Applicability of a Classical Perturbation Technique for Perturbation Parameters with Large Values

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In this letter we will illustrate and discuss some problems regarding the validity and accuracy of the perturbation-like methods applied to systems with weak and strong non-linearities.

Hu[1] studied the well-known Duffing equation:

$$x'' + \omega_0^2 x + \varepsilon x^3 = 0, \tag{1}$$

which has the initial conditions of

$$x(0) = A,\ x^{'}(0) = 0. \tag{2}$$

Hu[1] assumed the solution of Eq. (1) in the form of

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots . \tag{3}$$

The fundamental frequency ω^2 is given by

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{4}$$

Notice that in classical approaches of the theory of perturbation[7] an asymptotic series of $x(t)$ is sought in Eq. (3), but the fundamental frequency being sought is estimated through the following equation

$$\omega = \omega_0 + \varepsilon\omega^{(1)} + \varepsilon^2\omega^{(2)} + \cdots \tag{5}$$

instead of being estimated by using Eq. (4).

Equations (1) and (2) possess an exact solution, and hence a comparison of accuracy of Eqs. (4) and (5) can be carried out. Hu has shown numerically that Eq. (3), contrary to traditional application of Eq. (5), yields suitable results even for $0 \le \varepsilon \le \infty$.

Hu claims that he has derived a new perturbation technique that is valid for large parameters.[1] However, this should be treated rather as a particular case, and such a general statement for any other dynamical systems remains invalid. In order to explain the result obtained by Hu[1] we will recall the exact formula in what follows:

$$\omega^2 = \frac{\pi^2}{4}\left(\sqrt{1+\varepsilon A^2}\right)/K(m), \tag{6}$$

where

$$K(m) = \int_0^{\pi/2}\left(1 - m\sin^2\theta\right)^{-1/2} d\theta, \tag{7}$$

$$m = \frac{\varepsilon A^2}{2\left(1+\varepsilon A^2\right)}. \tag{8}$$

Since the following approximation holds[8]

$$K(m) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2 + \right. \\ \left. + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3 + \cdots\right], \tag{9}$$

and since for $0 \le m \le \frac{1}{2}$, Eq. (9) is convergent with a speed of geometrical progression convergence.

On the other hand, a solution representation in Eq. (4) allows avoiding the occurrence of the development of the expression $\sqrt{1+\varepsilon A^2}$ into a series whose radius of convergence is bounded.

Owing to some of the results presented by Hu[1] it is difficult to judge whether Eq. (4) has greater advantages when it is compared to Eq. (5) for cases different from these defined by Eqs. (1) and (2).

For instance, it is often required to study a non-autonomous Duffing equation of the form of

$$x'' + \omega_0^2 x + \varepsilon x^3 = F\cos\phi t, \tag{10}$$

or the autonomous in the form of

$$x'' + \omega_0^2 x + \varepsilon x^\alpha = 0, \quad \alpha > 0, \quad \alpha \ne 3, \tag{11}$$

with attached boundary conditions (see (2)), or, finally, the equation

$$x'' + \omega_0^2 x + \varepsilon x^\alpha + \varepsilon a x^\beta = 0, \quad \alpha \ne \beta, \quad a \equiv \text{const}, \tag{12}$$

and many other similar problems.

The main aim of this letter is to warn researchers that the title of Hu[1] promises more than has been shown.

In addition, let us give our point-of-view regarding the discussion included in Sanchez and He.[3,4] Sanchez's remark that the amplitude of oscillation of the Duffing equation is badly approximated by the perturbation technique for parameters with large values is not true. In order to show our statement, one may consider Fig. 1 given by Sanchez,[3] where initial condition $A = 1$ is not satisfied. In Sanchez[3] initial conditions (see

Eq. (2)) are taken and the assumed solution takes the form of

$$u(t) = A\cos\omega_0 t + \varepsilon\frac{A^3}{32\omega_0^2}\cos 3\omega_0 t + \\ +\varepsilon^2\Bigg(-\frac{1}{256\omega_0^4}\left(A^5 + 8\omega_0^2 A^3\right)\cos\omega_0 t + \\ +\frac{A^5}{1024\omega_0^4}\left(3\cos 3\omega_0 t + \cos 5\omega_0 t\right)\Bigg). \qquad (13)$$

Observe that Eq. (13) does not satisfy the initial conditions. In fact, it should have the following form:

$$u(t) = A\cos\omega_0 t + \varepsilon\frac{A^3}{32\omega_0^2}\left(\cos 3\omega_0 t - \cos\omega_0 t\right) + \\ +\varepsilon^2\frac{A^5}{256\omega_0^4}\left(-\cos\omega_0 t + \frac{1}{4}\left(3\cos 3\omega_0 t + \cos 5\omega_0 t\right)\right). \qquad (14)$$

One may also apply another approach that yields a more improved solution.[9] Namely, assume that

$$u(t) = B\cos\omega_0 t + \varepsilon\frac{B^3}{32\omega_0^2}\cos 3\omega_0 t + \\ +\varepsilon^2\left(\frac{B^5}{1024\omega_0^4}\left(3\cos 3\omega_0 t + \cos 5\omega_0 t\right)\right), \qquad (15)$$

where B is an unknown constant defined by the following algebraic solution

$$A = B + \varepsilon\frac{B^3}{32\omega_0^2} + \varepsilon^2\left(\frac{B^5}{256\omega_0^4}\right), \qquad (16)$$

which yields a reasonably good result.

In addition, one may estimate the problem regarding asymptotic and real errors that occur as a result of approximating solutions. Namely, from Eq. (1) for $\omega_0^2 = 1$ one gets

$$\omega^{(I)} = \sqrt{1 + \frac{3}{4}\varepsilon} + \mathrm{O}(\varepsilon^2). \qquad (17)$$

At this stage it is tempting but quite wrong to conclude that Eq. (17) can be substituted by its equivalent asymptotic approximation

$$\omega^{(II)} = 1 + \frac{3}{8}\varepsilon, \qquad (18)$$

which is often applied. In what follows we estimate the real errors introduced by Eqs. (17) and (18) with the exact frequency value (see Eq.(9)) of the form

$$\omega^{(\text{exact})} = \frac{\pi\sqrt{1+\varepsilon}}{2}\left[\int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - m^2\sin^2\theta}}\right]^{-1}, \qquad (19)$$

where $m^2 = \frac{\varepsilon}{2(1+\varepsilon)}$ (see Fig. 1).

Assuming $\varepsilon \to \infty$ one sees that $m^2 = 0.5$. Furthermore, using Eqs. (17) and (18) and integral value[8] one sees

$$\lim_{\varepsilon\to\infty}\frac{\omega^{(I)}}{\omega^{(\text{exact})}} = 0.9782, \quad \lim_{\varepsilon\to\infty}\frac{\omega^{(II)}}{\omega^{(\text{exact})}} = \infty. \qquad (20)$$

On the basis of the obtained results one may conclude that Eq. (17) can be applied for both small and large values of ε whereas Eq. (18) can only be applied for small values of ε.

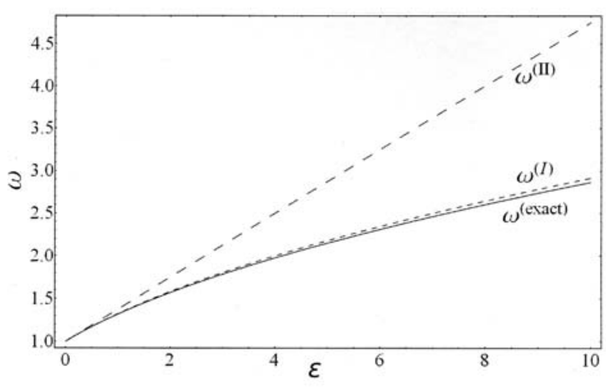

Figure 1. Comparison of the results yielded by Eqs. (17), (18) and (19).

One may also introduce the following principle of a minima solution singularity: an asymptotic solution should contain a minima set of singular operations. In other words, it is better to apply the series (Eq. (4)) and then determine ω instead of assuming it at the beginning of the series (Eq. (5)), although both series are asymptotically equivalent.

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Igor V. Andrianov was born in the Ukraine (FSU) in 1948. He obtained his Master of Applied Mechanics degree from the Dnepropetrovsk State University (DSU, Ukraine) in 1971, his PhD in structural mechanics from DSU in 1975, and his DSc in mechanics of solids from the Moscow State Institute of Electronics and Mathematics in 1990. During 1974-1977, he was a research scientist of DGU. He served as an associated professor from 1977 to 1990 and as a full professor of mathematics in the Dnepropetrovsk Civil Engineering Institute from 1990 to 1997. Now he is the research scientist of the RWTH Aachen University (Germany). Andrianov is the author or co-author of 12 books and more than 250 papers in peer-reviewed journals. He has presented papers at more than 150 international congresses, conferences, and seminars, and he has also supervised 21 PhD students. He was a professor of the Soros Professorship Program (1996). His research interests include mechanics of solids, nonlinear dynamics and asymptotic methods.