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Homogenization of rods and plates with weakenings

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ABSTRACT

Rods and plates with weakenings are studied using homogenization approach. For rods, two limiting cases amenable to asymptotic integration are considered. It is shown, among other things, that plates with periodic weakenings can be reduced to rods.

1. Introduction

Homogenization of rods and plates with weakenings is important from a technical point of view (Sergienko et al., 1991; Dejneka et al., 1995). Various numerical (Sergienko et al., 1991; Dejneka et al., 1995) and matrix algorithms (Mikhaylov, 1980; Molotkov, 1984) as well as saw-tooth function approaches (Pilipchuk and Starushenko, 1997) are used for homogenization. In the case of a large number of periodic nonhomogeneities, homogenization appears to be very promising, which has been clearly demonstrated using examples of bending of plates with weakenings (Bogan, 1999; Lewinsky and Telega, 2000). In this work, we consider the homogenization of the plane problem of elasticity for plate with periodic weakenings. We use asymptotic simplifications of the plane problem of elasticity proposed in Manevitch et al. (1979), Manevitch and Pavlenko (1991) and described in Awrejcewicz et al. (1998), Andrianov et al. (2004).

2. Rods with weakenings

We consider longitudinal oscillations of a rod comprised of periodically repeated elements (elementary rods) with different characteristics (see Fig. 1). The motions of the elements of the rod are governed by the following equations:

$$(\mathrm{EF})_i U_{i x_1 x_1} - (\bar{\rho} F)_i U_{itt} = f_i(x_1, t), \quad i = 1, 2, \tag{1}$$

where E_i are Young's moduli; F_i are areas of the rod cross-sections; $\bar{\rho}_i$ are the densities; $f_i(x_1,t)$ are the forces acting on the rod elements; U_i are the displacements, x_1 is the spatial coordinate; and t is the time.

The following conjugation conditions of neighborhood elements hold:

$$U_1 = U_2, T_1 = T_2 \text{ on the contact}, \tag{2}$$

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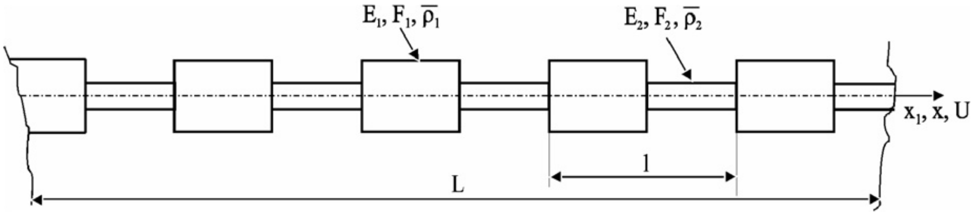

Fig. 1. Rod with weakenings.

where $T_i = (\mathrm{EF})_i U_{i x_1}, \quad i = 1, 2.$

We transform the relations (1) and (2) to the following form:

$$U_{ixx} - \rho_i U_{itt} = \varphi_i(x, t), \quad i = 1, 2, \tag{3}$$

$$U_1 = U_2; \quad U_{1x} = \varepsilon_1 U_2 \text{ on the contact}, \tag{4}$$

where $\rho_i = L^2 \bar{\rho}_i / E_i; \quad \varphi_i = L^2 f_i(x, t)/E_i; \quad x = x_1/L$; $\varepsilon_1 = (\mathrm{EF})_2/(\mathrm{EF})_1$; and L is the length of the rod.

In what follows we take $\varepsilon = \ell/L$, assuming that $\varepsilon << 1$ then we apply a multiple-scale method. After introduction of fast ($\xi = x/\varepsilon$) and slow (x) variables one gets

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial \xi}. \tag{5}$$

A typical periodically repeated cell is shown in Fig. 2. The desired functions are represented by the series

$$U_i = U_0(x, t) + \varepsilon^{\alpha_i} U_i^{(1)}(x, \xi, t) + \varepsilon^{\alpha_i + 1} U_i^{(2)}(x, \xi, t) + \ldots, \quad i = 1, 2, \tag{6}$$

where $U_i^{(k)}(x, \xi, t) = U_i^{(k)}(x, \xi + 1, t); i = 1, 2; \quad k = 1, 2, \ldots;$ and the parameters α_i will be defined later.

Observe that consider three key parameters ε, ε_1 and μ appear in the system. The first of them is small in comparison to the others and we chose ε as the basic one for an order estimation of ε_1 and μ. We introduce parameters β_1, β_2 and β_3 using the formulas

$$\varepsilon_1 \sim \varepsilon^{\beta_1}, (1 - \varepsilon_1) \sim \varepsilon^{\beta_2}, \quad \mu \sim \varepsilon^{\beta_3}, (1 - \mu) \sim \varepsilon^{\beta_4}. \tag{7}$$

A choice of the parameters of asymptotic integrations α_i, β_k ($i = 1,2$, $k = 1 \div 4$) is carried out using a routine procedure (Awrejcewicz et al., 1998; Andrianov et al., 2004). As a result one obtains two following limiting cases.

Case (a): $\alpha_1 = \alpha_2 = 2$, $\beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = \beta_4 = 0$. The choice of the parameters corresponds to elementary rods with approximately similar lengths and similar characteristics.

The dynamics of a cell $0 \leqslant \xi \leqslant 1$ is governed (in the first approximation) by the equation

$$\frac{\partial^2 U_i}{\partial \xi^2} = A_i(U_o), \quad i = 1, 2; \tag{8}$$

$$\text{for} \quad \xi = \mu \quad U_1^{(1)} = U_2^{(1)}; \tag{9}$$

$$U_{1\xi}^{(1)} = \varepsilon_1 U_{2\xi}^{(1)} - (1 - \varepsilon_1) U_{0x}; \tag{10}$$

$$U_1^{(1)}|_{\xi=0} = U_2^{(1)}|_{\xi=1}; \tag{11}$$

$$U_1^{(1)}|_{\xi=0} = \varepsilon_1 U_{2\xi}^{(1)}|_{\xi=0} - (1 - \varepsilon_1) U_{0x}, \tag{12}$$

where $A_i(U_0) = \varphi_i - U_{0xx} + \rho_1 U_{0tt}$, and conditions (11) and (12) follow from the periodicity condition (7).

Integrating (8) gives

$$U_i^{(1)} = C_i^{(1)}(x, t) + C_i^{(2)}(x, t)\xi + 0.5 A_i \xi^2, \quad i = 1, 2. \tag{13}$$

From Eqs. (9)–(13) one obtains the following homogenized equation

$$A_1 \mu + \varepsilon_1 (1 - \mu) A_2 = 0. \tag{14}$$

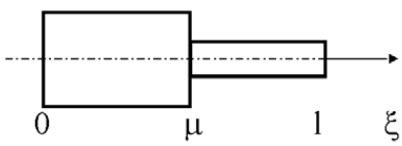

Fig. 2. A typical periodically repeated cell.

The relations (9)–(12) yield only the difference $C_1^{(1)} - C_2^{(1)}$, and we can arbitrarily set $C_2^{(1)} = 0$. The other constants are defined as follows:

$$\begin{aligned} C_1^{(1)} &= C_2^{(2)} + 0.5A_2; \\ C_1^{(2)} &= \varepsilon_1(C_2^{(2)} + A_2) - (1-\varepsilon_1)U_{0x}; \\ C_2^{(2)} &= \frac{A_1\mu^2 + A_2(1+2\varepsilon_1\mu - \mu^2) - (1-\varepsilon_1)\mu U_{0x}}{2[\mu(1-\varepsilon_1)-1]}. \end{aligned} \tag{15}$$

Now we briefly discuss the boundary conditions. Assume, for example, that we have boundary conditions

$$U = 0 \quad \text{for} \quad x = 0, 1.$$

Then for Eq. (14) one obtains

$$U_0 = 0 \quad \text{for} \quad x = 0, 1.$$

Case (b): $\alpha_1 = 3$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = 0$, $\beta_4 = 1$. This case corresponds to one of the most important for practice. Namely, weak short elements separated by long stiff ones are studied (Sergienko et al., 1991; Dejneka et al., 1995).

A solution to the cell problem yields the following homogenized equation:

$$A_1(U_0) = 0, \tag{16}$$

and the 'fast' corrector

$$U_2^{(1)} = U_{0x}(1-\xi). \tag{17}$$

Boundary conditions for Eq. (16) may be written as follows:

$$U_0 = 0 \quad \text{for} \quad x = 0, 1.$$

3. Plates with weakenings

Now we will deal with plates with periodic weakenings (see Fig. 3).

The complete system of plane orthotropic elasticity equations has the following form:

$$\begin{aligned} &B_1^{(i)}U_{ix_1x_1} + G^{(i)}U_{iy_1y_1} + (B_1^{(i)}\nu^{(i)} + G^{(i)})V_{ix_1y_1} - \rho_i U_{itt} = f_i(x_1, y_1, t); \\ &B_2^{(i)}V_{iy_1y_1} + G^{(i)}V_{ix_1x_1} + (B_2^{(i)}\nu^{(i)} + G^{(i)})U_{ix_1y_1} - \rho_i V_{itt} = F_i(x_1, y_1, t), \quad i = 1, 2, \end{aligned} \tag{18}$$

with the associated conjugate relations

$$\begin{aligned} &U_1 = U_2; \quad V_1 = V_2; \\ &T_x^{(1)} = T_x^{(2)}; \quad T_{xy}^{(1)} = T_{xy}^{(2)}, \end{aligned} \tag{19}$$

where

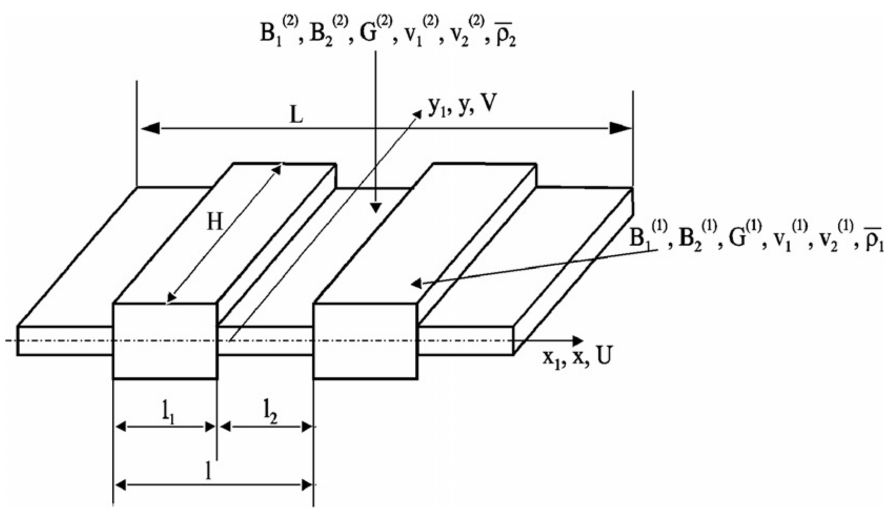

Fig. 3. Plate with periodic weakenings.

$$T_x^{(i)} = B_1^{(i)}(U_{ix_1} + \nu_1^{(i)} V_{iy_1});$$
$$T_{xy}^{(i)} = G^{(i)}(U_{iy_1} + V_{ix_1}), \quad i = 1, 2.$$

We will use the asymptotic procedure proposed in Manevitch et al. (1979), Manevitch and Pavlenko (1991) (see Awrejcewicz et al., 1998; Andrianov et al., 2004). We introduce the following small parameters $\chi_i = G^{(i)}/B_1^{(i)}$ and we assume $B_1^{(i)} \sim B_2^{(i)}; \quad \nu_k^{(i)} \sim \chi_1; i = 1,2; \quad k = 1,2.$

After asymptotic splitting in relation to χ_i, the following Laplace equations and conjugate conditions are obtained:

$$B_1^{(i)} U_{ix_1x_1} + G^{(i)} U_{iy_1y_1} - \bar{\rho}_i U_{itt} = f_i(x_1, y_1, t); \tag{20}$$
$$U_1 = U_2; B_1^{(1)} U_{1x_1} = B_1^{(2)} U_{2x_1} \text{ on a contact;} \tag{21}$$
$$B_2^{(i)} V_{iy_1y_1} + G^{(i)} V_{ix_1x_1} - \bar{\rho}_i V_{itt} = F_i(x_1, y_1, t); \tag{22}$$
$$V_1 = V_2; G^{(1)} V_{1x_1} = G^{(2)} V_{2x_1} \text{ on a contact.} \tag{23}$$

Homogenization of the BVP (22) and (23) leads to the following equations:

$$B_2 V_{0y_1y_1} + G V_{0x_1x_1} - \rho V_{0tt} = F_i(x_1, y_1, t), \tag{24}$$

where

$$B_2 = \frac{B_2^{(1)}\ell_1 + B_2^{(2)}\ell_2}{\ell}; \quad G = \frac{G^{(1)}\ell_1 + G^{(2)}\ell_2}{\ell}; \quad \rho = \frac{\bar{\rho}_1\ell_1 + \bar{\rho}_2\ell_2}{\ell}; \quad F = \frac{F_1\ell_1 + F_2\ell_2}{\ell}.$$

In order to analyze the BVP of (20) and (21) we introduce the following non-dimensional equations:

$$\begin{aligned} &U_{ixx} + \chi_i U_{iyy} - \rho_i U_{itt} = \varphi_i(x, y, t), \\ &U_1 = U_2; \quad U_{1x} = \varepsilon_1 U_{2x} \quad \text{on a contact,} \end{aligned} \tag{25}$$

where $\rho_i = \bar{\rho}_i L^2 / B_1^{(i)}; \quad \varphi_i = f_i L^2 / B_1^{(i)}; \quad x = x_1/L; \quad y = y_1/L;$ and $\varepsilon_1 = B_1^{(2)}/B_1^{(1)}$.

Observe that the BVP (25) is identical to that discussed earlier for a rod. All of the earlier results hold if one supposes

$$A_i(U_0) = \varphi_i - U_{0xx} - \chi_i U_{0yy} + \rho_i U_{0tt}.$$

For elements with similar stiffness and length the relations (13)–(15) hold, whereas for short and weak components the relations (16), and (17) are valid.

The boundary layers occurring in the neighborhood of the plate edges $y_1 = 0, H$ can be constructed using Kantorovich (Kantorovich and Krylov, 1958) methods.

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