Localization of Vibrations in Blade Assemblies

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Abstract: A simple model of blade interactions through an elastic disk is suggested in this paper. The elasticity of a disk is modeled using a ring on an elastic foundation. The linear discrete model of a bladed disk is obtained using the flexibility matrix. The asymptotic procedure is used to analyze the localization of vibration eigenmodes and to obtain eigenfrequencies. The eigenfrequencies loci veering phenomena are treated. The results of the asymptotic analysis are compared with the data of the direct numerical simulations.

Keywords: Bladed disk, localization of vibrations modes, perturbation methods.

1. INTRODUCTION

It is well known that the localization of one blade takes place in the bladed disk (Pearson, 1953; Dye and Henry, 1969; Benediksen, 2000). The imperfections of blades, which lead to small difference in blades eigenfrequencies, are the cause of the localization. The simplest model of free vibrations of bladed disk was considered by Wei and Pierre (1988). It is assumed in this paper that the blades are coupled only through the shroud. Dye and Henry (1969) described the interaction of blades through the disk by a discrete mechanical chain. The elastic beams with discrete masses at the end are attached to the discrete mass of this chain. Using this model the influence of imperfections on the stresses is analyzed. A discrete model of bladed disk vibrations was suggested by Afolabi (1985), where every blade is described by the subsystem with two degrees of freedom. The forced vibrations of a system with imperfections are considered. The applications of discrete models for the analysis of bladed disk vibrations are treated by Buslenko et al. (1991), where the disk is treated as a rigid body and the blade vibrations are described by the subsystem with one or two degrees of freedom. Moreover, good agreements between numerical and experimental results are presented in this paper. Zinkovski (1997) suggested the new discrete model of a bladed disk. Bauer and Shor (1965) used a multi-span rod on an elastic foundation to simulate the disk. Moreover, the beams are attached to this rod and the blade interaction through the disk is

considered using such a model. Kaza and Kielb (1984) described the disk using a circular plate with constant thickness and the blades are simulated using elastic beams. The dynamics of a bladed disk are described by the standing waves of the disk and by the traveling waves of the blades. Ottarson et al. (1994) stress that it is impossible to study the influence of the imperfections on the dynamics of the bladed disk using the finite element model. Therefore, a reduced model of the system was suggested by these authors.

The ring on an elastic foundation is used to model the interactions of blades through the disk in this paper. Every blade is simulated by a mechanical subsystem with one degree of freedom. The flexibility matrix is considered to obtain a discrete model of bladed disk vibrations. The asymptotic technique is used to obtain the eigenfrequencies and eigenmodes of vibrations. The influence of imperfections on the localization of free vibrations is investigated.

The novelty of this paper is the following. First, a new simple model of blade interactions through a disk is suggested. Second, the phenomenon of localization in the case of strong coupling of the blades through the disk is studied.

2. THE MODEL OF A BLADED DISK

The approaches suggested by Bauer and Shor (1965) and Zinkovski (1997) will be developed here to obtain the model for vibrations of a bladed disk (Figure 1). The linear vibrations of the system are considered here. It is assumed that all blades perform tangential vibrations according to the first eigenmode of a cantilever beam in the plane of the disk. The vibrations of every blade are described by a one-degree-of-freedom oscillator and the discrete mass of every oscillator is placed at the end of the blades. As shown by the results of the experimental analysis of vibrations (Vorobiev, 1988), the blades frequently perform motions according to the first vibrations mode and the frequency spectrum of a bladed disk is dense. The vibrations of the ith blade $W_i(x, t)$ are presented as

$$W_i(x, t) = w(x)q_i(t); \quad W_i(L, t) = q_i(t),$$

where L is a length of a beam, i is the number of a blade and $w(x)$ is a first vibrations mode. Then the reduced mass of a blade (Figure 1) is determined as

$$M = \int_0^L \rho A w^2(x)dx,$$

where ρ is the material density and A is the area of a beam cross section.

It is assumed, that the system has n blades. Now the interactions of these blades through the disk are considered. The disk is considered as a ring on linear elastic foundation and the influence coefficients of the system are calculated. The force F_i is applied to the ith discrete mass and the jth mass has displacement u_j. The influence coefficients α_{ji} for $i \neq j$ are determined from the equation:

$$F_i \alpha_{ji} = u_j; \quad \alpha_{ji} = L^2 \gamma_{ji},$$
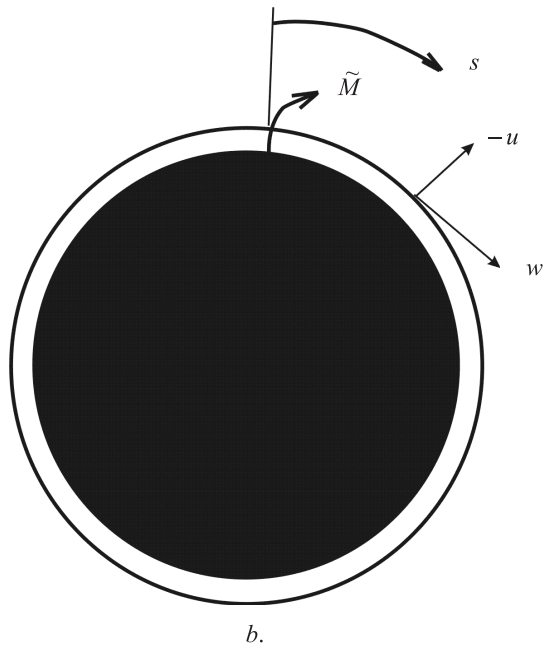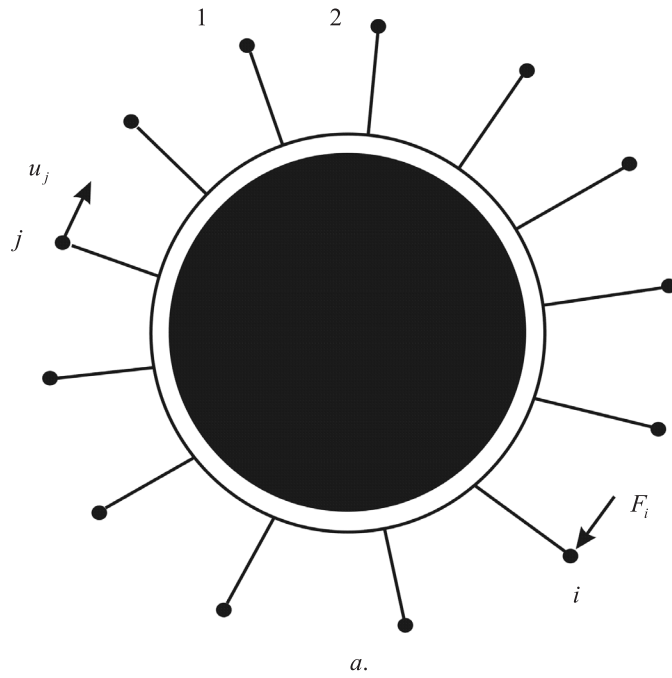
Figure 1. The model of a bladed disk.

where γ_{ji} is rotation angle of the ring in the point of the jth blade attachment under the action of unit moment applied at the point of ith blade attachment. In the case of $i = j$, the influence coefficients are determined as

$$u_i = \frac{F_i L^3}{3 E_1 J_1} + F_i L^2 \gamma_{ii} = \alpha_{ii} F_i, \tag{1}$$

where $E_1 J_1$ is the stiffness of the blade in the disk plane.

Thus, the inertia of the blades and elastic characteristics of both the blades and a ring on a linear elastic foundation are taken into account in the considered model. Therefore, to determine the influence coefficients, it is assumed, that a blade and the elastic ring work sequentially. The ring on an elastic foundation (Figure 1b) without blades is considered to calculate the elements of the matrix γ_{ji}. The equation for ring flexure has the form (Birger and Panovko, 1968):

$$E J \left(\frac{d^5 u}{ds^5} + \frac{2}{R^2} \frac{d^3 u}{ds^3} + \frac{1}{R^4} \frac{du}{ds} \right) + c \frac{du}{ds} = 0, \tag{2}$$

where $E J$ is the stiffness of the ring, c is the stiffness of the foundation and R is radius of the ring. The following boundary conditions are used:

$$
\begin{aligned}
\tilde{M} &= E J \left(\frac{d^2 u}{ds^2} + \frac{u}{R^2} \right)_{s=0} ; \\
u(0) &= u(2\pi R); \\
\left. \frac{du}{ds} \right|_{s=0} &= \left. \frac{du}{ds} \right|_{s=2\pi R} ; \\
w(0) &= w(2\pi R); \tag{3}
\end{aligned}
$$

where $w(s)$ is the tangential displacement of a ring and $\tilde{M}$ is the moment that is applied at $s = 0$. This moment is applied in order to calculate the influence coefficients.

The solution of equation 2 has the following form:

$$u = \frac{\tilde{M}}{E J} \left[\exp(as)(D_1 \cos bs + D_2 \sin bs) + \exp(-as)(D_3 \cos bs + D_4 \sin bs) \right] ;$$

$$w = \frac{\tilde{M}}{E J R(a^2 + b^2)} \left[(D_1 a - D_2 b) \exp(as) \cos bs + (D_1 b + D_2 a) \exp(as) \sin bs \right.$$

$$\left. - (D_3 a + D_4 b) \exp(-as) \cos bs + (D_3 b - D_4 a) \exp(-as) \sin bs \right] ;$$

$$a = \sqrt[4]{\frac{1}{R^4} + \frac{c}{E J}} \cos \left(\frac{\varphi_*}{2} \right) ;$$

$$b = \sqrt[4]{\frac{1}{R^4} + \frac{c}{EJ}} \sin\left(\frac{\varphi_*}{2}\right);$$

$$\varphi_* = \text{arctg}\left[R^2 \sqrt{\frac{c}{EJ}}\right]; \tag{4}$$

where $D_1, ..., D_4$ are constants of integration. Satisfying the boundary conditions equation 3, the system of four linear algebraic equations with respect to $D_1, ..., D_4$ is obtained. Solving this system, the values $D_1, ..., D_4$ are calculated. Then the rotation angle of the ring cross section $\beta(s)$ has the following form:

$$\beta(s) = \frac{\tilde{M}}{EJ}\left(L_1 \exp(as) \cos(bs) + L_2 \exp(as) \sin(bs)\right.$$
$$+ \left. L_3 \exp(-as) \cos(bs) + L_4 \exp(-as) \sin(bs)\right), \tag{5}$$

where

$$L_1 = D_1 a \chi_1 - D_2 b \chi_2; \quad L_2 = D_1 b \chi_2 + D_2 b \chi_1;$$

$$L_3 = -b D_4 \chi_2 - D_3 a \chi_1; \quad L_4 = D_3 b \chi_2 - D_4 a \chi_1;$$

$$\chi_1 = 1 + R^{-2}\left(a^2 + b^2\right)^{-1}; \quad \chi_2 = R^{-2}\left(a^2 + b^2\right)^{-1} - 1.$$

Now the calculation of the elements of the flexibility matrix is considered. Then the value $\tilde{M} = 1$ is taken. The distance between the blades with the numbers i and j is determined as

$$s_{ij} = \varsigma |i - j|,$$

where ς is the blades step. The values s_{ij} are substituted into equation 5 and the parameters γ_{ij} are calculated. If the influence coefficients α_{1j} are determined, then the other elements of the flexibility matrix are calculated using the following formulas (Bauer and Shor, 1965):

$$\alpha_{m,m+1-j} = \alpha_{1,j}; \quad j = 1, ..., m;$$

$$\alpha_{m,m-1+j_1} = \alpha_{1,j_1}; \quad j_1 = 2, ..., n + 1 - m; \quad m = 2, 3, ...$$

The equations of bladed disk vibrations have the following form:

$$q_i = -M \sum_{j=1}^{n} \alpha_{ij} \ddot{q}_j; \quad i = \overrightarrow{1, n}. \tag{6}$$

The dimensionless variables and parameters are used in the following form:

$$\tilde{\alpha}_{ij} = \delta^{-1} \alpha_{ij}; \quad \tau = \frac{t}{\sqrt{\delta M}} = pt, \tag{7}$$

where

$$\delta = \frac{L^3}{3E_1 J_1}$$

is a compliance of one blade and p is the frequency of partial bending vibrations of a tuned blade.

As the stiffness of the disk is much higher than the stiffness of one blade, the dimensionless values $\tilde{\alpha}_{ij}$ $(i \neq j)$ are small. Then in the future calculations, it is assumed, that

$$\tilde{\alpha}_{ij} = \varepsilon \alpha_{ij}^*; \quad \varepsilon \ll 1; \quad i \neq j.$$

In the limit $\varepsilon \to 0$, the system with the blades placed at the rigid disk is obtained.

It is assumed, that the blades have different first partial frequencies of bending vibrations, which have the following form:

$$p_i = \chi_i p; \quad i = 1, 2, ... \tag{8}$$

Thus, the parameters χ_i determine the detuning of frequencies. Then the elements of the flexibility matrix $\tilde{\alpha}_{ii}$ have the following form:

$$\tilde{\alpha}_{ii} = \chi_i^{-2} + \varepsilon \alpha_{ii}^*. \tag{9}$$

The equations of vibrations with respect to dimensionless variables and parameters can be presented as

$$\ddot{u}_i + \chi_i^2 u_i = -\varepsilon \sum_{j=1}^{n} \chi_i^2 \alpha_{ij}^* \ddot{u}_j; \quad i = \overrightarrow{1, n}. \tag{10}$$

3. ASYMPTOTIC ANALYSIS OF FREE VIBRATIONS

The vibrations of the system of equation 10 have the following form: $u_i = A_i \cos(\omega t + \varphi_i)$. Then the following eigenvalue problem is obtained:

$$D^{(0)} A = \omega^2 (\varepsilon \Delta D + E) A, \tag{11}$$

where $D^{(0)} = \text{diag}(\chi_1^2, ..., \chi_n^2)$, $A = (A_1, ..., A_n)^T$, E is the unity matrix and

$$\Delta D = \begin{bmatrix} \chi_1 \alpha_{11} & \chi_1 \alpha_{12} & \cdots & \chi_1 \alpha_{1n} \\ \chi_2 \alpha_{21} & \chi_2 \alpha_{22} & \cdots & \chi_2 \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \chi_n \alpha_{n1} & \chi_n \alpha_{n2} & \cdots & \chi_n \alpha_{nn} \end{bmatrix}.$$

Note that in the formula for ΔD and in the future analysis, the asterisks in the notation of α_{ij} are dropped.

Now the asymptotic analysis of equation 11 is carried out. The solutions of these equations are presented as

$$\lambda = \frac{1}{\omega^2} = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \cdots ;$$

$$A = A^{(0)} + \varepsilon A^{(1)} + \varepsilon^2 A^{(2)} + \cdots . \tag{12}$$

Performing the expansion with respect to ε of equation 11, the sequence of recursive equations is derived:

$$D^{(0)}A^{(0)} = \lambda^{(0)}A^{(0)}; \tag{13a}$$

$$\lambda^{(0)}A^{(1)} + \lambda^{(1)}A^{(0)} = D^{(0)}A^{(1)} + \Delta D A^{(0)}; \tag{13b}$$

$$\ldots\ldots\ldots$$

$$D^{(0)}A^{(k)} + \Delta D A^{(k-1)} = \lambda^{(0)}A^{(k)} + \sum_{j=1}^{k}\lambda^{(j)}A^{(k-j)}; \tag{13c}$$

$$\ldots\ldots\ldots\ldots$$

The solution of equation 13a is as follows:

$$\lambda_\nu^{(0)} = \chi_\nu^{-2}; \quad A_\nu^{(0)} = (\delta_{\nu 1}, \delta_{\nu 2}, ..., \delta_{\nu n})^{\mathrm{T}}; \quad \nu = \overrightarrow{1, n}, \tag{14}$$

where $\delta_{\nu n}$ is Kronecker delta.

The method for the analysis of equations 13 is treated by Pierre (1988) and Poberegnikov (2004). In the present paper, this method is used to study equations 13. Then the eigenvectors $A_\mu^{(k)}$ are presented as

$$A_\mu^{(k)} = \sum_{s=1}^{n}\alpha_{\mu s}^{(k)}A_s^{(0)}; \quad \mu = \overrightarrow{1, n}, \tag{15}$$

where $\alpha_{\mu s}^{(k)}$ are unknown constants. From the system 13c, the following formulas for eigenvectors are derived:

$$\lambda_\mu^{(k)} = \sum_{s=1}^{n}\alpha_{\mu s}^{(k-1)}\left(A_\mu^{(0)}, \Delta D A_s^{(0)}\right) - \sum_{j=1}^{k-2}\lambda_\mu^{(j)}\alpha_{\mu\mu}^{(k-j)}, \tag{16}$$

where $(\cdot, \cdot)$ is the scalar products of vectors. The equation $(A_k, A_k) = 1$ is used to obtain the parameters $\alpha_{kk}^{(j)}; j = 1, 2$ of the expansion 15. Using the last equation and the asymptotic analysis, we derive the following:

$$\alpha_{rr}^{(k)} = -0.5 \sum_{\mu=1}^{k-1} \sum_{s=1}^{n} \alpha_{rs}^{(\mu)} \alpha_{rs}^{(k-\mu)}. \tag{17}$$

The values $\alpha_{\mu r}^{(k)}$; $\mu \neq r$ are determined in the following way:

$$\alpha_{\mu r}^{(k)} = \frac{1}{\chi_r^{-2} - \chi_\mu^{-2}} \left(\sum_{j=1}^{k-1} \lambda_\mu^{(j)} \alpha_{\mu r}^{(k-j)} - \sum_{s=1}^{n} \alpha_{\mu s}^{(k-1)} \chi_r^{-1} \alpha_{rs} \right);$$

$$\mu \neq r; \quad \mu = \overrightarrow{1, n}; \quad r = \overrightarrow{1, n}. \tag{18}$$

Using the above-presented analysis, the first, the second and the third approximations of the eigenvalues and eigenvectors are derived. They are presented in the appendix.

4. NUMERICAL SIMULATIONS OF FREE VIBRATIONS

The parameters of a bladed disk from Bladh et al. (2003) are used for the numerical analysis. The parameters of the detuning blades are taken as follows:

$$J_1 = 0.29 \times 10^{-10} \text{ m}^4; \quad E_1 = 2.1 \times 10^{11} \text{ Nm}^{-2}; \quad l = 0.12 \text{ m},$$

and the parameters of the disk are

$$J = 0.56 \times 10^{-8} \text{ m}^4; \quad E = E_1; \quad R = 0.1 \text{ m}.$$

For the numerical analysis the parameters of the frequency detuning are set in the following form:

$$\chi_i^2 = 1 - \tilde{r}_i,$$

where $\tilde{r}_i = \delta r_i$; $i = \overrightarrow{1, n}$. Thus, the values of detuning are determined by δ. The following numerical values of detuning are used in the calculations: $r_1 = 0.1$; $r_2 = -0.1$; $r_3 = 0.05$; $r_4 = -0.08$; $r_5 = 0.09$. Note, that if $\delta = 0$, the bladed disk is tuned.

The eigenvalue problem 11 is solved for the numerical analysis of linear vibrations. First, the tuned system is solved numerically by calculating the eigenvalue problem 11. Figure 2 shows the modes of free vibrations, where the frequencies satisfy the following inequalities: $\omega_1 < \omega_2 < \cdots$. The phenomenon of localization is not observed in this tuned system.

Now the vibrations of the detuned system are studied. Figure 3 shows the results of the calculations with $\delta = 1$. In this figure the eigenmodes of vibrations, which are obtained by the direct numerical simulation, are shown in the left column and at the right column the results of the asymptotic analysis (equations 18 and 19) are presented. As you can see from Figure 3, these results are close. The frequencies, which are obtained from the solution of equation 11 and from asymptotic analysis, are presented in Table 1. As you can see from Table 1, the results of numerical and asymptotic analysis are very close. Note, that for
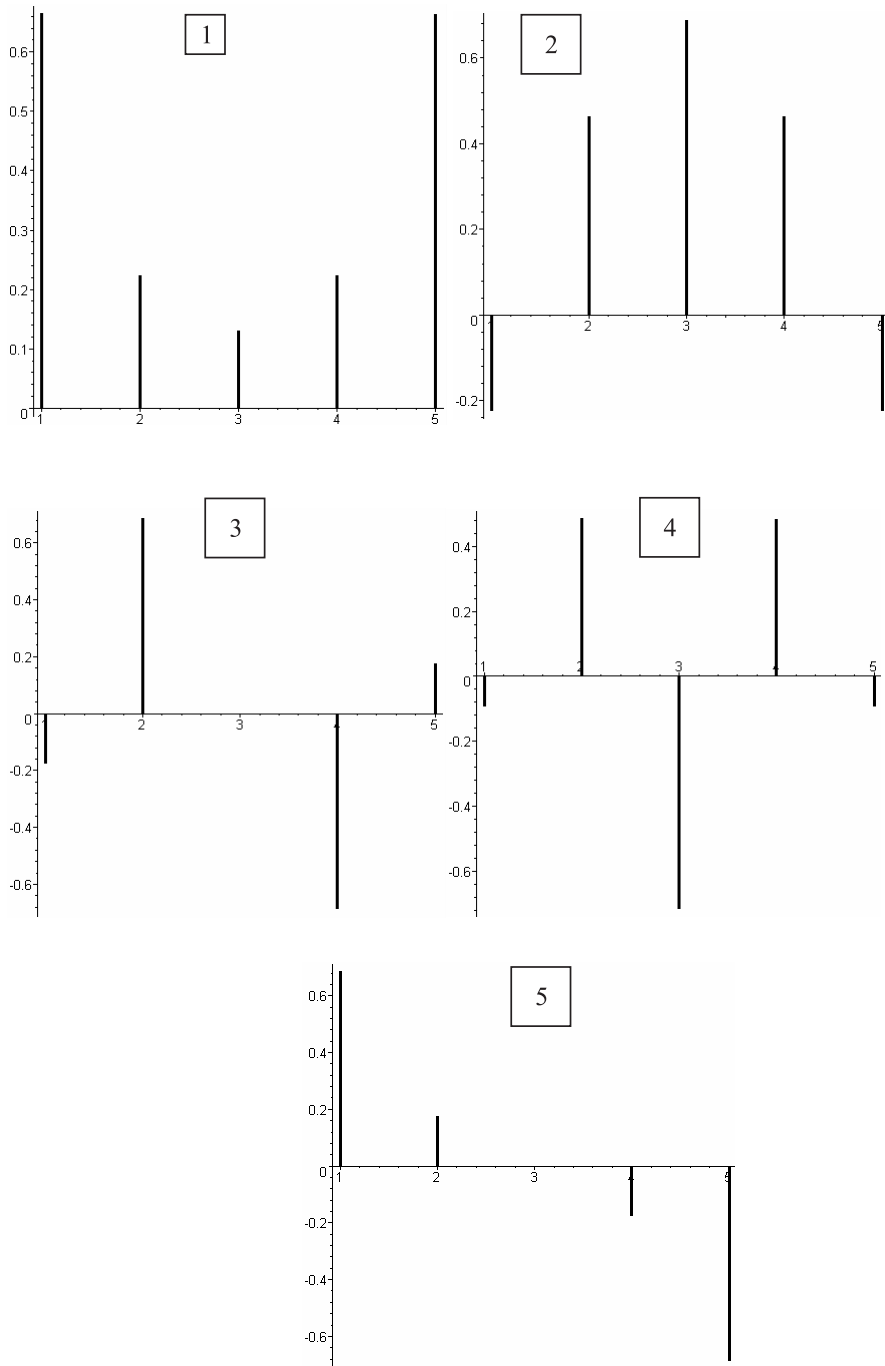
Figure 2. The eigenmodes of the tuned system. The numbers of modes are shown. These modes correspond to the following frequencies: $\omega_1 = 0.9877440599$, $\omega_2 = 0.9998544232$, $\omega_3 = 1.000190387$, $\omega_4 = 1.000249010$, $\omega_5 = 1.011630942$.
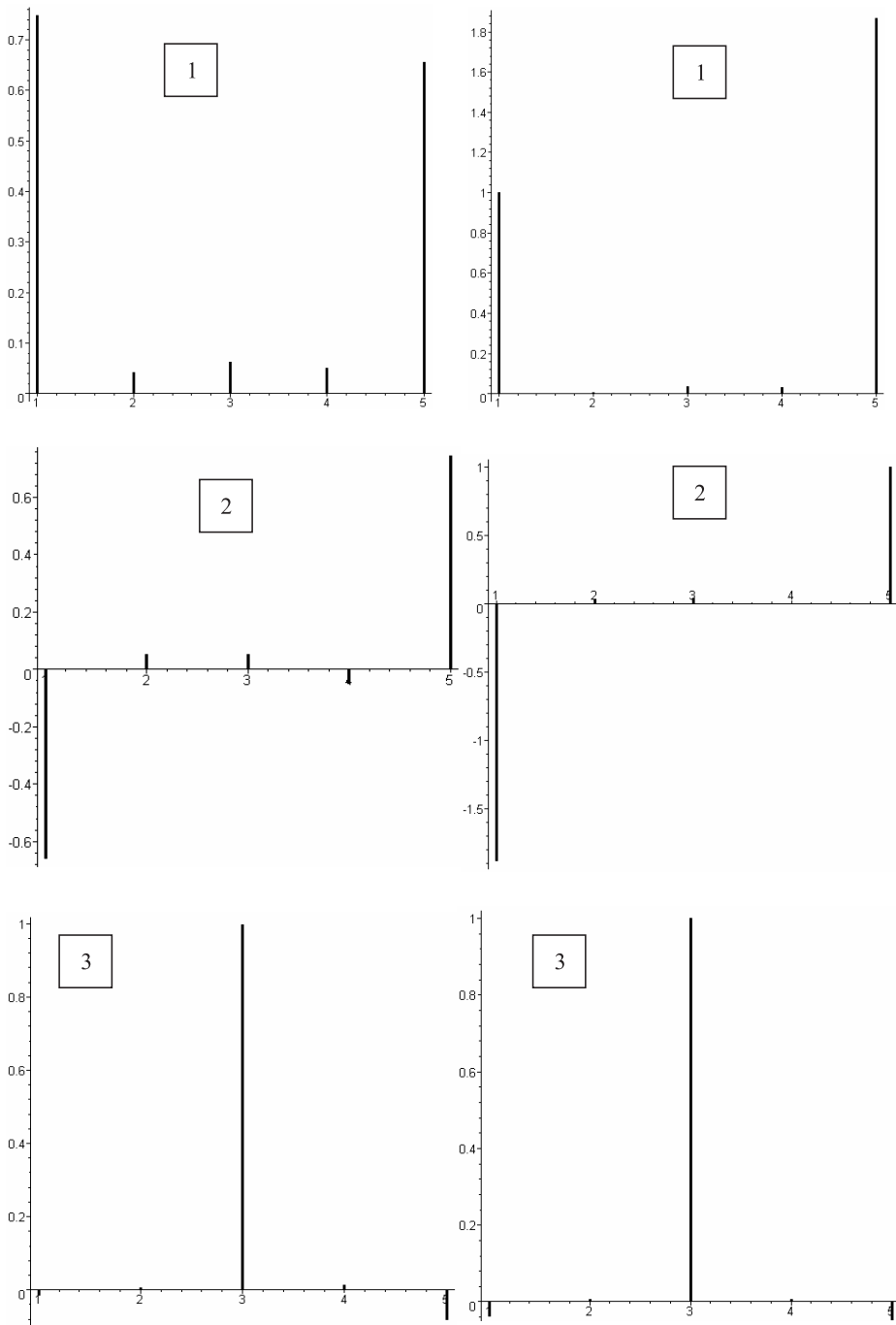
Figure 3. The eigenmodes of the system with imperfections $\delta = 1$. The data of the direct calculations of eigenmodes are presented in the left-hand column and the results of asymptotic analysis are given in the right-hand column.
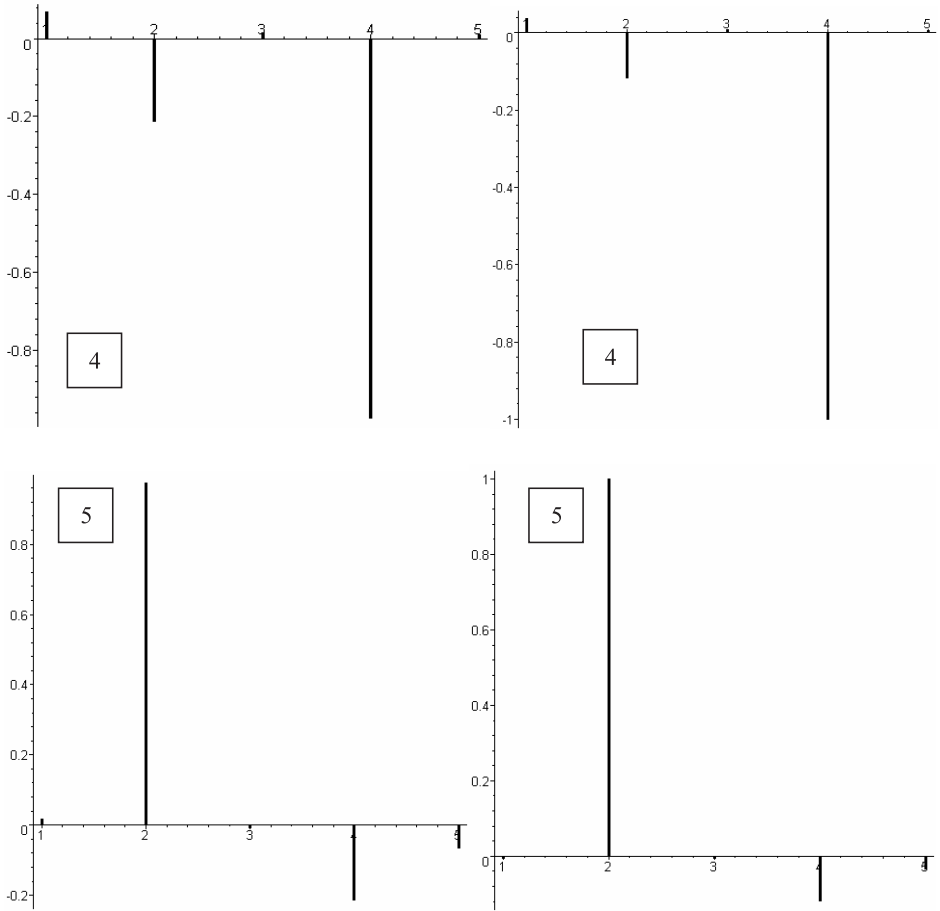
Figure 3. The eigenmodes of the system with imperfections $\delta = 1$. The data of the direct calculations of eigenmodes are presented in the left-hand column and the results of asymptotic analysis are given in the right-hand column. *(Continued)*

Table 1. The results of the frequency analysis, which are obtained analytically and numerically.

	Numerical calculations of ω_i				
	0.964875	0.98536	0.987198	1.0192	1.0244
Asymptotic analysis of ω_i	0.97420	0.9768	0.98719	1.019	1.024

the above-presented parameters, the first approximations with respect to ε for eigenvalues and eigenvectors are close to the results of the direct numerical simulations. However, the results of the second approximations with respect to ε are far from the results of the direct calculations. A similar case is described by Pierre (1988).

The results, which are presented in Figure 3, correspond to the phenomenon of strong localization. The first mode of vibrations has large amplitudes of two adjacent masses and all of the remaining discrete masses have small vibration amplitudes. The second vibration mode shows similar behavior. The third, the fourth and the fifth vibration modes correspond to the strong localization, in this case only one mass has vibrations with large amplitudes. In the previous papers (Wei and Pierre, 1988; Buslenko et al., 1991; Zinkovski, 1997) the strong localization for the case of weak coupling between the blades was considered. In this paper, this phenomenon for the essential coupling between the blades is considered.

The results of the vibrations mode calculations for $\delta = -1$ are shown in Figure 4. The first, the second and the fifth modes correspond to the phenomenon of the strong localization.

The dependences of the frequencies and the vibrations modes on initial imperfections, which are determined by the parameter δ, are analyzed. Figure 5a shows the dependence of five vibration frequencies on δ. Figure 5b shows the scaled-up dependence of three vibration frequencies on δ. As follows from the detailed analysis of Figure 5, the eigenfrequencies come very close to each other, but they do not intersect. This phenomenon is called the veering of eigenvectors (Afolabi, 1985). In this case the basic alteration of eigenmodes is observed in several systems. The calculation of eigenmode vibrations at $\delta = -0.2$ and $\delta = 0.2$ is carried out numerically by solving the eigenvalue problem 11. We analyze the possibility of such an alteration in our system. Figure 6 shows the results of such calculations. The vibration eigenmodes for $\delta = -0.2$ and $\delta = 0.2$ are shown in the left and right columns, respectively. As you can see, in the case of such variation of initial imperfections the first eigenfrequency is not rebuilt. The second mode of vibration is essentially rebuilt. The second vibration mode is not localized at $\delta = -0.2$, but at $\delta = 0.2$ this vibration mode is localized essentially. The third vibration mode is rebuilt essentially; however, in these cases it is not localized. The fourth and the fifth vibrations modes are localized at $\delta = -0.2$.

5. CONCLUSION

The phenomenon of eigenmodes localization in enough simple model of blades interaction through the disk has been investigated in this paper.

The asymptotic formulas for eigenfrequencies and eigenmodes of bladed disk are derived. The results of the first approximation with respect to ε for eigenmodes and eigenfrequencies are close to the data of the direct numerical simulations.

The influence of initial imperfections on mode localization is investigated in this paper. The phenomenon of eigenfrequencies veering, which was observed in simpler models, is discovered when initial imperfections are varied.
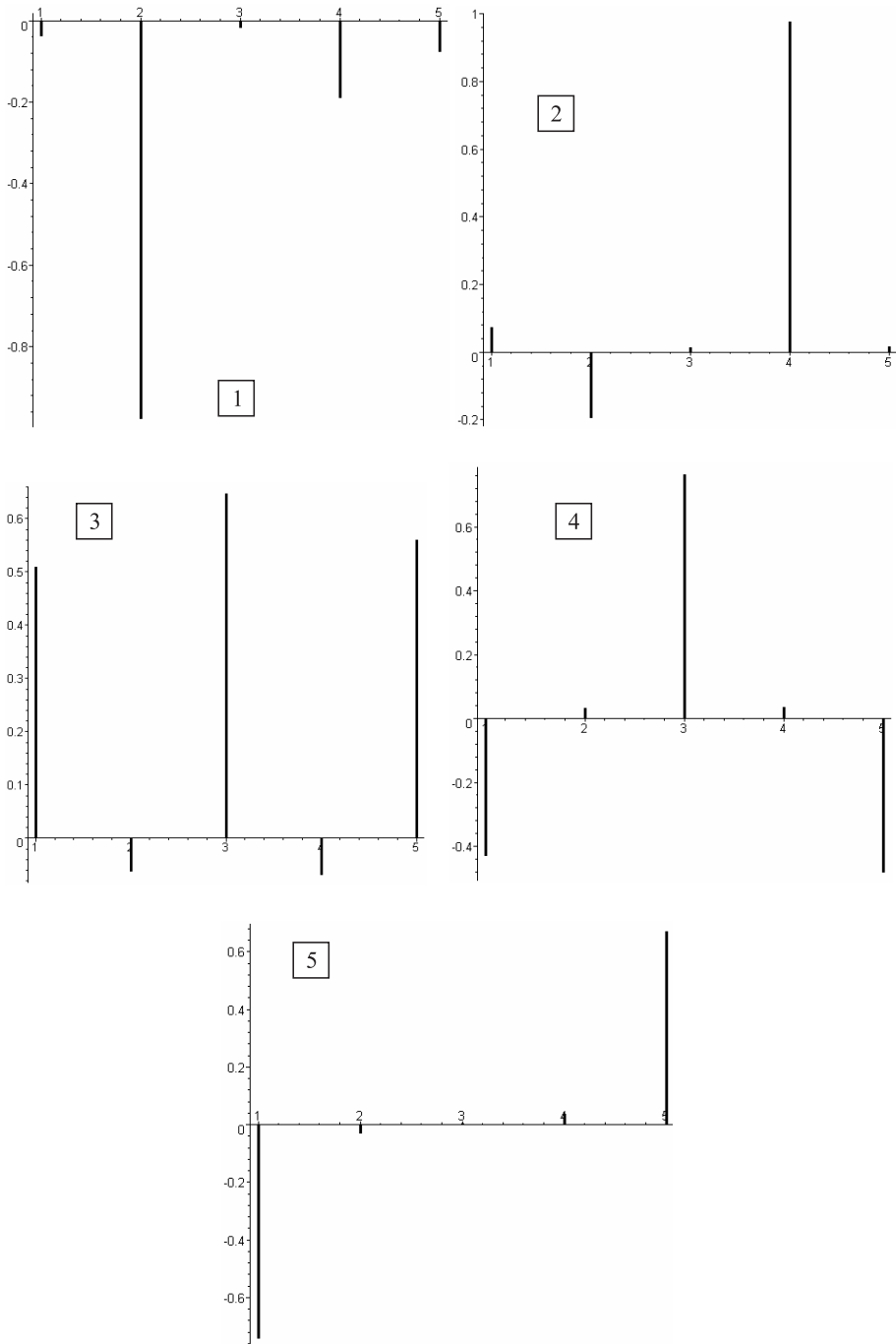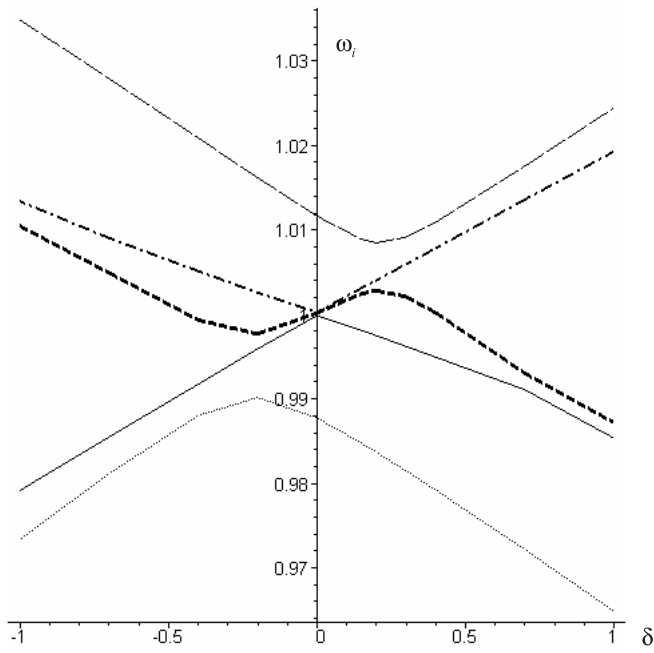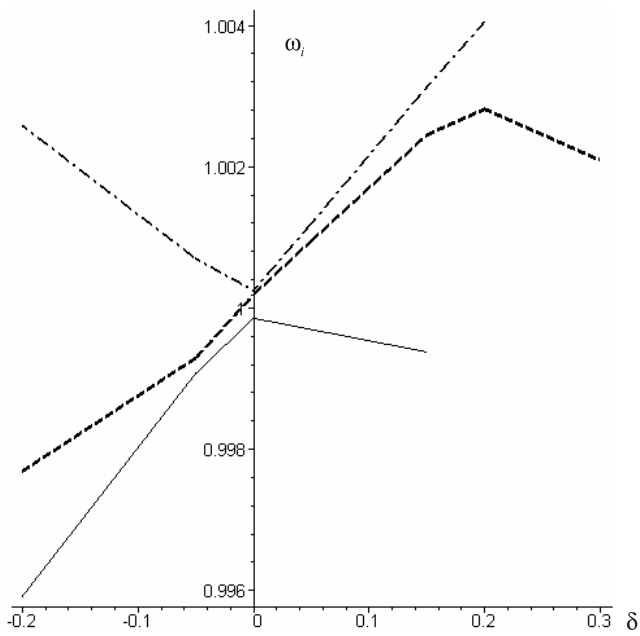
Figure 4. The vibrations eigenmodes at $\delta = -1$.

Figure 5. The dependence of the eigenmodes on δ.

Figure 6. The eigenmodes of system vibrations with imperfections. The results of the calculations with $\delta = -0.2$ and $\delta = 0.2$ are presented at the left- and right-hand columns, respectively. The eigenmodes of vibrations are numbered in the figure.
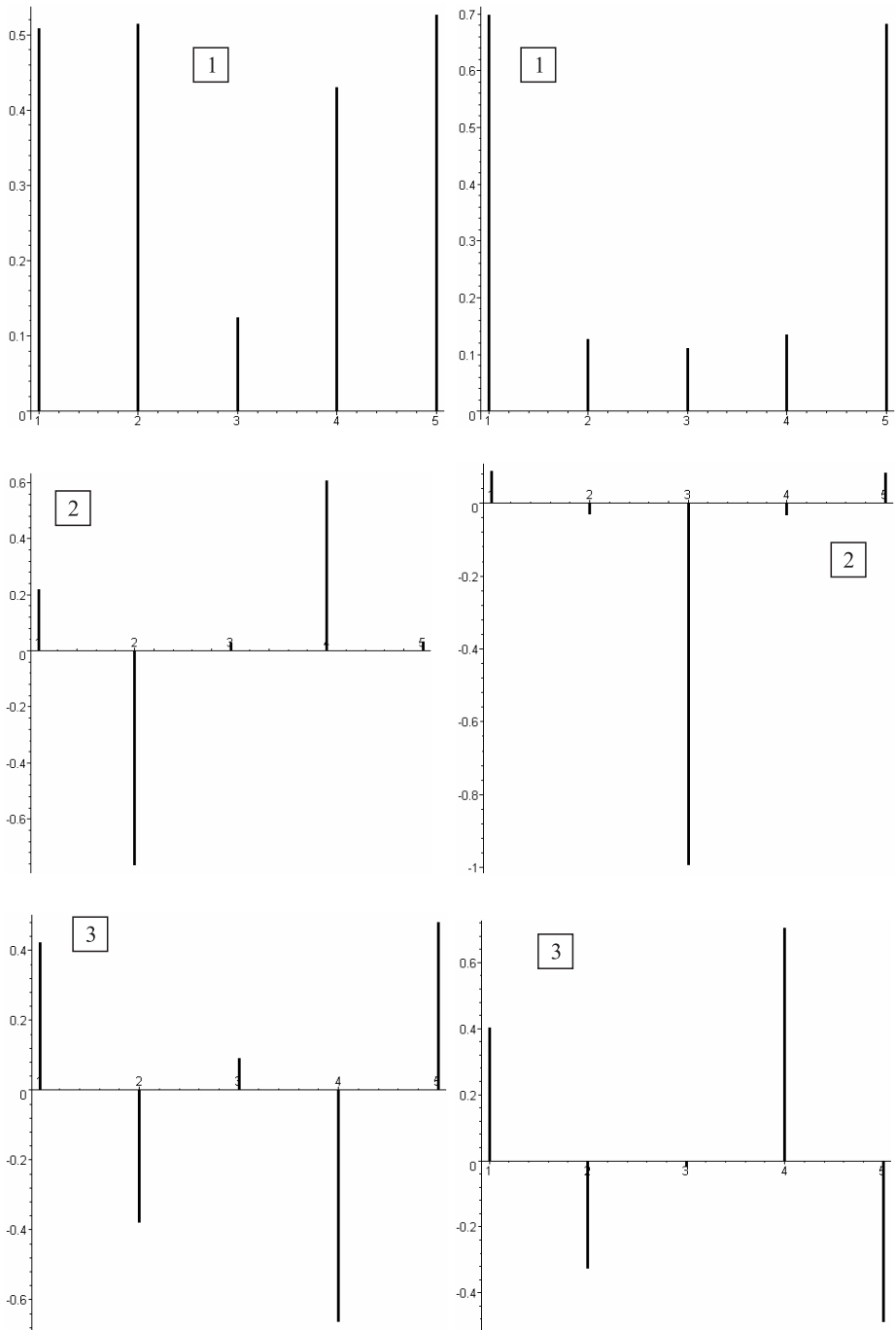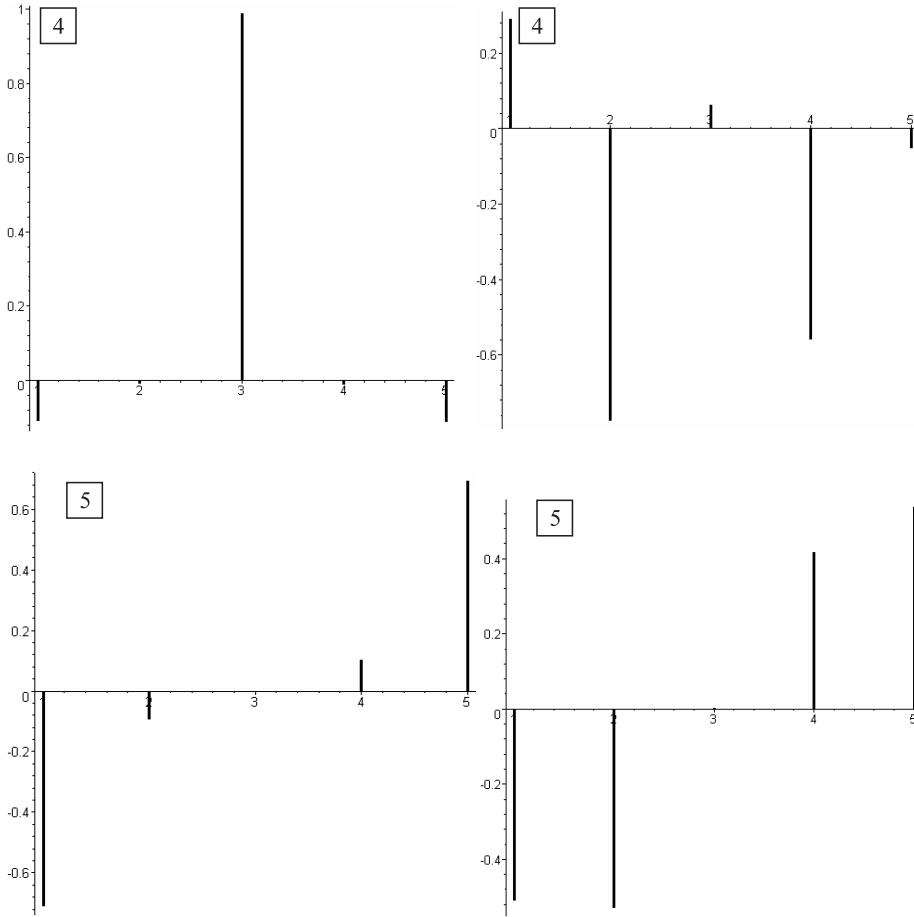
Figure 6. The eigenmodes of system vibrations with imperfections. The results of the calculations with $\delta = -0.2$ and $\delta = 0.2$ are presented at the left- and right-hand columns, respectively. The eigenmodes of vibrations are numbered in the figure. *(Continued)*

APPENDIX

The first, the second and the third approximations of the eigenvalues and eigenvectors are derived here:

$$\lambda_k^{(1)} = \left(A_k^{(0)}, \Delta D A_k^{(0)} \right) = \chi_k^{-1}\alpha_{kk}; \quad \alpha_{kk}^{(1)} = 0; \quad k = \overrightarrow{1, n};$$

$$\alpha_{\mu r}^{(1)} = \frac{\left(A_r^{(0)}, \Delta D A_\mu^{(0)} \right)}{\lambda_\mu^{(0)} - \lambda_r^{(0)}} = \frac{\chi_r^{-1}\alpha_{r\mu}}{\chi_\mu^{-2} - \chi_r^{-2}}; \quad \mu \neq r; \quad \mu = \overrightarrow{1, n}; \quad r = \overrightarrow{1, n};$$

$$\lambda_\mu^{(2)} = \sum_{\substack{s=1\\s\neq\mu}}^{n} \frac{\alpha_{s\mu}^2}{\chi_\mu^{-1}\chi_s - \chi_s^{-1}\chi_\mu}; \quad \mu = \overrightarrow{1,n};$$

$$\alpha_{kk}^{(2)} = -0.5\sum_{\substack{s=1\\s\neq k}}^{n} \alpha_{ks}^{(1)2} = -0.5\sum_{\substack{s=1\\s\neq k}}^{n} \frac{\chi_s^{-2}\alpha_{sk}^2}{\left(\chi_k^{-2} - \chi_s^{-2}\right)^2};$$

$$\alpha_{\mu r}^{(2)} = -\frac{\chi_r^{-1}\chi_\mu^{-1}\alpha_{r\mu}\alpha_{\mu\mu}}{(\chi_\mu^{-2} - \chi_r^{-2})^2} - \sum_{\substack{s=1\\s\neq\mu}}^{n} \frac{\chi_s^{-1}\chi_r^{-1}\alpha_{s\mu}\alpha_{rs}}{\left(\chi_\mu^{-2} - \chi_s^{-2}\right)\left(\chi_r^{-2} - \chi_\mu^{-2}\right)};$$

$$r \neq \mu; \quad r = \overrightarrow{1,n}; \quad \mu = \overrightarrow{1,n};$$

$$\lambda_\mu^{(3)} = -\sum_{\substack{s=1\\s\neq\mu}}^{n} \frac{\chi_s^{-1}\chi_\mu^{-2}\alpha_{s\mu}^2\alpha_{\mu\mu}}{\left(\chi_\mu^{-2} - \chi_s^{-2}\right)^2} - \sum_{\substack{s=1\\s\neq\mu}}^{n}\sum_{\substack{v=1\\v\neq\mu}}^{n} \frac{\chi_\mu^{-1}\chi_v^{-1}\chi_s^{-1}\alpha_{v\mu}\alpha_{sv}\alpha_{\mu s}}{\left(\chi_\mu^{-2} - \chi_v^{-2}\right)\left(\chi_s^{-2} - \chi_\mu^{-2}\right)}; \quad \mu = \overrightarrow{1,n};$$

$$\alpha_{kk}^{(3)} = -\sum_{s=1}^{n} \alpha_{ks}^{(1)}\alpha_{ks}^{(2)} = \sum_{\substack{s=1\\s\neq k}}^{n} \frac{\chi_s^{-2}\chi_k^{-1}\alpha_{sk}^2\alpha_{kk}}{\left(\chi_k^{-2} - \chi_s^{-2}\right)^3} - \sum_{\substack{s=1\\s\neq k}}^{n}\sum_{\substack{j=1\\j\neq k}}^{n} \frac{\chi_s^{-2}\chi_j^{-1}\alpha_{sk}\alpha_{jk}\alpha_{sj}}{\left(\chi_k^{-2} - \chi_j^{-2}\right)\left(\chi_s^{-2} - \chi_k^{-2}\right)^2};$$

$$k = \overrightarrow{1,n};$$

$$\alpha_{\mu r}^{(3)} = \frac{\chi_\mu^{-2}\alpha_{\mu\mu}^2\chi_r^{-1}\alpha_{r\mu}}{\left(\chi_\mu^{-2} - \chi_r^{-2}\right)^3} + \sum_{\substack{s=1\\s\neq\mu}}^{n}\sum_{\substack{v=1\\v\neq\mu}}^{n} \frac{\chi_r^{-1}\chi_v^{-1}\chi_s^{-1}\alpha_{rs}\alpha_{v\mu}\alpha_{sv}}{\left(\chi_\mu^{-2} - \chi_v^{-2}\right)\left(\chi_s^{-2} - \chi_\mu^{-2}\right)\left(\chi_r^{-2} - \chi_\mu^{-2}\right)}$$

$$+ \sum_{\substack{s=1\\s\neq\mu}}^{n} \frac{\chi_s^{-1}\chi_r^{-1}\alpha_{s\mu}}{\left(\chi_\mu^{-2} - \chi_s^{-2}\right)^2\left(\chi_r^{-2} - \chi_\mu^{-2}\right)^2} \left\{\alpha_{rs}\chi_\mu^{-1}\alpha_{\mu\mu}\left(\chi_r^{-2} - 2\chi_\mu^{-2} + \chi_s^{-2}\right)\right.$$

$$+ \left.\alpha_{r\mu}\alpha_{s\mu}\left[0.5\chi_s^{-1}\left(\chi_r^{-2} - \chi_\mu^{-2}\right) - \chi_\mu^{-1}\left(\chi_\mu^{-2} - \chi_s^{-2}\right)\right]\right\}.$$

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