered $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ laminated square plate.

Method	$\overline{w}_{max}$	$\overline{\sigma}_x^+(0;\alpha)\vert\overline{\sigma}_x^-(0;\alpha)$	$\overline{\sigma}_x^+(0;\alpha)\vert\overline{\sigma}_x^+(0;\alpha)$	$\overline{\sigma}_y^+(0;\alpha)\vert\overline{\sigma}_y^-(0;\alpha)$	$\overline{\sigma}_y^+(0;\alpha)\vert\overline{\sigma}_y^-(0;\alpha)$
			$\theta^\circ = 30^\circ, 60^\circ$		
Present4	4.763	$-6.783^+\vert 6.783^-$	$-9.348^+\vert 9.348^-$	$-22.81^+\vert 22.81^-$	$-8.755^+\vert 8.755^-$
Present10	4.706	$-7.861^+\vert 7.861^-$	$-10.65^+\vert 10.65^-$	$-26.04^+\vert 26.04^-$	$-12.79^+\vert 12.79^-$
Present20	4.702	$-7.795^+\vert 7.795^-$	$-10.59^+\vert 10.59^-$	$-25.81^+\vert 25.81^-$	$-12.86^+\vert 12.86^-$
Present40	4.701	$-7.795^+\vert 7.795^-$	$-10.59^+\vert 10.59^-$	$-25.80^+\vert 25.80^-$	$-12.87^+\vert 12.87^-$
ANSYS^{181}4	4.989	$-1.424^+\vert 1.424^-$	$-2.234^+\vert 2.234^-$	$-7.500^+\vert 7.500^-$	$-0.583^+\vert 0.583^-$
ANSYS^{181}10	4.819	$-5.433^+\vert 5.433^-$	$-6.425^+\vert 6.425^-$	$-18.56^+\vert 18.56^-$	$-7.537^+\vert 7.437^-$
ANSYS^{181}20	4.796	$-6.901^+\vert 6.901^-$	$-8.627^+\vert 8.627^-$	$-22.87^+\vert 22.87^-$	$-11.01^+\vert 11.01^-$
ANSYS^{181}40	4.791	$-7.647^+\vert 7.647^-$	$-9.877^+\vert 9.877^-$	$-25.06^+\vert 25.06^-$	$-12.91^+\vert 12.91^-$
ANSYS^{281}4	4.777	$-7.908^+\vert 7.908^-$	$-9.602^+\vert 9.602^-$	$-25.94^+\vert 25.94^-$	$-12.59^+\vert 12.59^-$
ANSYS^{281}10	4.788	$-8.386^+\vert 8.386^-$	$-10.98^+\vert 10.98^-$	$-27.24^+\vert 27.24^-$	$-14.69^+\vert 14.69^-$
ANSYS^{281}20	4.789	$-8.397^+\vert 8.397^-$	$-11.16^+\vert 11.16^-$	$-27.26^+\vert 27.26^-$	$-14.85^+\vert 14.85^-$
ANSYS^{281}40	4.789	$-8.396^+\vert 8.396^-$	$-11.20^+\vert 11.20^-$	$-27.26^+\vert 27.26^-$	$-14.87^+\vert 14.87^-$
			$\theta^\circ = 45^\circ$		
Present4	4.854	$-10.56^+\vert 10.56^-$	$-14.67^+\vert 14.67^-$	$-14.67^+\vert 14.67^-$	$-10.56^+\vert 10.56^-$
Present10	4.777	$-11.79^+\vert 11.79^-$	$-15.89^+\vert 15.89^-$	$-15.89^+\vert 15.89^-$	$-11.79^+\vert 11.79^-$
Present20	4.767	$-11.75^+\vert 11.75^-$	$-15.79^+\vert 15.79^-$	$-15.79^+\vert 15.79^-$	$-11.75^+\vert 11.75^-$
Present40	4.764	$-11.75^+\vert 11.75^-$	$-15.79^+\vert 15.79^-$	$-15.79^+\vert 15.79^-$	$-11.75^+\vert 11.75^-$
ANSYS^{281}4	4.728	$-11.63^+\vert 11.63^-$	$-15.11^+\vert 15.11^-$	$-15.11^+\vert 15.11^-$	$-11.63^+\vert 11.63^-$
ANSYS^{281}10	4.744	$-12.92^+\vert 12.92^-$	$-16.62^+\vert 16.62^-$	$-16.62^+\vert 16.62^-$	$-12.92^+\vert 12.92^-$
ANSYS^{281}20	4.746	$-13.01^+\vert 13.01^-$	$-16.74^+\vert 16.74^-$	$-16.74^+\vert 16.74^-$	$-13.01^+\vert 13.01^-$
ANSYS^{281}40	4.746	$-13.02^+\vert 13.02^-$	$-16.76^+\vert 16.76^-$	$-16.76^+\vert 16.76^-$	$-13.02^+\vert 13.02^-$

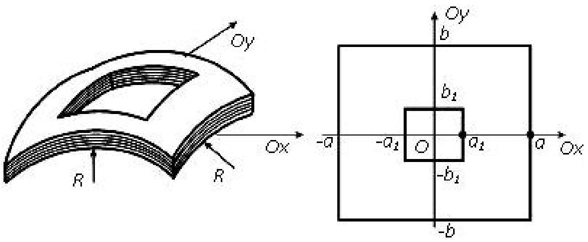


Fig. 1 A form of a shallow shell.

border and II-type (FC) – a shell clamped only on the internal contour and free on the external border. The solution structure for the first type of boundary condition is taken according to (3), where the function ω takes the following form: $\omega = (f_1 \wedge_0 f_2) \wedge_0 (f_3 \vee_0 f_4)$. In the second case the boundary conditions are natural, thus, due to usage of the Ritz method, it's possible to apply basic functions satisfying only the kinematical conditions. In this case the following solution structure can be applied:

$$u = \omega_1 P_1, \quad v = \omega_1 P_2, \quad w = \omega_1 P_3, \quad \Psi_1 = \omega_1 P_4, \quad \Psi_2 = \omega_1 P_5,$$

where $\omega_1 = (f_3 \vee_0 f_4)$. Here functions $f_i, i = 1..4$ are defined as follows:

$$f_1 = \frac{1}{2a}(b^2 - y^2) \geq 0, \quad f_2 = \frac{1}{2a}(a^2 - x^2) \geq 0, \quad f_3 = \frac{1}{b_1}(y^2 - b_1^2) \geq 0,$$

$$f_4 = \frac{1}{2a_1}(x^2 - a_1^2) \geq 0.$$

The signs $\wedge_0$ and $\vee_0$ denote R-conjunction and R-disjunction [4], respectively.

The appropriate quantity of basic functions corresponding to the used meshes and the number of linear algebraic equations in (6) for both RFM and FEM are presented in Table 2 (dimensions for shells are marked in the italics).

Table 2 Meshes and corresponding dimensions of a problem.

Used meshes		Dimension of a problem
Present	20×20	2645/*2645*
	40×40	9245/*9245*
	60×60	19845/*19845*
ANSYS[281]	20	5928/*7338*
	40	24960/*27210*
	60	57096/*60144*

It should be noted that results obtained by RFM combined with both the polynomial and the spline approximation are almost the same for plates with a square planform. But if we deal with concave domains or domains with cut-outs the spline-approximation allows getting more stable results convergence. The values of the maximum deflections and stresses for plates with boundary condition of the I-type (CC) are presented in Table 3.

Table 3 Maximum deflections $\overline{w}$ and stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ of the clamped (CC) laminated $(\theta°, -\theta°, \theta°, -\theta°, \theta°)$ plates (Fig. 1).

Method	$\overline{w}_{max}$	$\overline{\sigma}_x^+(\alpha_1;0)\vert\overline{\sigma}_x^-(\alpha_1;0)$	$\overline{\sigma}_x^+(\alpha;0)\vert\overline{\sigma}_x^+(\alpha;0)$	$\overline{\sigma}_y^+(\alpha_1;0)\vert\overline{\sigma}_y^-(\alpha_1;0)$	$\overline{\sigma}_y^+(\alpha;0)\vert\overline{\sigma}_y^-(\alpha;0)$
			$\theta^0 = 30°, 60°$		
Present20	0.916	$-5.154^+\vert 5.154^-$	$-5.070^+\vert 5.070^-$	$-6.438^+\vert 6.438^-$	$-6.407^+\vert 6.407^-$
Present40	0.919	$-5.222^+\vert 5.222^-$	$-5.079^+\vert 5.079^-$	$-6.494^+\vert 6.494^-$	$-6.420^+\vert 6.420^-$
Present60	0.920	$-5.231^+\vert 5.231^-$	$-5.081^+\vert 5.081^-$	$-6.506^+\vert 6.506^-$	$-6.423^+\vert 6.423^-$
ANSYS[281]20	0.944	$-5.418^+\vert 5.418^-$	$-5.432^+\vert 5.432^-$	$-7.564^+\vert 7.564^-$	$-7.823^+\vert 7.823^-$
ANSYS[281]40	0.948	$-5.509^+\vert 5.509^-$	$-5.498^+\vert 5.498^-$	$-7.655^+\vert 7.655^-$	$-7.901^+\vert 7.901^-$
ANSYS[281]60	0.948	$-5.528^+\vert 5.528^-$	$-5.511^+\vert 5.511^-$	$-7.674^+\vert 7.674^-$	$-7.916^+\vert 7.916^-$
			$\theta° = 45°$		
Present20	0.779	$-5.201^+\vert 5.201^-$	$-5.511^+\vert 5.511^-$	$-3.828^+\vert 3.828^-$	$-4.117^+\vert 4.117^-$
Present40	0.780	$-5.109^+\vert 5.109^-$	$-5.518^+\vert 5.518^-$	$-3.755^+\vert 3.755^-$	$-4.123^+\vert 4.123^-$
Present60	0.781	$-5.123^+\vert 5.123^-$	$-5.520^+\vert 5.520^-$	$-3.765^+\vert 3.765^-$	$-4.125^+\vert 4.125^-$
ANSYS[281]20	0.831	$-5.459^+\vert 5.459^-$	$-6.218^+\vert 6.218^-$	$-4.233^+\vert 4.233^-$	$-4.945^+\vert 4.945^-$
ANSYS[281]40	0.836	$-5.545^+\vert 5.545^-$	$-6.276^+\vert 6.276^-$	$-4.291^+\vert 4.291^-$	$-4.988^+\vert 4.988^-$
ANSYS[281]60	0.838	$-5.561^+\vert 5.561^-$	$-6.288^+\vert 6.288^-$	$-4.302^+\vert 4.302^-$	$-4.996^+\vert 4.996^-$

The comparison of the presented results with appropriate results, computed by ANSYS FEM package is also carried out. The analysis of the comparison confirms the verification of the presented results. But it should be noted that the convergence of results for stresses obtained by the RFM is better than the convergence of similar results obtained by FEM.

In Table 4 the comparison of the obtained results with similar ones obtained by FEM (ANSYS) for plates with boundary condition of the II-type (FC) is presented.

The results for shallow spherical shells with different types of boundary conditions and lamina schemes are presented in Tables 5 and 6. Let us put the dimensionless value of the shells curvature equal to 0.2, that is $\frac{R}{2a}$ or $k = 0.2/(2a)$. The convergence of obtained results with an increase in number of partitions of the source region had been studied for both approaches. It should be noted that the convergence of the FEM for shallow shells is worse than for plates. It is observed in the second significant digit for deflections (I and II type of the boundary conditions) and stresses (only I type of the boundary conditions). The convergence of the stresses for the second type of the boundary conditions (FC) is observed only in the first significant digit. One can also observe, that the deflections obtained with the FEM for $\theta = 30°$ and $\theta = 60°$ are different in spite of the symmetry.

The presented convergence for shallow shells shows high levels of disagreement, especially for II-type (FC) of the boundary condition (Table 6). The reason for that in our opinion is the combined effect of curvature and boundary conditions. It should be noted that the dimension of a problem using ANSYS[281] for shallow shells is up to 3 times higher than one using RFM (Table 2), which leads to higher levels of CPU time and memory requirements.

Table 4 Maximum deflections $\overline{w}$ and stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ of the laminated $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ plates with boundary conditions of the second type (FC).

Method	$\overline{w}_{max}$	$\overline{\sigma}_x^+(\alpha_1;0) \vert \overline{\sigma}_x^-(\alpha_1;0)$	$\overline{\sigma}_x^+(\alpha;0) \vert \overline{\sigma}_x^+(\alpha;0)$	$\overline{\sigma}_y^+(\alpha_1;0) \vert \overline{\sigma}_y^-(\alpha_1;0)$	$\overline{\sigma}_y^+(\alpha;0) \vert \overline{\sigma}_y^-(\alpha;0)$
			$\theta^\circ = 30^\circ, 60^\circ$		
Present60	27.21	$-59.94^+ \vert 59.94^-$	$0.363^+ \vert -0.363^-$	$-83.79^+ \vert 83.79^-$	$-3.055^+ \vert 3.055^-$
ANSYS[281]60	27.63	$-61.22^+ \vert 61.22^-$	$0.400^+ \vert -0.400^-$	$-88.72^+ \vert 88.72^-$	$-3.116^+ \vert 3.116^-$
			$\theta^\circ = 45^\circ$		
Present60	19.44	$-51.42^+ \vert 51.42^-$	$0.918^+ \vert -0.918^-$	$-39.25^+ \vert 39.25^-$	$-5.365^+ \vert 5.365^-$
ANSYS[281]60	19.85	$-53.20^+ \vert 53.20^-$	$0.950^+ \vert -0.950^-$	$-41.27^+ \vert 41.27^-$	$-5.333^+ \vert 5.333^-$

Table 5 Maximum deflections $\overline{w}$ and stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ of the clamped (CC) laminated $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ spherical shells shown in Fig. 1.

Method	$\overline{w}_{max}$	$\overline{\sigma}_x^+(\alpha_1;0) \vert \overline{\sigma}_x^-(\alpha_1;0)$	$\overline{\sigma}_x^+(\alpha;0) \vert \overline{\sigma}_x^+(\alpha;0)$	$\overline{\sigma}_y^+(\alpha_1;0) \vert \overline{\sigma}_y^-(\alpha_1;0)$	$\overline{\sigma}_y^+(\alpha;0) \vert \overline{\sigma}_y^-(\alpha;0)$
			$\theta^\circ = 60^\circ$		
Present20	0.863	$-4.150^+ \vert 5.586^-$	$-4.054^+ \vert 5.523^-$	$-4.755^+ \vert 7.411^-$	$-4.702^+ \vert 7.407^-$
Present40	0.865	$-4.207^+ \vert 5.656^-$	$-4.061^+ \vert 5.532^-$	$-4.795^+ \vert 7.474^-$	$-4.713^+ \vert 7.420^-$
Present60	0.866	$-4.213^+ \vert 5.665^-$	$-4.063^+ \vert 5.534^-$	$-4.803^+ \vert 7.489^-$	$-4.715^+ \vert 7.424^-$
ANSYS[281]20	0.952	$-6.407^+ \vert 8.089^-$	$-5.361^+ \vert 6.934^-$	$-4.619^+ \vert 6.099^-$	$-7.076^+ \vert 9.403^-$
ANSYS[281]40	0.966	$-5.455^+ \vert 6.396^-$	$-5.503^+ \vert 6.913^-$	$-4.217^+ \vert 5.309^-$	$-7.451^+ \vert 9.490^-$
ANSYS[281]60	0.972	$-6.509^+ \vert 7.971^-$	$-5.150^+ \vert 6.323^-$	$-6.023^+ \vert 7.876^-$	$-7.880^+ \vert 9.794^-$
			$\theta^\circ = 45^\circ$		
Present20	0.751	$-3.784^+ \vert 6.118^-$	$-3.902^+ \vert 6.583^-$	$-2.638^+ \vert 4.652^-$	$-2.761^+ \vert 5.071^-$
Present40	0.752	$-3.723^+ \vert 6.003^-$	$-3.905^+ \vert 6.590^-$	$-2.590^+ \vert 4.558^-$	$-2.764^+ \vert 5.077^-$
Present60	0.753	$-3.733^+ \vert 6.020^-$	$-3.907^+ \vert 6.592^-$	$-2.597^+ \vert 4.570^-$	$-2.765^+ \vert 5.079^-$
ANSYS[281]20	0.767	$-5.693^+ \vert 8.982^-$	$-5.459^+ \vert 7.957^-$	$-1.867^+ \vert 3.005^-$	$-3.691^+ \vert 5.459^-$
ANSYS[281]40	0.779	$-4.799^+ \vert 7.310^-$	$-5.122^+ \vert 7.736^-$	$-2.382^+ \vert 3.777^-$	$-3.515^+ \vert 5.424^-$
ANSYS[281]60	0.774	$-5.292^+ \vert 8.241^-$	$-4.857^+ \vert 7.110^-$	$-2.675^+ \vert 4.308^-$	$-3.853^+ \vert 5.751^-$
			$\theta^\circ = 30^\circ$		
Present20	0.864	$-4.099^+ \vert 6.979^-$	$-4.309^+ \vert 7.751^-$	$-1.188^+ \vert 2.107^-$	$-1.266^+ \vert 2.365^-$
Present40	0.866	$-3.483^+ \vert 6.119^-$	$-4.346^+ \vert 7.811^-$	$-0.998^+ \vert 1.839^-$	$-1.277^+ \vert 2.385^-$
Present60	0.866	$-3.605^+ \vert 6.285^-$	$-4.353^+ \vert 7.825^-$	$-1.035^+ \vert 1.891^-$	$-1.280^+ \vert 2.389^-$
ANSYS[281]20	0.708	$-8.236^+ \vert 8.839^-$	$-5.101^+ \vert 8.275^-$	$-0.766^+ \vert 0.799^-$	$-1.269^+ \vert 2.085^-$
ANSYS[281]40	0.726	$-3.892^+ \vert 9.308^-$	$-6.293^+ \vert 8.016^-$	$-0.657^+ \vert 1.686^-$	$-1.625^+ \vert 2.080^-$
ANSYS[281]60	0.723	$-4.443^+ \vert 7.437^-$	$-5.515^+ \vert 7.575^-$	$-0.771^+ \vert 1.321^-$	$-1.698^+ \vert 2.353^-$

The behavior of deflections $\overline{w}$ and stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ in the section $y = 0$ for different values $\theta = 30^\circ$, $\theta = 45^\circ$, $\theta = 60^\circ$ is presented in Fig. 2.

Let us analyze the effects of geometrical factors and lamina scheme on the stress-strain state. The effect of curvature is observed, when we compare results in Tables 1, 3–6 for plates and shells. The maximum deflections for all types of boundary conditions and lamina schemes decrease at increasing curvature. Nevertheless, it is more obvious for the case of CC shells. For FC shells the change of curvature in the examined range $k \in [0; 0.2/(2a)]$ does not make essential influence on deflections. The increase in the curvature decreases values of stresses on a clamped border only for CC shells, but slightly increases for FC shells. The maximum values of deflections of the shells are observed for angle-ply $(30^\circ, -30^\circ, 30^\circ, -30^\circ, 30^\circ)$ and $(60^\circ, -60^\circ, 60^\circ, -60^\circ, 60^\circ)$ for different boundary conditions.

The extreme values of stresses in the section $y = 0$ are reached on the clamped border and have biggest values for $\theta = 60^\circ$. The smallest values are reached for $\theta = 30^\circ$. For example at the point $x/(2a) = 0.15, y = 0$ the absolute value of $\overline{\sigma}_y^+$ for $\theta = 60^\circ$ is 8 times higher than for $\theta = 30^\circ$. Thus, one can choose the angle stacking layers using the proposed calculations.

Table 6 Maximum deflections $\overline{w}$ and stresses $\overline{\sigma}_x, \overline{\sigma}_y$ of the laminated $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ spherical shells with boundary conditions of the second type (FC), see Fig. 1.

Method	$\overline{w}_{max}$	$\overline{\sigma}_x^+(\alpha_1;0)\vert\overline{\sigma}_x^-(\alpha_1;0)$	$\overline{\sigma}_x^+(\alpha;0)\vert\overline{\sigma}_x^+(\alpha;0)$	$\overline{\sigma}_y^+(\alpha_1;0)\vert\overline{\sigma}_y^-(\alpha_1;0)$	$\overline{\sigma}_y^+(\alpha;0)\vert\overline{\sigma}_y^-(\alpha;0)$
			$\theta^\circ = 60^\circ$		
Present20	26.21	$-60.76^+\vert 54.50^-$	$0.166^+\vert -0.545^-$	$-86.52^+\vert 75.78^-$	$0.598^+\vert 7.102^-$
Present40	26.39	$-61.36^+\vert 55.03^-$	$0.182^+\vert -0.561^-$	$-86.67^+\vert 75.85^-$	$0.774^+\vert 6.972^-$
Present60	26.43	$-61.61^+\vert 55.26^-$	$0.175^+\vert -0.555^-$	$-87.11^+\vert 76.27^-$	$0.808^+\vert 6.949^-$
ANSYS[28]20	16.12	$-69.26^+\vert 63.18^-$	$1.079^+\vert -1.394^-$	$-52.51^+\vert 49.18^-$	$3.654^+\vert 15.02^-$
ANSYS[28]40	16.00	$-49.77^+\vert 43.87^-$	$1.023^+\vert -1.449^-$	$-37.59^+\vert 32.88^-$	$4.059^+\vert 16.59^-$
ANSYS[28]60	16.46	$-64.62^+\vert 58.14^-$	$1.156^+\vert -1.803^-$	$-61.62^+\vert 55.65^-$	$5.593^+\vert 19.25^-$
			$\theta^\circ = 45^\circ$		
Present20	19.02	$-52.45^+\vert 49.59^-$	$0.716^+\vert -1.121^-$	$-40.19^+\vert 37.79^-$	$-2.613^+\vert 7.864^-$
Present40	19.08	$-51.84^+\vert 49.05^-$	$0.704^+\vert -1.104^-$	$-39.67^+\vert 37.34^-$	$-2.638^+\vert 7.908^-$
Present60	19.09	$-51.98^+\vert 49.19^-$	$0.703^+\vert -1.103^-$	$-39.78^+\vert 37.45^-$	$-2.641^+\vert 7.914^-$
ANSYS[28]20	15.06	$-87.20^+\vert 84.12^-$	$0.877^+\vert -1.352^-$	$-29.16^+\vert 28.16^-$	$-1.930^+\vert 7.230^-$
ANSYS[28]40	15.45	$-64.99^+\vert 60.96^-$	$0.739^+\vert -1.209^-$	$-32.26^+\vert 29.99^-$	$-1.882^+\vert 8.151^-$
ANSYS[28]60	15.41	$-80.73^+\vert 76.14^-$	$0.774^+\vert -1.183^-$	$-41.93^+\vert 39.43^-$	$-2.863^+\vert 8.283^-$
			$\theta^\circ = 30^\circ$		
Present20	26.22	$-45.63^+\vert 46.99^-$	$0.814^+\vert -1.167^-$	$-13.75^+\vert 14.19^-$	$-6.443^+\vert 14.29^-$
Present40	26.39	$-33.36^+\vert 34.20^-$	$0.792^+\vert -1.138^-$	$-9.878^+\vert 10.15^-$	$-6.545^+\vert 14.48^-$
Present60	26.43	$-35.67^+\vert 36.58^-$	$0.787^+\vert -1.133^-$	$-10.61^+\vert 10.90^-$	$-6.563^+\vert 14.52^-$
ANSYS[28]20	18.37	$-52.29^+\vert 50.43^-$	$1.230^+\vert -1.543^-$	$-4.452^+\vert 4.269^-$	$-1.402^+\vert 6.535^-$
ANSYS[28]40	20.39	$-120.3^+\vert 115.6^-$	$0.479^+\vert -0.986^-$	$-22.04^+\vert 21.09^-$	$-2.653^+\vert 8.581^-$
ANSYS[28]60	20.38	$-85.53^+\vert 79.96^-$	$0.442^+\vert -0.669^-$	$-15.46^+\vert 14.43^-$	$-3.005^+\vert 8.440^-$

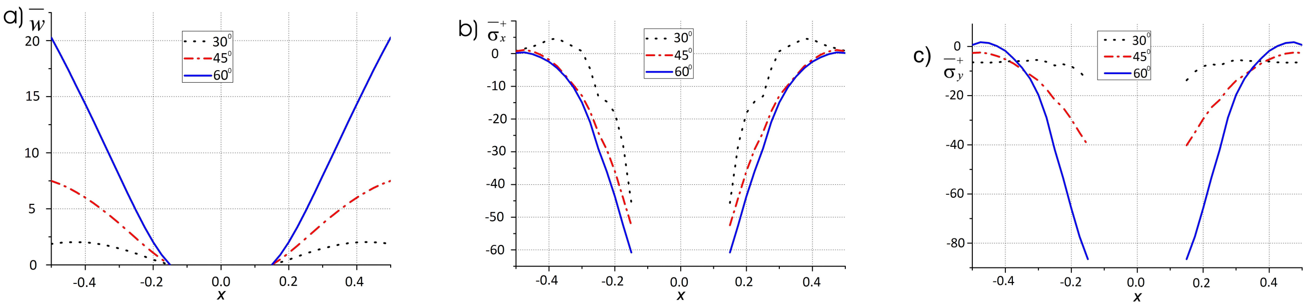


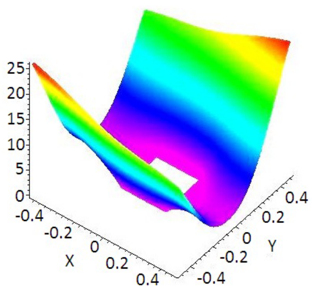
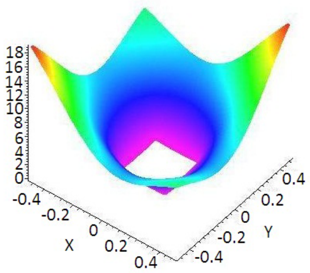
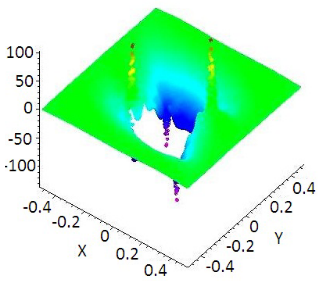
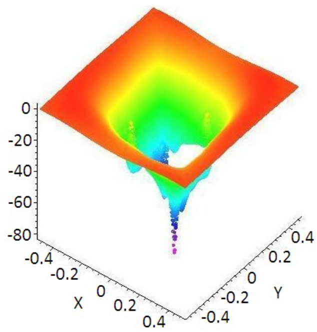
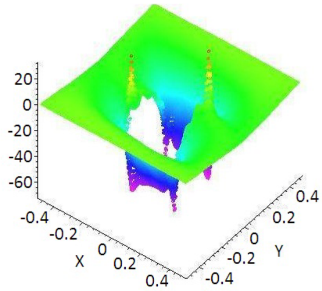
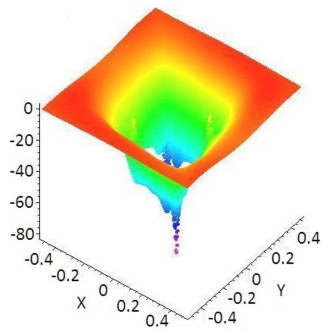
Fig. 2 (online colour at: www.zamm-journal.org) Values of deflections $\overline{w}$ and stresses $\overline{\sigma}_x, \overline{\sigma}_y$ of a five-layers $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ shallow spherical shell ($k = 0.2a$) in the section $y = 0$.

Results presented in Table 7 are the plots of deflections and stresses of the clamped (FC) five layered $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ shallow spherical shell ($k = 0.2a$) with different angles of lamination.
The concentration of stresses is observed on a clamped part of a border, which confirm our initial choice of the examined points in the Tables 3–6.

5 Conclusions

In the present work an effective numerically-analytical method of investigation of the stress-strain state of shallow laminated shells and plates with noncanonical shape and different boundary conditions is proposed. This approach is based on combined application of the varionational Ritz method, R-functions theory, and spline-approximation. Influence of mechanical and geometrical parameters on a static response of shallow laminated shells and plates with square hole is studied. A comparison of the present results with appropriate results, obtained by FEM, including the convergence study for each

Table 7 (online colour at: www.zamm-journal.org) Deflections and stresses of a five layered $(\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ, \theta^\circ)$ shallow spherical shell, clamped on the internal contour and free on the external boundary $(k = 0.2a)$ with different angles of lamination.

θ	30°	45°
$\overline{w}$		
$\overline{\sigma}_x^+$		
$\overline{\sigma}_y^+$		

method (FEM and RFM) is presented. It is observed that for shallow shells the convergence of FEM results is unstable, while RFM provides stable results for both plates and shells. The investigation of dynamic behavior of laminated shallow shells with complex shape is to be examined further.

References

[1] I. Kreja, R. Schmidt, and J. N. Reddy, Finite elements based on a first-order shear deformation moderate rotation theory with applications to the analysis of composite structures, Int. J. Non-Linear Mech. **32**, 1123–1142 (1997).
[2] H. Altenbach, Theories for laminated and sandwich plates (a review), Mech. Compos. Mater. **34**(3), 243–252 (1998).
[3] O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu, The Finite Element Method: Its Basis and Fundamentals (Butterworth-Heinemann, Amsterdam, 2005) p. 752.
[4] V. L. Rvachev and L. V. Kurpa, R-Functions in Problems of the Theory of Plates, in Russian (Naukova Dumka, Kiev, 1987).
[5] L. V. Kurpa, R-functions Method for Linear Bending and Vibration of Shallow Shells Problems Solution, in Russian (National Technical University of Ukraine Kharkov, Polytechnic Institute, Kharkov, 2009).
[6] L. V. Kurpa, K. I. Lyubitska, and A. V. Shmatko, Solution of vibration problems for shallow shells of arbitrary form by the R-function method, J. Sound Vib. **279**, 1071–1084 (2005).

[7] L. V. Kurpa and A. A. Osetrov, Study of free vibrations of shallow shells using R-functions method and spline-approximation, in Russian, Math. Methods Physicomechanical Fields (UAN) **50**, 83–93 (2007).
[8] L. V. Kurpa, G. Pilgun, and M. Amabili, Nonlinear vibrations of shallow shells with complex boundary: R-functions method and experiments, J. Sound Vib. **306**, 580–600 (2007).
[9] L. V. Kurpa, T. V. Shmatko, and O. G. Onufrienko, Research of nonlinear vibrations of orthotropic plates with a complex form, Math. Probl. Eng. **2006**, 125–138 (2005).
[10] V. L. Rvachev and T. I. Sheiko, R-functions in boundary value problems in mechanics, Appl. Mech. Rev. **48**, 155–185 (1995).
[11] J. Awrejcewicz, L. V. Kurpa, and O. S. Mazur, Proceedings of the 9th Conference on Dynamical Systems – Theory and Applications, Lodz, Poland, Stability investigation of nonlinear vibrations of plates by R-functions method, edited by J. Awrejcewicz, P. Olejnik, and J. Mrozowski Lodz, 2007, pp. 961–970.
[12] J. Awrejcewicz, L. V. Kurpa, and O. S. Mazur, Research of Stability and Nonlinear Vibrations by R-functions Method, in: Modeling, Simulation and Control of Nonlinear Engineering Dynamical System. State-of-the-Art, Perspectives and Applications, edited by J. Awrejcewicz (Springer, Berlin, Heidelberg, 2009) pp. 179–190.
[13] K. Hollig, U. Reif, and J. Wipper, Weighted extended B-spline approximation of Dirichlet problems, SIAM J. Numer. Anal. **39**, 442–462 (2001).
[14] S. A. Ambartsumian, General Theory of Anisotropic Shells (Nauka, Moscow, 1974).
[15] J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, The Theory of Splines and Their Applications (Academic Press, New York, 1967).
[16] Y. Saad, Iterative Methods for Sparse Linear Systems (Society for Industrial and Applied Mathematics, Philadelphia, 2002).

Book Review

J. Chakrabarty, Applied Plasticity, 2nd Ed. – Mechanical Engineering Series (Ed. by Frederick F. Ling) Springer, New York, Dordrecht, Heidelberg, London 2010, 776 pp., 225 figs, 3 tabs., Hardcover, EUR 79.95 (net price), CHF 115.00, USD 99.00, GBP 71.99, ISBN 978-0-387-77673-6

Several years ago (1987) the author published his famous *Theory of Plasticity* – in that time one of the best textbooks for people which are more interested in applications. Many simple examples, discussions concerning problems coming from the practice, etc. were included and the textbook becomes very popular.

The new Springer textbook is named *Applied Plasticity*. In this sense it is an excellent continuation of the previous books of the author. It must be underlined that new theoretical developments are not highlighted since they are often far from the needs of practice. On the other hand the book reflects the basic knowledge of an Advanced Course in the transition of BSc to MSc programs.

The book is split into 9 chapters

– Fundamental Principles
– Problems in Plane Stress
– Axisymmetric and Related Problems
– Plastic Bending of Plates
– Plastic Analysis of Shells
– Plastic Anisotropy
– Plastic Buckling
– Dynamic Plasticity
– The Finite Element Method

Each chapter is added by problems – but there are no solutions. The last one is a disadvantage since for students the "self-education" is necessary and complete solutions can be very helpful.

For further developments (new editions, etc.) is will be nice to include as a minimum a literature survey containing more information on new developments in the theory of plasticity, especially within theoretical frameworks (Continuum Plasticity, Crystal Plasticity, Multi-Scale Approaches among others).

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