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RESEARCH PAPER

Analytical study of the interface in fibre-reinforced 2D composite material

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Abstract Imperfect bonding between the constitutive components can greatly affect the properties of the composite structures. An asymptotic analysis of different types of imperfect interfaces arising in the problem of 2D fibre-reinforced composite materials are proposed. The performed study is based on the asymptotic reduction of the governing biharmonic problem into two harmonic problems. All solutions are obtained in a closed analytical form. The obtained results can be used for the calculation of pull-out and push-out tests, as well as for the investigation of the fracture of composite materials.

Keywords Fibre composite material · Asymptotic approach · Weak interface

1 Introduction

Thin coatings at the interfaces of the constituents of a composite material can make a substantial difference in the functional characteristics and reliability of composites. The optimum use of stiffness and strength properties of composites directly depends on the effectiveness of the transfer of load from the inclusions to the matrix, proceeding through the coatings. Furthermore, in the heterogeneous materials the greatest concentrations of local stresses occur, as a rule, on the interfaces between the constituents and, thus, the strength of coatings is one of the key factors, determining the load bearing capacity of composite as a whole. The fracture of coatings leads to the development of dislocations and cracks, which in the majority of the cases gives rise to the rapid destruction of entire material.

The problems of analysis of the composites with the coatings were examined by many authors, see, e.g., Achenbach and Zhu [1], Chen and Liu [2], Hashin [3], Jasiuk and Kouider [4], Lagache et al. [5], Lucas da Silva et al. [6,7] and Manevitch [8]. The analysis of the limiting cases of soft and rigid coatings is given by Benveniste and Miloh [9].

It should be noted that the interaction between the neighbouring fibres can cause the significant variation of physical fields in the composite on the microlevel. Increase in the rigidity of fibers and their volume fraction (i.e., the decrease of distances between the neighbouring fibres) leads to an increase in the local stresses on the interface of constituents. In this case the application of many known analysis methods can be limited by the difficulties of computational nature. Thus, analytical approaches based on representing stress fields in the form of expansions in various infinite series, can experience a deficiency in the convergence. Numerical methods require an increase in the mesh density and, accordingly, a significant increase in computing time (Mishuris and Öchsner [10]). Mentioned difficulties justify the introduction of a model of the interface which simplifies the computation of the solution and furnishes a good approximation. The interface between fibre and matrix can play an

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important role in determining the properties of the composite material. The effect of the lost interface stiffness is very often one of the reasons for the composite fracture.

Usually, stresses are continuous across the interface, while the displacements may be continuous or discontinuous. In the former case the interface is called "strong", whereas in the latter case it is called "weak". A weak interface described by the spring-layer model is dealt with, which assumes that the interfacial stress is a function of the gap in the displacements. This model was initially proposed by Golland and Reissner [11]. Asymptotic justification of spring-layer model was proposed by many authors, e.g., Geymonat et al. [12], Klarbring [13], Krasucki and Lenci [14,15], Lenci [16]. As a rule they dealt with infinite domains, but for real composite materials it is very important to monitor behaviour near the boundaries.

The main aim of the present work is to propose a new asymptotic technique for the calculation of properties of fibre composites with interface. Obtained approximate analytical expressions deserve an interest in practical applications, because they allow to detect the effects of the loss of interface stiffness due to a damage or a defect.

The used asymptotic techniques are described briefly. Problems involving anisotropy in general are more difficult to solve than the isotropic ones. On the other hand, as it has been shown by Kosmodamianskii [17], and independently by Manevitch et al. [8] and by Everstine and Pipkin [18] (see also Spencer [19], and Christensen [20]), in the elastic case the strong anisotropy may allow to construct solutions with the help of an asymptotic approach using as a small parameter ε as the ratio of rigidities in the different directions. A special asymptotic technique using expansions with respect to ε gives a possibility to reduce the input biharmonic boundary value problem of the generalized plane stress problem to two harmonic boundary value problems (Manevitch and Pavlenko [8,21,22]). It has also been shown that even in the isotropic case the error involved in the first approximation is rather low.

The paper is organized as follows. In Sect. 2 the governing relations used in the sequel are described. The problem for a single fibre perfectly bonded with matrix is solved in Sect. 3. In Sect. 4 the case of spring-layer model is investigated. An improved model of fibre, matrix and interface interaction is proposed in Sect. 5. Finally, the obtained results are discussed briefly in Sect. 6.

2 Governing relations

Let us briefly discuss the asymptotic approach proposed by Manevitch and Pavlenko [21,22] (see also Andrianov et al. [23]). The generalized plane stress problem for an isotropic matrix (Fig. 1) can be written as follows

$$\begin{aligned} &E^*hV_{yy} + GhV_{xx} + (\nu E^* + G)hU_{xy} = 0, \\ &E^*hV_{xx} + GhU_{yy} + (\nu E^* + G)hV_{xy} = 0, \end{aligned} \qquad (1)$$

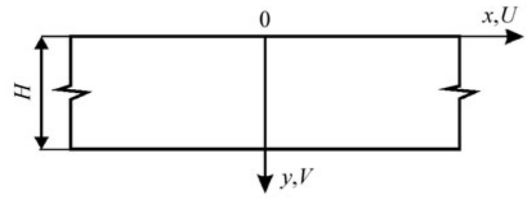

Fig. 1 Isotropic matrix under consideration

where U and V are the displacements in x and y directions, respectively; $E^* = E(1-\nu^2)^{-1}$, h, E, G and ν are the thickness of the matrix, the Young's modulus, the shear modulus and the Poisson's ratio of the matrix, respectively, and subscripts x and y denote the corresponding partial derivatives.

At $y = 0, H$, the following boundary conditions (BC) are assumed

$$T_1 = T_1^{(0)}, \quad T_{12} = T_{12}^{(0)}, \qquad (2)$$

where $T_1^{(0)}(x)$, $T_{12}^{(0)}(x)$ are the boundary normal and tangential forces, respectively,

$$T_1 = E^*hV_y + \nu E^*hU_x, \quad T_{12} = Gh(V_x + U_y). \qquad (3)$$

In the first approximation boundary value problems (BVPs) (1)–(3) can be reduced to the following simplified form

$$E^*hV_{yy} + GhV_{xx} = 0, \qquad (4)$$

$$E^*hU_{xx} + GhU_{yy} = 0, \qquad (5)$$

$$T_1 = E^*hV_y, \quad T_2 = E^*hU_x, \quad T_{12} = Gh(V_x + U_y), \qquad (6)$$

$$V_y = T_1^{(0)}/(E^*h), \quad \text{at} \quad y = 0, H, \qquad (7)$$

$$V_x + U_y = T_{12}^{(0)}/Gh, \quad \text{at} \quad y = 0, H. \qquad (8)$$

The quantity $0.5(1-\nu)$ is used as a small parameter here (Andrianov et al. [24]).

Now the error of the proposed approximate solution is estimated roughly. Variational formulation of the generalized plane stress problem equations (1)–(3) can be reduced to the searching of the stationary point of the Lagrange functional $\varphi(\boldsymbol{u})$ (Slivker [25])

$$\varphi(\boldsymbol{u}) = \frac{Eh}{2(1-\nu^2)} \int\limits_{\Omega} \left[U_x^2 + V_y^2 + 2\nu U_x V_y + \frac{1-\nu}{2}(U_y + V_x)^2 \right] \mathrm{d}\Omega - \int_{\Gamma} \left(VT_1^{(1)} + UT_{12}^{(1)} \right) \mathrm{d}s, \qquad (9)$$

where Ω, Γ denote domains occupied by the governing body and their boundary, respectively, and $\boldsymbol{u}$ is the displacement vector.

Lagrange functional for approximate BVPs (4)–(8) follows

$$\varphi_1(\boldsymbol{u}) = \frac{Eh}{2(1-\nu^2)} \int_{\Omega} \left[U_x^2 + V_y^2 + \frac{1-\nu}{2}(U_y + V_x)^2 \right] \mathrm{d}\Omega$$

$$-\int_{\Gamma}\left(VT_1^{(1)} + UT_{12}^{(1)}\right)\mathrm{d}s. \quad (10)$$

From Eqs. (9) and (10), one obtains

$$\Delta\varphi(\boldsymbol{u}) \equiv \varphi(\boldsymbol{u}) - \varphi_1(\boldsymbol{u})\Delta\mathrm{L}(\boldsymbol{u})$$
$$= \frac{Eh}{2(1-\nu^2)}\int_{\Omega}[2\nu U_x V_y + (1-\nu)U_y V_x]\mathrm{d}\Omega. \quad (11)$$

Equation (11) can be used for the estimation of the approximate solution accuracy. For free edges functions U_y and V_x have the opposite signs near the boundaries, and hence $\Delta\varphi(\boldsymbol{u}) < 0$. At the stationary point, one has $\varphi(\boldsymbol{u}) = -\Pi(\boldsymbol{u})/2$, where $\Pi(\boldsymbol{u})$ is the potential energy of deformation. So, the potential energy of deformation of the approximate system is less than potential energy of deformation of the governing system.

3 Interaction of fibre and matrix (perfect bonding)

In this section, the study is based on the following basic approximate assumptions: the fibre is considered as a 1D elastic continuum, and the bending stiffness of the fibre is neglected; the matrix is considered as a 2D elastic continuum according to the conventional theory of the generalized plane stress; the bond between the matrix and the fibre is supposed to be perfect and continuous; the fibre's attachment is modelled as an ideal line-contact.

The analysis starts with a problem for a single fibre (Fig. 2). Suppose that the loading are applied only to the fibre and the boundaries of matrix are free from stresses. Following the approach presented so far the governing equations and boundary conditions can be written as follows

$$[E^*h + E_1F\delta(x)]V_{yy} + GhV_{xx} = 0, \quad (12)$$

$$[E^*h + E_1F\delta(x)]V_y = -P\delta(x), \quad \text{at } y = 0, H, \quad (13)$$

$$V \to 0, \quad \text{at } |x| \to \infty, \quad (14)$$

where $\delta(\cdot)$ is the Dirac delta-function, E_1 is the Young's modulus of the fibre, F is the cross-section area of the fibre.

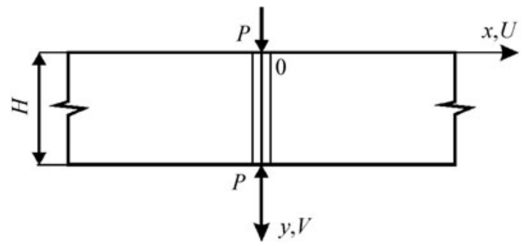

Fig. 2 Perfect bonded fibre

Using the finite integral transform (Tranter [26])

$$\overline{V}(x,s) = \int_0^H V(x,y)\cos\frac{\pi s y}{H}\mathrm{d}y, \quad (15)$$

from the BVPs (12) and (13), one obtains

$$-\left(\frac{\pi s}{H}\right)^2[E^*h + E_1F\delta(x)]\overline{V}(x,s) + Gh\overline{V}_{xx}(x,s) = P_0\delta(x), \quad (16)$$

where $P_0 = P[(-1)^s - 1]$.

Applying to Eq. (16) the exponential Fourier transform

$$\widetilde{V}(q,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{V}(x,s)\exp(-\mathrm{i}qx)\mathrm{d}x, \quad (17)$$

one obtains

$$\widetilde{V}(q,s) = \frac{A}{\sqrt{2\pi}[(\pi s/H)^2E^*h + Ghq^2]}, \quad (18)$$

where $A = -P_0 - \dfrac{\pi^2E_1Fs^2}{H^2}\overline{V}(0,s)$.

Applying the Fourier transform (17) to Eq. (18), one gets

$$\overline{V}(x,s) = A_1\exp\left(-\frac{\pi\alpha s|x|}{H}\right), \quad (19)$$

where $A_1 = \dfrac{AH}{2\pi\sqrt{E^*Ghs}}$, $\alpha = \sqrt{E^*/G}$.

Assuming $x = 0$ in Eq. (19), an equation for function $\overline{V}(0,s)$ is obtained as

$$\overline{V}(0,s) = A_1. \quad (20)$$

Determining function $\overline{V}(0,s)$ from Eq. (20) and using the inverse finite integral transform

$$V(x,y) = \frac{2}{H}\sum_{j=1}\cos\frac{\pi jy}{H}\overline{V}(x,j), \quad (21)$$

from Eq. (19), one finally obtains

$$V(x,y) = P_1H\sum_{j=1,3,5,\cdots}\frac{\exp(-\pi\alpha j|x|/H)}{j(j+gH)}\cos\frac{\pi jy}{H}, \quad (22)$$

where $g = \dfrac{2h\sqrt{E^*G}}{\pi E_1F}$, $P_1 = \dfrac{4P}{\pi^2E_1F}$.

The force in the fibre is defined as the following

$$T(y) = E_1FV_y|_{x=0}. \quad (23)$$

It is not possible to differentiate the right hand part of Eq. (22) with respect to y term by term at $x = 0$. In order to improve this drawback, the author add to and subtract from the function $V(0,y)$ the series

$$P_1H\sum_{j=1,3,5,\cdots}\frac{1}{j^2}\cos\frac{\pi jy}{H}. \quad (24)$$

Calculating the sum (24) by means of formula 5.4.2.12 of Ref. [27] and using Eq. (23), one derives

$$T(y) = -P + \frac{4PgH}{\pi}\sum_{j=1,3,5,\cdots}\frac{\sin(\pi jy/H)}{j(j+gH)}. \quad (25)$$

The contact force is

$$\tau(y) = 0.5T_y. \quad (26)$$

The right hand part of Eq. (25) can not be differentiated term by term with respect to y at $x = 0$, that is why add to and subtract from the right hand part of Eq. (25) the series [28]

$$\frac{4PgH}{\pi} \sum_{j=1,3,5,\cdots} \frac{1}{j^2} \sin \frac{\pi j y}{H}. \quad (27)$$

Calculating the sum (27) by means of formula 5.4.2.11 of Ref. [27], one obtains

$$T(y) = -P - 2Pg \int_0^y \ln\left(\tan \frac{\pi z}{2H}\right) \mathrm{d}z - \frac{4Pg^2H^2}{\pi} \sum_{j=1,3,5,\cdots} \frac{\sin(\pi j y/H)}{j^2(j + gH)}. \quad (28)$$

Using Eqs. (26) and (28), one derives

$$\tau(y) = -2Pg \ln\left(\tan \frac{\pi y}{2H}\right) - \pi^2 P g^2 \left(\frac{H}{2} - y\right) + 4Pg^3H^2 \sum_{j=1,3,5,\cdots} \frac{\cos(\pi j y/H)}{j^2(j + gH)}. \quad (29)$$

The contact force $\tau(y)$ has the logarithmic singularities at the points $y = 0$ and $y = H$.

Now turn to the limit $H \to \infty$. One can not do this for the displacements, but it is possible to do this for the stresses (Parton and Perlin [29]). Equation (25) for $H \to \infty$ yields

$$T(y) = -P - 2Pg \int_0^\infty \frac{\sin(qy)}{q[q + \pi g]} \mathrm{d}q.$$

After calculating this integral by means of formula 2.5.5.5 of Ref. [27], one obtains

$$T(y) = -\frac{2P}{\pi}[\mathrm{Ci}(\pi q y) \sin(\pi q y) - \mathrm{si}(\pi q y) \cos(\pi q y)], \quad (30)$$

where $\mathrm{si}(\cdot)$ and $\mathrm{Ci}(\cdot)$ are the familiar sine and cosine integrals, respectively, $\mathrm{si}(y) = -\pi/2 + \mathrm{Si}(y)$, $\mathrm{Ci}(y) = \gamma + \ln y + \int_0^y t^{-1}(\cos t - 1)\mathrm{d}t$, $\mathrm{Si}(y) = \int_0^y t^{-1} \sin t \mathrm{d}t$, γ is the Euler constant, $\gamma = 0.577\,215\,664\,9\cdots$ (see Abramowitz and Stegun [30], Chapter 5).

Using Eqs. (26) and (30), one obtains

$$\tau(y) = Pg[\mathrm{Ci}(\pi q y) \cos(\pi q y) - \mathrm{si}(\pi q y) \sin(\pi q y)]. \quad (31)$$

At $y \to 0$ the contact force involves a logarithmic singularity.

The obtained solution can be easily generalized for Melan's problem (Fig. 3), which has the exact solution (Muki and Sternberg [31,32])

$$T(y) = -\frac{P}{\pi}[-\cos(\lambda y)\mathrm{si}(\lambda y) + \sin(\lambda y)\mathrm{Ci}(\lambda y)], \quad (32)$$

where

$$\lambda = 4Eh[(3 - \nu)(1 + \nu)E_1F]^{-1}. \quad (33)$$

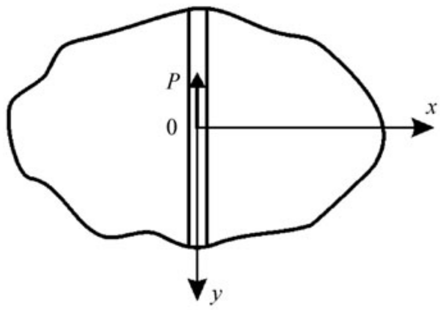

Fig. 3 Melan's problem for infinite matrix and fibre

The proposed approach gives solution of the Melan's problem in the form (32), where the parameter λ has to be replaced by parameter λ_1 of the form

$$\lambda_1 = \frac{\sqrt{2}Eh}{(1 + \nu)\sqrt{1 - \nu E_1 F}}. \quad (34)$$

In Figs. 4a and 4b, functions $T_i = 0.5\pi T(y_1)/P$ are depicted, which are calculated using Eq. (32) with parameter (33) ($i = 1$) and (34) ($i = 2$), $Eh^2/(E_1F) = 0.01$, $y_1 = y/h$.

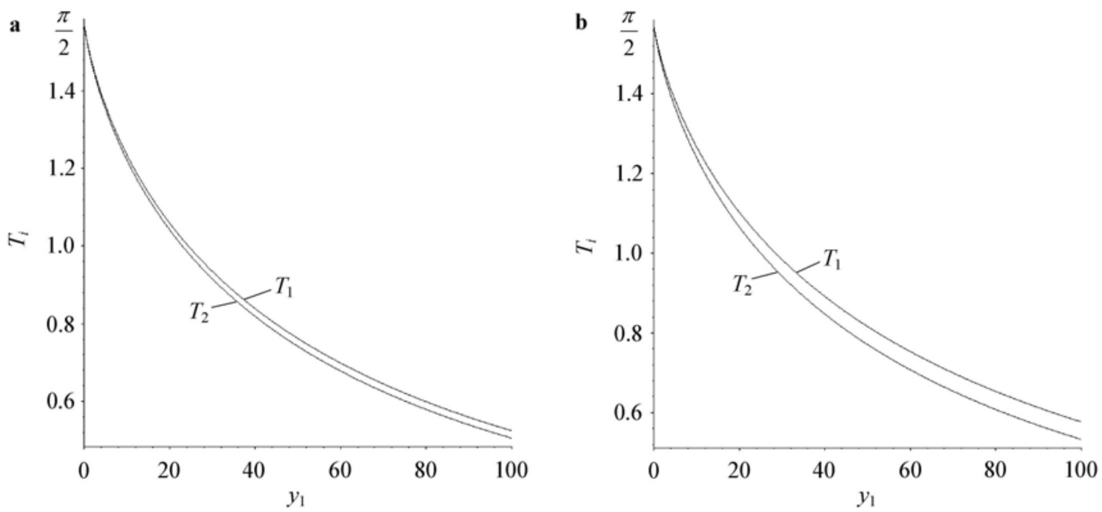

Fig. 4 Comparison of exact and approximate solution of Melan's problem. **a** $\nu = 0$; **b** $\nu = 0.3$

From Figs. 4a and 4b, one can conclude that a force in the fibre calculated using the approximate method is slightly less than the exact value. It means that stresses occurring in matrix are large in the approximate solution.

4 Interaction of fibre and matrix-spring-layer model

In this section, the spring-layer model is talked about, for the interface presented in Fig. 5, initially proposed by Golland and Reissner [11] and analysed by many authors, for example, see Geymonat et al. [12], Klarbring [13], Krasucki and Lenci [14,15], Lenci [16] and references cited therein.

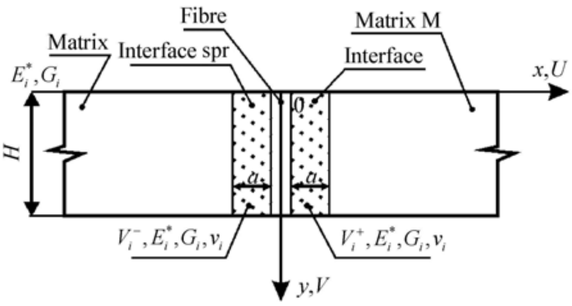

Fig. 5 Fibre-reinforced composite with interface

Assume that the interface guarantees perfect bonding in normal direction, and only tangential sliding is possible. In this case the spring-layer model is described by the following equation (Geymonat et al. [12])

$$\tau(y) = k[V_f(y) - V(0, y)], \tag{35}$$

where $V_f(y)$ is the displacement of the fibre.

The parameter k summarizes the mechanical characteristics of the interface and can be computed from the elastic moduli of the interface (Geymonat et al. [12]) as follows: $k = G_i h/a$. The governing BVP for the matrix with the single fibre is cast to the form

$$E^* h V_{yy} + G h V_{xx} = -2\tau(y)\delta(x), \tag{36}$$

$$E_1 F V_{fyy} = 2\tau(y), \tag{37}$$

$$V_y = 0, \quad \text{at } y = 0, H, \tag{38}$$

$$E_1 F V_{fy} = -P, \quad \text{at } y = 0, H. \tag{39}$$

In addition, Eq. (14) should be added. Applying the finite integral transform (15) to Eqs. (35)–(39), one could obtain

$$-\left(\frac{\pi s}{H}\right)^2 E^* h \overline{V}(x, s) + G h \overline{V}_{xx}(x, s) = -2\tau_s \delta(x), \tag{40}$$

$$\overline{\tau}(s) = k[\overline{V}_f(s) - \overline{V}(0, s)], \tag{41}$$

$$-\left(\frac{\pi s}{H}\right)^2 E_1 F \overline{V}_f(s) = 2\overline{\tau}(s) - P_0, \tag{42}$$

where

$$\overline{\tau}(s) = \int_0^H \tau(y) \cos \frac{\pi s y}{H} \mathrm{d}y. \tag{43}$$

Applying the integral transform (17) to Eq. (40), one obtains

$$\widetilde{V}(q, s) = \frac{2\overline{\tau}(s)}{[E^* h(\pi s/H)^2 + G h q^2]}. \tag{44}$$

Using the transform (17), from Eq. (44) one gets

$$\overline{V}(x, s) = \frac{\overline{\tau}(s) H}{\pi s h \sqrt{E^* G}} \exp\left(-\frac{\pi \alpha s |x|}{H}\right). \tag{45}$$

Supposing $x = 0$ in Eq. (45), one obtains

$$\overline{V}_f(s) = \overline{\tau}(s)\varphi(s), \tag{46}$$

where $\varphi(s) = \left[\frac{1}{k} + \frac{H}{\pi s h \sqrt{E^* G}}\right]$.

Then from Eq. (42), one gets

$$\overline{\tau}(s) = \frac{P_0}{2 + (\pi s/H)^2 E_1 F \varphi(s)}.$$

Using the finite inverse integral transform (21), one derives

$$\tau(y) = 2Pg \sum_{j=1,3,5,\cdots} \frac{\cos(\pi j y/H)}{gH + j + k_1 j^2}, \tag{47}$$

where $k_1 = \frac{\pi h \sqrt{E^* G}}{kH}$.

Observe that for $k_1 \to 0$, after some transformations, one obtains Eq. (22) from Eq. (47).

For $H \to \infty$, Eq. (47) yields

$$\tau(y) = \pi P_1 k \int_0^\infty \frac{\cos(ys)}{D_1(s)} \mathrm{d}s, \tag{48}$$

where $D_1(s) = s^2 + 2g_2 s + g_3$, $g_2 = \frac{k}{2\sqrt{E^* G h}}$, $g_3 = \frac{2k}{E_1 F}$.

If condition $\frac{k E_1 F}{8 E^* G h^2} > 1$ is satisfied, the polynomial $D_1(s)$ has the real roots of the form

$$-\alpha_i = g_2 + (-1)^{i+1} \sqrt{g_2^2 - g_3}, \qquad i = 1, 2. \tag{49}$$

Then, formula 2.5.5.5 in Ref. [27], Eqs. (48) and (49) yield

$$\tau(y) = pg[\mathrm{Ci}(\alpha_1 y)\cos(\alpha_1 y) + \mathrm{si}(\alpha_1 y)\sin(\alpha_1 y) - \mathrm{Ci}(\alpha_2 y)\cos(\alpha_2 y) - \mathrm{si}(\alpha_2 y)\sin(\alpha_2 y)]. \tag{50}$$

For $k \to \infty$ (perfect bonding) from Eq. (50), one obtains Eq. (33).

Note that the spring-layer model prevents singularities in the solution.

5 Improved model of fibre, matrix and interface interaction

Now analyse the interface problem (Fig. 5) in more detail.

On the first step, artificial spring-layer between interface and fibre with coefficient k_2 are introduced as

$$\tau(y) = k_2[V_f(y) - V_i(+0, y)]. \quad (51)$$

The interface is perfect and is bonded with a matrix. The governing BVP reads ($x \geq 0$, for domain $x \leq 0$ there are same results)

$$E_i^* V_{iyy} + G_i V_{ixx} = 0, \quad (52)$$

$$E^* V_{yy} + G V_{xx} = 0, \quad (53)$$

$$V = V_i, \quad \text{at } x = a, \quad (54)$$

$$G V_x = G_i V_{ix}, \quad \text{at } x = a, \quad (55)$$

$$V_y = V_{iy} = 0, \quad \text{at } y = 0, H, \quad (56)$$

$$E_1 F V_{fy} = -P, \quad \text{at } y = 0, H, \quad (57)$$

$$V \to 0, \quad \text{at } x \to \infty. \quad (58)$$

Applying the integral transform (15) to Eqs. (52) and (53), one obtains

$$\overline{V}_{ixx} - \left(\frac{\pi s \alpha_1}{H}\right)^2 \overline{V}_i = 0, \quad (59)$$

$$\overline{V}_{xx} - \left(\frac{\pi s \alpha}{H}\right)^2 \overline{V} = 0, \quad (60)$$

where $\alpha_1 = \sqrt{E_i^*/G_i}$. Solutions of Eqs. (58) and (60) are

$$\overline{V} = C(s) \exp\left[\frac{\pi s \alpha}{H}(a - x)\right], \quad x \geq a. \quad (61)$$

Integrating Eq. (59), one obtains

$$V_i = C_1(s) \exp\left(-\frac{\pi s \alpha_1}{H} x\right) + C_2(s) \exp\left(\frac{\pi s \alpha_1}{H} x\right), \quad 0 \leq x \leq a. \quad (62)$$

Equations (54) and (55) yield

$$C = \frac{2}{1+\beta} \exp\left(\frac{-\pi \alpha_1 s a}{H}\right) C_1, \qquad C_2 = \frac{1-\beta}{1+\beta} \exp\left(-\frac{2\pi \alpha_1 s a}{H}\right) C_1, \quad (63)$$

where $\beta = \dfrac{\sqrt{E^* G}}{\sqrt{E_i^* G_i}} \geq 1$.

The equilibrium condition for the fibre has the form (37). Applying the integral transform (15) to Eqs. (51), (55), (37) and taking into account Eq. (58), one obtains

$$\overline{\tau}(s) = k_2[\overline{V}_f(s) - (C_1 + C_2)], \quad (64)$$

$$-\left(\frac{\pi s}{H}\right)^2 E_1 F \overline{V}_f(s) = 2\overline{\tau}(s) - P_0, \quad (65)$$

$$\pi s \sqrt{E_i^* G_i} h (C_1 - C_2) = \overline{\tau}(s) H. \quad (66)$$

Equations (64)–(66) yield

$$\overline{V}_f = C_1 + C_2 + \frac{\overline{\tau}(s)}{k_2}, \quad (67)$$

$$C_1 = \overline{\tau}(s) \psi(s), \quad (68)$$

$$\overline{\tau}(s) = \frac{P_0 g_i H}{g_i H + s\psi_1(s) + k_3 s^2}, \quad (69)$$

where

$$\psi(s) = \frac{H(1+\beta)}{\pi s h \sqrt{E_i^* G_i} \psi_3(s)},$$

$$\psi_1(s) = \frac{\psi_2(s)}{\psi_3(s)},$$

$$\psi_2(s) = 1 + \beta - (1-\beta) \exp\left(-\frac{2\pi \alpha_1 s a}{H}\right),$$

$$\psi_3(s) = 1 + \beta + (1-\beta) \exp\left(-\frac{2\pi \alpha_1 s a}{H}\right),$$

$$g_i = \frac{2h\sqrt{E_i^* G_i}}{\pi E_1 F},$$

$$k_3 = \frac{\pi h \sqrt{E_i^* G_i}}{k_2 H}.$$

Using inverse finite integral transform of Eq. (21), one obtains

$$\tau(y) = 2 P g_i \sum_{j=1,3,5,\cdots} \frac{\cos(\pi j y / H)}{g_i H + j \psi_1(j) + k_3 j^2}. \quad (70)$$

For $E_i^* = E^*$, $G_i = G$, $k_2 = k$, $a = 0$ Eq. (70) goes over into Eq. (47).

After some transformation, one could obtain for $k_2 = \infty$

$$T(y) = -P - 2Pg_i \int_0^y \ln\left(\tan \frac{\pi z}{2H}\right) dz - \frac{4 P g_i H}{\pi} \times \sum_{j=1,3,5,\cdots} \frac{[g_i H + j(\psi_1(j) - 1)] \sin(\pi j y/H)}{j^2 [j \psi_1(j) + g_i H]}, \quad (71)$$

$$\tau(y) = -2Pg_i \ln\left(\tan \frac{\pi z}{2H}\right) - \pi^2 P g_i^2 \left(\frac{H}{2} - y\right) + 4Pg_i \sum_{j=1,3,5,\cdots} \{[g_i^2 H^2 + g_i H j(\psi_1(j) - 1) - j^2(\psi_1(j) - 1)] \cos(\pi j y/H)\} / \{j^2 [j\psi_1(j) + g_i H]\}. \quad (72)$$

For $H \to \infty$ Eqs. (71) and (72) yield

$$T(y) = -P - \frac{2P}{\pi} \int_0^\infty \frac{[\pi g_i + q(\psi_1(q) - 1)] \sin(qy)}{q[q\psi_1(q) + \pi g_i]} dq, \quad (73)$$

$$\tau(y) = \frac{2P}{\pi} \int_0^\infty \frac{[\pi g_i + q(\psi_1(q) - 1)] \cos(qy)}{q\psi_1(q) + \pi g_i} dq. \quad (74)$$

For $E_i^* = E^*$, $G_i = G$, $a = 0$ and Eqs. (71)–(74) go over into Eqs. (28)–(31).

At $y \to 0$ the contact force involves a logarithmic singularity.

It is known that failure of the assemblage starts in the external boundary of the adhesive and then continues along the interface (Lucas da Silva et al. [6,7]). To increase the resistance of the joint, in practice it is common to avoid (or at least to reduce) the singularity (Adams and Harris [33]). However, sometimes for fracture analysis it is important to detect and monitor the singularities. Proposed approximation does not give correct singularities. Correct singularities in points $(\pm a, 0)$ coincide with singularities in angle point of edge-bonded elastic quarter planes loaded at the boundary (Bogy [34–36]).

The obtained solutions (71) and (74) have logarithmic singularities in the contact forces at the ends of the fibre. On the other hand the exact analysis of the asymptotic behaviour of the contact forces based on the governing biharmonic problem reveals the presence of a power singularity (Muki and Sternberg [31], Sternberg [37]). More exactly, Muki and Sternberg [31] obtained

$$\tau(y) = O(y^{-\rho}), \quad \text{as } y \to 0, \tag{75}$$

where ρ is the unique real root on the interval $(0, 1)$ of the transcendental equation

$$\cos[(1-\rho)\pi] - \frac{2(1+\nu_i)}{3-\nu_i}(1-\rho)^2 + \frac{8-(3-\nu_i)(1+\nu_i)}{(3-\nu_i)(1+\nu_i)} = 0.$$

The above mentioned drawback of the MP approach can be overcome by matching the approximate solutions (71) (or (74)) with singular solution (76), using the technique described by Parton and Perlin [29]. The singular solution could be represented in the form

$$\tau(y) = Cy^{-\rho}, \tag{76}$$

where C is an unknown constant.

Suppose that in some unknown point approximate solutions (71) (or (74)) and singular solution (76) must have the same value and the same first derivatives. Then, it obtains two equations to determine the unknown constants C and y_s.

6 Conclusions

The diffusion of the load from a fibre to a surrounding matrix has been analysed for the 2D case. All solutions have been obtained in the closed analytical form. The method proposed by Kosmodamianskii [17] and, independently by Manevitch and Pavlenko [21,22] are applied. The comparison of the approximate solutions based on this approach with known analytical results obtained by Muki and Sternberg [31,32] show the acceptable accuracy of the proposed asymptotic simplifications. Different models have been considered. First, the classical case of the perfect bonding of the fibre and the matrix is analyzed and the problems for a single fibre in the matrix-strip and for a periodic system of fibres are solved. Also the solutions for half-plane problems are obtained. In these cases, the solutions presented in this paper are compared with results obtained by Manevitch and Pavlenko [21,22]. Then the problems for which analytical solution are unknown are considered. In particular, the influence of the interface stiffness on the diffusion of load from a single fibre and from a periodic system of fibres to the matrix (the concept of weak interface was used) is estimated. The obtained results can be applied for the calculation of pull-out and push-out tests, as well as for the investigation of the fracture of composite materials. In civil engineering, the obtained solutions can describe the behaviour of piles or piers embedded in soil media exhibiting a linear elastic response in the working-load range. As mentioned by Sinclair [38,39], for axially symmetric configurations one expects that, in the local vicinity of greatest interest, a state of plane strain dominates response. So, the obtained results in this paper can be used for 3D problem for fibre-reinforced composites.

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