Nonlinear Dyn (2008) 53: 19–30
DOI 10.1007/s11071-007-9292-4

ORIGINAL PAPER

Nonlinear oscillations of an elastic two-degrees-of-freedom pendulum

Jan Awrejcewicz · Alexander G. Petrov

Received: 2 April 2007 / Accepted: 18 July 2007 / Published online: 5 September 2007

Abstract Nonlinear oscillations of the vertical plane swinging spring pendulum in the resonance case are studied (frequencies ratio regarding horizontal and vertical directions is equal to 1 : 2). Square and cubic terms of the Hamiltonian are taken into account. Novel normal form method, i.e., the so called invariant normalization is applied to solve the stated problem.

Full system of integrals exhibits equations of the normal form, and solution for the pendulum coordinates is expressed via elementary functions. Frequencies of modes of oscillations are proportional to the first power of amplitude, and not to the second power as it is exhibited by one dimensional Duffing oscillator. Amplitudes of the modes are changed periodically, and energy from one mode is transited to energy of the second one, whereas the period of oscillations depends on the initial conditions. It is illustrated that asymptotic solution with small amplitudes approximates well numerical solution of the governing equations. In addition, an example of a periodic stable solution with constant amplitudes of the oscillation modes is given. Stability of this solution is proved.

J. Awrejcewicz (✉)
Department of Automatics and Biomechanics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland
e-mail: k16@p.lodz.pl

J. Awrejcewicz (✉)
e-mail: awrejcew@p.lodz.pl

A.G. Petrov
Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia
e-mail: petrov@ipmnet.ru

Keywords Swinging pendulum · Hamiltonian normal form · Perturbation · Stability

1 Introduction

In order to investigate nonlinear oscillations of an elastic pendulum, the Poincaré–Birkhoff normal form method is applied. For this purpose [1–3], a Hamiltonian of the investigated system is split into two parts: (i) a square form called non-perturbed and (ii) a remaining part that contains terms of power three. Applied canonical transformations enable a significant simplification of the studied system and the obtained Hamiltonian system of the third order is integrated. It means that an asymptotic solution to the investigated strongly non-linear problem can be obtained. Note that traditional methods of normalization are rather difficult to use and they require a quite large number of transformations [3–9]. A sought change of variables is realized via either guiding functions or a guiding Hamiltonian.

In this work, we apply the invariant normal form proposed by Zhuravlev [10, 11], which does not require (contrary to classical approaches) splitting either to autonomous–non-autonomous or resonance–non-resonance cases, and is realized only by one general approach. Asymptotics of a normal form are defined through successive computations of quadratures. Contrary to the referenced method [10, 11], the proposed approach relies on the application of a parameterized guiding function [12–14]. In this work, our earlier developed method [15] is applied to further analysis of non-linear oscillations of a swinging elastic pendulum initiated also in references [16, 17], where only a linear term of force-elongation function has been used. Now, a non-linear dependence between force and elongation of the spring (pendulum length) is taken into account. In this general case, a normal form of a Hamiltonian with accuracy of the third order terms with respect to amplitude is derived for the studied 1 : 2 frequency ratio. The obtained normal form equations are integrated.

In the applied approximation, only linear and quadratic terms are taken into account. Although this approximation does not allow to follow the Duffing effect, i.e. dependence of oscillation frequency versus amplitude, but it is sufficient to study non-linear resonances interaction of vertical and horizontal modes.

In the obtained asymptotic solution, the frequencies of the modes differ from the corresponding linear case on amount of a quantity proportional to amplitude of oscillations. Therefore, the Duffing phenomenon is mainly realized via the resonance interactions of the corresponding modes of oscillations.

2 Hamiltonian normal form

Following the method given in reference [18], a normal Hamiltonian form of two-degrees-of-freedom system will be derived. We are going to study the case of resonance 1 : 2.

Let $(\mathbf{q},\mathbf{p}) \stackrel{\text{def}}{=} (q_1, q_2, p_1, p_2)$ are dependent variables and let $H = H(\mathbf{q},\mathbf{p})$ is the Hamiltonian function governed by the equations

$$\dot{q}_i = \partial H/\partial p_i, \qquad \dot{p}_i = -\partial H/\partial q_i, \qquad i = 1, \ldots, n, \tag{1}$$

where dot denotes d/dt. Let $\mathbf{q} = \mathbf{p} = 0$ is a fixed point of system (1), and the function $H = H(\mathbf{q},\mathbf{p})$ is analytical. The function H can be developed into series regarding $\mathbf{q}$, $\mathbf{p}$, which begins with quadratic terms, whereas the series of the functions of the right-hand sides of (1) begin from linear terms. Let R is the matrix of linear part of (1). Eigenvalues $\lambda_1, \ldots, \lambda_4$ of matrix R can be split into pairs $\lambda_{j+n} = -\lambda_j$, $j = 1, 2$. Introducing a linear canonical change of coordinates

$$(\mathbf{q},\mathbf{p})^* = B(\mathbf{x},\mathbf{y})^*, \tag{2}$$

where * denotes transposition, matrix R can be always reduced to the following complex normal form $C = B^{-1}RB$, where its eigenvalues $\lambda_1, \ldots, \lambda_4$ lie on a diagonal. Therefore, $H(\mathbf{q},\mathbf{p}) = \tilde{H}(\mathbf{x},\mathbf{y})$. Let a formal non-linear complex coordinates transformations exist of the following form

$$(\mathbf{x},\mathbf{y}) = (\mathbf{u},\mathbf{v}) + \mathbf{N}(\mathbf{u},\mathbf{v}), \tag{3}$$

where $\mathbf{N} \stackrel{\text{def}}{=} (N_1, \ldots, N_4)$, and $N_j(\mathbf{u},\mathbf{v})$ are power series without free and linear terms. Therefore, the Hamiltonian function $\tilde{H}(\mathbf{x},\mathbf{y})$ is transformed into the form

$$h(\mathbf{u},\mathbf{v}) \stackrel{\text{def}}{=} \sum h_s \mathbf{w}^s, \tag{4}$$

where $\mathbf{w} \stackrel{\text{def}}{=} (w_1, \ldots, w_4) \stackrel{\text{def}}{=} (\mathbf{u},\mathbf{v})$, $\mathbf{s} = (s_1, \ldots, s_4)$, $\mathbf{w}^s \stackrel{\text{def}}{=} w_1^{s_1} \cdots w_4^{s_4} \stackrel{\text{def}}{=} u_1^{s_1} u_2^{s_2} v_1^{s_3} v_2^{s_4}$. The formally introduced Hamiltonian (4) is called a complex normal form, if:

1. The corresponding Hamiltonian system possesses a matrix of its linear part of diagonal form with elements $\lambda_1, \lambda_2, -\lambda_1, -\lambda_2$.
2. In series (4), only resonance terms of the form

$$s_1\lambda_1 + s_2\lambda_2 - s_3\lambda_1 - s_4\lambda_4 = 0 \tag{5}$$

are included.

In our example, we consider the resonance case regarding frequencies $\lambda_1 = 2i$, $\lambda_2 = i$, and in this case, a normal form condition takes the form: $s_1 \cdot 2 + s_2 \cdot 1 - s_3 \cdot 2 - s_4 \cdot 1 = 0$.

It has been shown in [19] that for an arbitrary system (1) there exists formal change (3) transforming Hamiltonian $H(\mathbf{q},\mathbf{p})$ into its normal form (4), (5). Owing to reference [2], if the initial system is real, then there exists also real normal form, which can be transformed to its complex normal form (4), (5) via standard linear transformation of coordinates. Special cases of the mentioned normal form include those proposed by Birkhoff [2], Cherry [20], and Gustavson [21]. Birkhoff [2] considered the case, where all eigenvalues λ_j are incommersable, i.e. (4) for integers s_j possesses only integer-valued solution $s_1 = s_2 = \cdots = s_n = 0$. In the latter case, the series (4) has the form $q_1 p_1, q_2 p_2, \ldots, q_n p_n$, and each of the product terms represents a formal integral form of the corresponding Hamiltonian system.

Cherry [20] considered the case, when the eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ are distinct, which has been also studied by Gustavson [21]. Belitskii [22] proposed the modified normal form, where Jordan cells of the linear part matrix are used to reduce the number of nonlinear terms. Note that examples of other normal forms are illustrated and discussed in reference [3].

3 Computation of normal forms

Three different methods are so far proposed to compute canonical normal transformations (3) and normal forms (4), (5): (i) using guiding functions; (ii) via Li series; (iii) via parametrization.

In order to introduce a reader to this subject, we briefly present all of them.

(i) In order to compute a normal form, the Jacobi guiding function is first introduced ([2–4], [6, 10, 20, 21]). In this case, the vector series $\mathbf{N}(\mathbf{u}, \mathbf{v})$ is sought via the guiding function $g(\mathbf{x}, \mathbf{v}) = x_1 v_1 + \cdots + x_2 v_2 + \cdots$ regarding the variables $\mathbf{x} = (x_1, x_2)$ and $\mathbf{v} = (v_1, v_2)$, where

$$\begin{aligned} u_j &= \partial g/\partial v_j = x_j + \cdots, \\ y_j &= \partial g/\partial x_j = v_j + \cdots, \quad j = 1, 2. \end{aligned} \tag{6}$$

If the guiding series $g(\mathbf{x}, \mathbf{v})$ is found, then in order to apply transformation (3) one has to express x_j via $\mathbf{u}, \mathbf{v}$ in (6), i.e. inversion of the power series for u_j is required. In practice, it is tedious task, but it works always (without any restriction given to matrix R).

(ii) Application of Li series is associated usually with the scaling

$$\begin{aligned} \mathbf{q} &= \varepsilon \mathbf{q}', & \mathbf{p} &= \varepsilon \mathbf{p}', & \mathbf{x} &= \varepsilon \mathbf{x}', \\ \mathbf{y} &= \varepsilon \mathbf{y}', & t' &= \varepsilon^2 t, & \mathbf{w} &= \varepsilon \mathbf{w}'. \end{aligned} \tag{7}$$

Hamiltonian $\tilde{H}(\mathbf{x}, \mathbf{y})$ normal form $h(\mathbf{w})$ and Li generator $G(\mathbf{w})$ are represented by the following series with respect to ε:

$$\begin{aligned} \tilde{H}(\mathbf{x}', \mathbf{y}') &= \sum_{k=0}^{\infty} \varepsilon^k \tilde{H}_k(\mathbf{x}', \mathbf{y}'), \\ h(\mathbf{w}') &= \sum_{k=0}^{\infty} \varepsilon^k h_k(\mathbf{w}'), \\ G(\mathbf{w}') &= \sum_{k=0}^{\infty} \varepsilon^k G_k(\mathbf{w}'). \end{aligned}$$

Normal transformation of coordinates and normal form $h(\mathbf{w}')$ are sought via the following Li series

$$\begin{aligned} \mathbf{z}' &= \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} L_G^k \mathbf{w}' \overset{\text{def}}{=} \mathbf{w}' + \varepsilon\{\mathbf{w}', G\} \\ &\quad + \frac{\varepsilon^2}{2!}\{\{\mathbf{w}', G\}, G\} + \cdots, \\ h(\mathbf{w}') &= \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} L_G^k \tilde{H}(\mathbf{w}') \overset{\text{def}}{=} \tilde{H}(\mathbf{w}') + \varepsilon\{\tilde{H}, G\} \\ &\quad + \frac{\varepsilon^2}{2!}\{\{\tilde{H}, G\}, G\} + \cdots, \end{aligned}$$

where

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right),$$

and L_G^k denotes the kth Poisson bracket regarding function G. Functions h_k and G_{k-1} define a sequence of solution of the homoclinic equation

$$h_k(\mathbf{w}) = \{\tilde{H}_0(\mathbf{w}), G_{k-1}(\mathbf{w})\} + M_k(\mathbf{w}), \tag{8}$$

where the function M_k is known via computation of a previous step. Equation (8) is solved as a system of linear equations with respect to h_k and G_{k-1}. This method has been applied by Hori [23] and Deprit [24]. In the discussed case, there are also no restrictions on matrix R. On the other hand, Zhuravlev [10, 11] has proposed solving of homoclinic (8) by integration. If the matrix R is diagonalized, then $\{H_0, G_{k-1}\} = dG_{k-1}/dt$ and the derivative follows from $\dot{\mathbf{q}} = \partial H_0/\partial \mathbf{p}$, $\dot{\mathbf{p}} = -\partial H_0/\partial \mathbf{q}$. Therefore, h_k are averaged functions of M_k along the system solution, whereas the function G_{k-1} is a constant in integral $\int_0^t M_k\,dt$ taken with minus sign. Two first approximations of function $M_k(\mathrm{w})$ have the following form

$$M_1 = H_1,$$
$$M_2 = H_2 + \{H_1, G_1\} + \frac{1}{2}\{\{H_0, G_1\}, G_1\}. \qquad (9)$$

(iii) Owing to references [12, 13], instead of the generating function G, a function $\Psi(\mathbf{x},\mathbf{y})$ and a parametrical canonical change of variables $\mathbf{q},\mathbf{p}\to\mathbf{u},\mathbf{v}$ are applied

$$\mathbf{q} = \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \qquad \mathbf{u} = \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}},$$
$$\mathbf{p} = \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}, \qquad \mathbf{v} = \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}. \qquad (10)$$

Note that two first approximations of G and Ψ coincide, whereas next are distinct. It should be emphasized that approaches (ii) and (iii) sufficiently reduce computations, which will be demonstrated by our example.

4 Algorithm of invariant normalization

In what follows, we present briefly an algorithm devoted to normal form computation of variant (iii). It can be treated as modification of the Zhuravlev approach, and a normal form in this case is found via direct integration. However, it works in either resonance or non-resonance cases. It can be also directly applied to the case of a non-autonomous Hamiltonian [14].

In the beginning, some fundamental definitions and background of the applied normal form algorithm will be introduced. Function $f(t)$ presented as a sum of harmonics of the form

$$f(t) = f_0 + \sum_i (a_i \cos\omega_i t + b_i \sin\omega_i t),$$

is called a quasi-periodic function.

Let us introduce two linear operators $L(f)$ and $L_1(f)$ defined on the manifold of quasi-periodic functions $f(t)$ of the form

$$L\big(f(t)\big) = f_0, \qquad L_1\big(f(t)\big) = \sum_i b_i/\omega_i. \qquad (11)$$

Let the following Hamiltonian

$$H(\mathbf{q},\mathbf{p}) = H_0(\mathbf{q},\mathbf{p}) + F(\mathbf{q},\mathbf{p}) + \cdots$$

be reduced to its normal form. In our example, it is sufficient to apply the approximation, where both squared $H_0(\mathbf{q},\mathbf{p})$ and cubic $F(\mathbf{q},\mathbf{p})$ Hamiltonian forms are taken into account. In order to find the appropriate normal form $\bar{H}(\mathbf{Q},\mathbf{P}) = H_0(\mathbf{Q},\mathbf{P}) + \bar{F}(\mathbf{Q},\mathbf{P})$, i.e. the cubic form $\bar{F}(\mathbf{Q},\mathbf{P})$, satisfying condition (6), and a canonical change of variables, the following operations should be carried out.

1. Solve the Cauchy problem for non-perturbed Hamiltonian H_0 and present it in the following form

$$\mathbf{q} = \mathbf{q}(t,\mathbf{X},\mathbf{Y}), \qquad \mathbf{p} = \mathbf{p}(t,\mathbf{X},\mathbf{Y}),$$
$$\mathbf{q}(0,\mathbf{X},\mathbf{Y}) = \mathbf{X}, \qquad \mathbf{p}(0,\mathbf{X},\mathbf{Y}) = \mathbf{Y}. \qquad (12)$$

2. Find normal form $\bar{F}$ and function Ψ

$$\bar{F}(\mathbf{Q},\mathbf{P}) = LF\big(\mathbf{q}(t,\mathbf{Q},\mathbf{P}), \mathbf{p}(t,\mathbf{Q},\mathbf{P})\big),$$
$$\bar{\Psi}(\mathbf{Q},\mathbf{P}) = L_1 F\big(\mathbf{q}(t,\mathbf{Q},\mathbf{P}), \mathbf{p}(t,\mathbf{Q},\mathbf{P})\big). \qquad (13)$$

Note that in the proposed approach formulas (13) define a normal form supplemented by condition (6), as well as a function defining a parametric canonical change of variables. The constructed normal form yields a solution to the Hamiltonian system as follows. Let $\mathbf{X} = \mathbf{X}(t,\mathbf{Q}_0,\mathbf{P}_0)$, $\mathbf{Y} = \mathbf{Y}(t,\mathbf{Q}_0,\mathbf{P}_0)$ be a solution associated with Hamiltonian $F(\mathbf{X},\mathbf{Y})$, i.e.

$$\dot{\mathbf{X}} = \frac{\psi\mathbf{F}}{\psi\mathbf{Y}}, \qquad \dot{\mathbf{Y}} = -\frac{\psi F}{\psi\mathbf{X}},$$
$$\mathbf{X}(0) = Q_0, \qquad \mathbf{Y}(0) = \mathbf{P}_0.$$

Applying Zhuravlev's theorem [10, 11] and substituting the solution into (12) associated with non-perturbed Hamiltonian, one gets a solution in reference to Hamiltonian $\bar{H}(\mathbf{Q},\mathbf{P})$ of the form

$$\mathbf{Q} = \mathbf{q}\big(t, \mathbf{X}(t,\mathbf{Q}_0,\mathbf{P}_0), \mathbf{Y}(t,\mathbf{Q}_0,\mathbf{P}_0)\big),$$
$$\mathbf{P} = \mathbf{p}\big(t, \mathbf{X}(t,\mathbf{Q}_0,\mathbf{P}_0), \mathbf{Y}(t,\mathbf{Q}_0,\mathbf{P}_0)\big).$$

Then, a solution expressed via initial variables is yielded with a help of a canonical change of variables $\mathbf{Q},\mathbf{P}\to\mathbf{q},\mathbf{p}$ expressed by the following parametric form

$$\mathbf{q} = \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \qquad \mathbf{Q} = \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}},$$
$$\mathbf{p} = \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}, \qquad \mathbf{P} = \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}. \qquad (14)$$

Removing parameters $\mathbf{x}, \mathbf{y}$ one gets the following explicit form of variables transformation with the accuracy of cubic terms:

$$\begin{aligned} \mathbf{q}(\mathbf{Q}, \mathbf{P}) &= \mathbf{Q} - \Psi_{\mathbf{P}}(\mathbf{Q}, \mathbf{P}), \\ \mathbf{p}(\mathbf{Q}, \mathbf{P}) &= \mathbf{P} + \Psi_{\mathbf{Q}}(\mathbf{Q}, \mathbf{P}). \end{aligned} \tag{15}$$

In our next step, we are going to apply the illustrated and discussed algorithm to construct an asymptotic solution of a spring-type swinging pendulum in the resonance case.

5 Swinging pendulum

We consider a pendulum with two-degrees-of-freedom, i.e. a point mass hanging on a massless spring swinging in a vertical plane (Fig. 1). Note that this problem has been considered by Vitt and Gorelik [25] in order to illustrate internal resonance behaviour. In addition, this problem has been studied using classical perturbation approaches for instance in references [8, 26].

In the majority of investigations, only partial results are reported due to applied complicated methods of analysis. Furthermore, it is extremely difficult to apply the mentioned techniques to study resonance oscillations. In what follows, we solve this problem in a relatively simple manner using the introduced invariant normalization approach with the application of a parametric change of variables.

The following notation is introduced: k—spring stiffness; l—spring length in mass equilibrium position; lx, ly—mass coordinates; lR—spring length, where

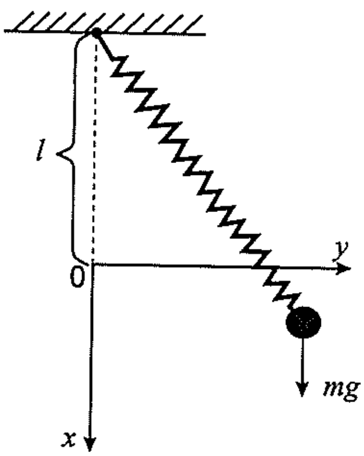

Fig. 1 Scheme of elastic two degrees of freedom pendulum

$$R = \sqrt{(1+x)^2 + y^2}.$$

The Descartes system of coordinates originates in point O, mass equilibrium position and axes x, y are oriented vertically and horizontally, respectively (see Fig. 1).

Spring tension is defined via the following formula

$$T = \frac{k}{l_0}(lR - l_0) + \frac{\delta k}{l_0^3}(lR - l_0)^3,$$

where l_0 is the length of non-stretched spring.

Potential and kinetic energies have the forms

$$\begin{aligned} E_p &= -mglx + \frac{k}{2l_0}\left[(lR - l_0)^2 - (l - l_0)^2\right] \\ &\quad + \frac{\delta k}{4l_0^3}\left[(lR - l_0)^4 - (l - l_0)^4\right], \\ E_c &= \frac{m}{2}\left[\left(\frac{dx}{dt'}\right)^2 + \left(\frac{dy}{dt'}\right)^2\right] \\ &= \frac{mgl}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right], \end{aligned}$$

where t' is the non-dimensional time; $t = \omega t'$ denotes dimensional time, and $\omega = \sqrt{g/l}$ is the frequency of linear oscillations regarding the vertical axis.

Introducing dimensionless impulses: $u = \dot{x}$ and $v = \dot{y}$, the following Hamilton function $H = (E_c + E_p)/(mgl)$ is defined

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2) - x + \frac{k}{2mg}\frac{l_0}{l}\frac{(lR - l_0)^2 - (l - l_0)^2}{l_0^2} \\ &\quad + \frac{\delta k}{4mg}\frac{l_0}{l}\frac{(lR - l_0)^4 - (l - l_0)^4}{l_0^4}. \end{aligned}$$

Note that a constant in H is taken to satisfy $H(0, 0, 0) = 0$.

The studied equation takes the following Hamiltonian form

$$\frac{dx}{dt} = \frac{\partial H}{\partial u}, \qquad \frac{du}{dt} = -\frac{\partial H}{\partial x},$$
$$\frac{dy}{dt} = \frac{\partial H}{\partial v}, \qquad \frac{dv}{dt} = -\frac{\partial H}{\partial y}.$$

The following Hamiltonian linear, square and cubic terms are distinguished

$$H = H_1 + H_2 + H_3, \quad H_1 = \left(-1 + K\lambda\left(1 + \delta\lambda^2\right)\right)x,$$

$$H_2 = \frac{1}{2}\left(u^2 + v^2 + \omega_1^2 x^2 + \omega_2^2 y^2\right),$$

$$H_3 = K\left[\frac{1}{2}\left(1 + \delta\lambda^2(2\lambda + 3)\right)xy^2 + \delta\lambda(\lambda + 1)^2 x^3\right],$$

$$K = \frac{k}{mg}, \quad \lambda = \left(\frac{l}{l_0} - 1\right),$$

$$\omega_1^2 = K(\lambda + 1)\left(1 + 3\delta\lambda^2\right), \quad \omega_2^2 = K\lambda\left(1 + \delta\lambda^2\right),$$

where ω_1 and ω_2 are the non-dimensional frequencies of linear oscillations regarding x and y, respectively.

Owing to equilibrium condition the linear term equals zero, and hence

$$-1 + K\lambda\left(1 + \delta\lambda^2\right) = 0 \quad \Rightarrow \quad \omega_2^2 = 1. \tag{16}$$

We are focused on oscillations investigation on equilibrium neighbourhood for large t and in the resonance condition case $\omega_1^2 = 4\omega_2^2$. The latter condition yields

$$\begin{aligned}(\lambda + 1)&\left(1 + 3\delta\lambda^2\right) - 4\lambda\left(1 + \delta\lambda^2\right) \\ &= 1 - 3\lambda + \delta\lambda^2(3 - \lambda) = 0,\end{aligned} \tag{17}$$

and hence one may present λ in the form of the series regarding δ:

$$\lambda = \frac{1}{3} + \frac{8}{81}\delta + \frac{40}{729}\delta^2 + \frac{728}{19\,683}\delta^3 + \cdots. \tag{18}$$

Remark In the taken approximation, only square and cubic terms appear in the Hamiltonian. In other words, in equations of motion only linear and quadratic terms exist. It means that the applied approximation is not sufficient to study the Duffing phenomenon exhibited by dependence of oscillations frequency versus amplitude of non-linear oscillator. However, it is sufficient to study non-linear resonance interaction of two obtained vertical and horizontal modes. In the obtained asymptotic solution, the frequencies of the modes are proportional to the first power of oscillation amplitudes owing to the resonance phenomenon. Amplitudes of the modes are changed in a periodic manner.

For small amplitudes, the constructed asymptotic solution governs the process analytically with a high accuracy, what is shown by numerical simulation. Now we are aimed on construction of the normal form. In the case of resonance (5), it can be obtained via the algorithm described in Sect. 4, which essentially simplifies the computations.

6 Resonance normal form

Applying equilibrium condition (16) and the resonance condition (17) our Hamiltonian takes the simplified form

$$H = H_0 + F,$$

$$H_0 = (1/2)\left(u^2 + v^2 + 4x^2 + y^2\right),$$

$$F = (3/2)xy^2 + Ax^3,$$

$$A = \frac{16\delta}{9} + \frac{16\delta^2}{243} + \frac{208\delta^3}{6561} + O\left(\delta^4\right).$$

Now we apply the earlier described algorithm. First, a general solution of non-perturbed system with Hamiltonian H_0 is found:

$$\begin{aligned} x(t) &= X\cos 2t + \frac{U}{2}\sin 2t, \\ y(t) &= Y\cos t + V\sin t, \\ u(t) &= U\cos 2t - 2X\sin 2t, \\ v(t) &= V\cos t - Y\sin t. \end{aligned} \tag{19}$$

Solution to the non-perturbed system is substituted into $R_1 = F(x(t), y(t))$. As a result one gets a quasi-periodic function $R_1(t, X, Y, U, V)$ regarding time, and applying the operators L and L_1 one finds normal form and function Ψ associated with the first approximation of the forms:

$$\begin{aligned} \bar{F}_1 &= L\left(R_1(t, X, Y, U, V)\right) \\ &= \frac{3}{8}\left(-V^2X + UVY + XY^2\right), \\ \Psi &= \frac{3}{64}\left(4XYV + 3UV^2 + 5Y^2U\right) \\ &\quad + A\left(\frac{1}{4}X^2U + \frac{1}{24}U^3\right). \end{aligned} \tag{20}$$

Observe that the resonance normal form does not depend on ε, i.e. on the non-linear damping system integral (it is exactly the same as in the case of linear damping).

The system is integrated using the Birkhoff variables of the form

$$z_1 = (1/\sqrt{2})U + \sqrt{2}iX, \qquad z_2 = V + iY. \tag{21}$$

Relations (19) present a canonical transformation with valency $2i$. Normal form of the first approximation is $\tilde{H} = \tilde{H}_0 + \tilde{F}$, where:

$$\tilde{H}_0 = i(2z_1\bar{z}_1 + z_2\bar{z}_2),$$

$$\tilde{F} = -\frac{3\sqrt{2}}{16}(Z_1\bar{Z}_2^2 - \bar{Z}_1 Z_2^2).$$

Equations associated with the non-perturbed part of the Hamiltonian read

$$\dot{z}_1 = 2iz_1, \qquad \dot{z}_2 = iz_2,$$

and they have the following solutions

$$z_1 = Z_1 e^{2it}, \qquad z_2 = Z_2 e^{it}. \tag{22}$$

Equations associated with the perturbed part of the Hamiltonian are as follows

$$\dot{Z}_1 = \frac{\partial\tilde{F}}{\partial\bar{Z}_1} = \frac{3\sqrt{2}}{16}Z_2^2,$$
$$\dot{Z}_2 = \frac{\partial\tilde{F}}{\partial\bar{Z}_2} = -\frac{3\sqrt{2}}{8}Z_1\bar{Z}_2. \tag{23}$$

They possess two integrals: $H_0 = \text{const}$ and $F = \text{const}$. The first integral presents the law of energy conservation

$$2|Z_1|^2 + |Z_2|^2 = C^2, \tag{24}$$

where $2|Z_1|^2$ is the first energy (horizontal oscillations), and $|Z_2|^2$ represents the second mode energy (vertical oscillations).

In order to obtain the equation for the second mode energy, it is differentiated twice and taking into account (23) one gets

$$\frac{d|Z_2|^2}{dt} = \dot{Z}_2\bar{Z}_2 + \dot{\bar{Z}}_2 Z_2 = -\frac{3\sqrt{2}}{8}(Z_1\bar{Z}_2^2 + \bar{Z}_1 Z_2^2),$$

$$\frac{d^2|Z_2|^2}{dt^2} = -\frac{3\sqrt{2}}{8}(2\dot{Z}_1\bar{Z}_2^2 + 4\dot{Z}_2\bar{Z}_1 Z_2)$$
$$= \frac{9}{32}(-|Z_2|^4 + 4|Z_1|^2|Z_2|^2).$$

The energy conservation law yields the second mode energy of the following form

$$\frac{d^2|Z_2|^2}{dt^2} = \frac{9}{32}(-3|Z_2|^4 + 2C^2|Z_2|^2).$$

Note that taking $\xi = |Z_2|^2$ as a variable, the last equation can be interpreted as a motion of a material point with coordinate ξ subjected to force action generated by potential $-\Pi(\xi)$ (motion in a potential well).

The equation has the following energy integral

$$\frac{1}{2}\left(\frac{d}{dt}\xi\right)^2 + \Pi = E, \quad \Pi = \frac{9}{32}(\xi^3 - C^2\xi^2). \tag{25}$$

In Fig. 2, a graph of potential energy $\Pi(\xi)$ for $C = 1$ is reported. Values of the second mode energy $|z_2|^2$ are bounded by intersection points of continuous and dashed curves. Potential energy achieves its minimum $\xi = (2/3)C^2$ for $-(1/24)C^6$.

7 Periodic solutions

In monograph [27], a periodic solution to the considered system is obtained using the Lyapunov method. Below, we illustrate how to find it using (23) and integrals (24), (25). Recall that a periodic solution corresponds to the minimum point $\xi = |Z_2|^2 = (2/3)C^2$, $|Z_1|^2 = (1/6)C^2$ (see Fig. 2). Therefore, a solution to (23) may be sought in the following form

$$Z_1 = \sqrt{(1/6)}Ce^{i\alpha t},$$
$$Z_2 = \sqrt{(2/3)}Ce^{i(\beta t + \gamma)}. \tag{26}$$

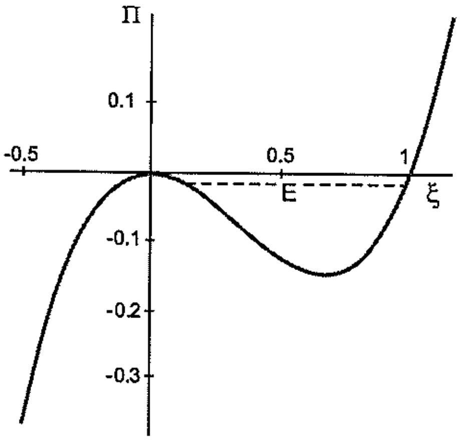

Fig. 2 Effective potential energy $\Pi(\xi)$

Substitution of (26) into (23) yields

$$\alpha = \pm(\sqrt{3}/4)C, \qquad \beta = \pm(\sqrt{3}/8)C,$$
$$\gamma = \pm\pi/4.$$

In order to get a solution to equations governing normal form $H_0 + \tilde{F}$, a theorem of superposition of solutions should be applied. Namely, solutions (26) should be substituted into solution (22). Finally, a family of periodic solutions of cubic power amplitude accuracy and with two arbitrary parameters t_1 and C is obtained

$$z_1 = \frac{C}{\sqrt{6}} e^{i(2\pm C\sqrt{3}/4)(t-t_1)},$$

$$z_2 = \sqrt{\frac{2}{3}} C e^{i[(1\pm C\sqrt{3}/8)(t-t_1)\pm\pi/4]}.$$

Applying variables x, y, u, v, the discussed solution takes the form

$$\begin{aligned} x &= \big(1/\sqrt{12}\big)C\sin\big(2\pm C\sqrt{3}/4\big)(t-t_1), \\ y &= \sqrt{2/3}C\sin\big[\big(1\pm C\sqrt{3}/8\big)(t-t_1)\pm\pi/4\big]. \end{aligned} \tag{27}$$

Note that using the Lyapunov approach in monograph [27], many complicated formulas have been applied. In contrast, our approach is direct and simple in application. Furthermore, it yields explicitly (without any additional study) stability estimation of the obtained periodic solution. Namely, the Lagrange theorem on equilibrium stability in the point of potential energy (25) minimum justifies it directly. Additionally, so far a stability of the periodic solution (27) has not been investigated.

8 Oscillations with a small perturbation of periodic solution

We are aimed now on finding an analytical solution to the studied problem applying small perturbation to solution (27) (this problem has been analysed only numerically so far). Note that the analysis of a perturbed solution corresponds to the study of linear oscillations of function $\xi(t)$ governed by (25) for small deviation of energy E in comparison to its minimum $E = -(1/24)C^6$. Frequency $\tilde{\omega}$ and periods T_0 of linear oscillations are as follows

$$\tilde{\omega} = \sqrt{\Pi''\big((2/3)C^2\big)} = \frac{3}{4}C, \qquad T_0 = \frac{8\pi}{3C}, \tag{28}$$

and amplitude solutions have the following forms

$$\xi = |z_2|^2 = C^2\left[\frac{2}{3} + 2\alpha\cos\frac{3}{4}C(t-t_2)\right],$$

$$|z_1|^2 = C^2\left[\frac{1}{6} - \alpha\cos\frac{3}{4}C(t-t_2)\right].$$

The computation leads to a general solution with four arbitrary constants $C, t_1, |\alpha| \ll 1, t_2$

$$z_1 = \frac{C}{\sqrt{6}}\left(1 - 3\alpha\cos\frac{3}{4}C(t-t_2)\right)e^{i(2\pm C\sqrt{3}/4)(t-t_1)},$$

$$z_2 = \sqrt{\frac{2}{3}}C\left(1 + \frac{3}{2}\alpha\cos\frac{3}{4}C(t-t_2)\right) \times e^{i[(1\pm C\sqrt{3}/8)(t-t_1)\pm\pi/4]}.$$

Separation of real and imaginary parts of z_1 and z_2 yields the solution with respect to co-ordinates x and y of the form

$$\begin{aligned} x &= (1/\sqrt{12})C\left(1 - 3\alpha\cos\frac{3}{4}C(t-t_2)\right) \\ &\quad\times\sin\big(2\pm C\sqrt{3}/4\big)(t-t_1), \\ y &= \sqrt{2/3}C\left(1 + \frac{3}{2}\alpha\cos\frac{3}{4}C(t-t_2)\right) \\ &\quad\times\sin\big[\big(1\pm C\sqrt{3}/8\big)(t-t_1)\pm\pi/4\big]. \end{aligned} \tag{29}$$

The obtained solution governs oscillational process with frequencies close to 1 and 2, and essentially smaller modulation frequency being proportional to the vibrational amplitude C.

If perturbation parameter $\alpha = 0$, then solution (29) yields periodic solution (27).

9 Oscillations for a finite energy value

For an arbitrary energy deviation E with respect to its minimum $-(1/24)C^6 < E < 0$ one may integrate (25), and hence the period of functions oscillation $\xi(t)$ can be found. For this purpose, the following transformation is introduced

$$\xi = C^2\eta(\tau), \quad \tau = (3/8)Ct, \quad \frac{32}{9C^6}E = \eta_0^3 - \eta_0^2.$$

Equation (23) is transformed to the following form

$$\frac{1}{4}\left(\frac{d\eta}{d\tau}\right)^2 + \eta^3 - \eta^2 = \eta_0^3 - \eta_0^2. \tag{30}$$

Its solution is

$$\tau = \frac{1}{2}\int \frac{d\eta}{\sqrt{P(\eta)}},$$
$$P(\eta) = \eta_0^3 - \eta_0^2 - \eta^3 + \eta^2 = (\eta - \eta_0)(\eta_1 - \eta)(\eta - \eta_2), \tag{31}$$

where η_0, η_1 and η_2 are the roots of polynomial $P(\eta)$:

$$\eta_1 = \frac{1}{2} - \frac{1}{2}\eta_0 + \frac{1}{2}\sqrt{(1-\eta_0)(1+3\eta_0)},$$
$$\eta_2 = \frac{1}{2} - \frac{1}{2}\eta_0 - \frac{1}{2}\sqrt{(1-\eta_0)(1+3\eta_0)}.$$

An analytical solution is given by the elliptic functions. Period T of function $\xi(t)$ is expressed via the elliptic integral $K(k)$ of the form

$$T = \frac{8\tau_0}{3C}, \quad \tau_0 = \int_{\eta_0}^{\eta_1} \frac{d\eta}{\sqrt{P(\eta)}} = \frac{2}{\sqrt{\eta_1 - \eta_2}}K(k),$$
$$k = \sqrt{\frac{\eta_1 - \eta_0}{\eta_1 - \eta_2}}, \quad K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}. \tag{32}$$

In the case of functions $K(k)$, the following known asymptotic series for $k \ll 1$ and $k' = \sqrt{1-k^2} \ll 1$ can be applied:

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + O(k^4)\right),$$
$$K(k) = \ln(4/k') + \frac{1}{4}\left(\ln(4/k') - 1\right)k'^2 + O\left(k'^4 \ln k'\right). \tag{33}$$

Observe that when oscillations start from the equilibrium position, the initial conditions are as follows: $x = x_0,\ u = 0,\ y = y_0,\ v = 0$. Magnitude τ_0 can be presented as the function of one argument $\varepsilon = |y_0/x_0|$. For this purpose we introduce quantities x_0 and y_0, and T depends on them in the following form

$$C = \sqrt{4x_0^2 + y_0^2}, \qquad \eta_0 = \frac{y_0^2}{C^2} = \frac{\varepsilon^2}{4(1+\varepsilon^2/4)},$$
$$\eta_1 = \frac{1+\sqrt{1+\varepsilon^2}}{2(1+\varepsilon^2/4)}, \qquad \eta_2 = \frac{1-\sqrt{1+\varepsilon^2}}{2(1+\varepsilon^2/4)},$$
$$k^2 = \frac{1}{2} + \frac{1}{2\sqrt{1+\varepsilon^2}} - \frac{\varepsilon^2}{4\sqrt{1+\varepsilon^2}},$$
$$k'^2 = \frac{1}{2} - \frac{1}{2\sqrt{1+\varepsilon^2}} + \frac{\varepsilon^2}{4\sqrt{1+\varepsilon^2}}. \tag{34}$$

Substituting (34) into (31), the following formula for period estimation is obtained

$$T = \frac{8\tau_0}{3C}, \quad \tau_0(\varepsilon) = 2\frac{(1+\varepsilon^2/4)^{1/2}}{(1+\varepsilon^2)^{1/4}}K(k). \tag{35}$$

A variation argument ε in interval $0 < \varepsilon \le 2\sqrt{2}$ forces variation of argument k from 1 to 0. The value of $\varepsilon = 2\sqrt{2}$ corresponds to $k = 0$ and the minimal value of $\tau_0 = \pi$. The obtained value corresponds to period T_0 associated with potential energy minimum (28).

A further increase of ε forces k^2 to be negative, and hence formula (35) can not be applied.

Application of both dependence (34) for $k'(\varepsilon)$ and asymptotics for $K(k)$ yields the following asymptotic series for the period for $\varepsilon = |y_0/x_0| \ll 1$:

$$\tau_0(\varepsilon) = \ln\left(32/\varepsilon^2\right) + (3/8)\varepsilon^2 + (3/256)\left(-17 + 5\ln\left(32/\varepsilon^2\right)\right)\varepsilon^4 + O\left(\varepsilon^6 \ln\varepsilon\right). \tag{36}$$

Observe that the main term of series (36) is in agreement with the asymptotics of the period found in reference [27].

In Fig. 3, the relation $\tau_0(\varepsilon)$ governed by formulas (35) (solid curve) and (36) (dashed curve) is compared. Both dependencies well coincide for (are in good agreement) $\varepsilon = |y_0/x_0| < 1.5$.

On the other hand, when ratio y_0/x_0 tends to 0, then period T approaches infinity. In this case, we get a non-periodic solution called separatrice. In the next

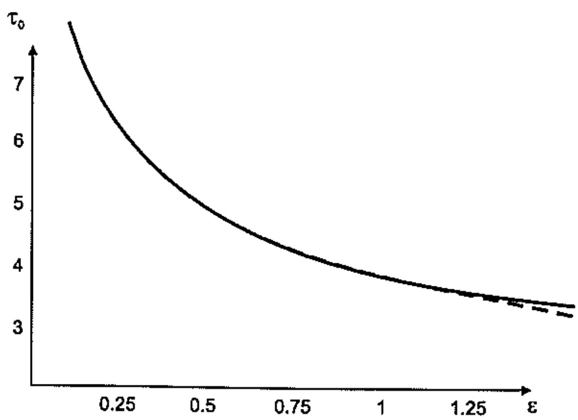

Fig. 3 Comparison of the relation $\tau_0(\epsilon)$ computed by formula (24) (*solid curve*) and formula (25) (*dashed curve*)

sections, separatrice solutions as well as periodic solution in the neighbourhood of separatrices are presented.

10 Break of a vertical oscillation

In the limiting case $E \to 0$, (36) has an exact solution $\eta = \operatorname{sech}^2(\tau)$ and one gets

$$|z_1| = (C/\sqrt{2})\tanh(\tau), \qquad |z_2| = C\operatorname{sech}(\tau), \qquad \tau = (3/8)Ct. \tag{37}$$

This solution is realised physically if the pendulum is strongly stretched vertically and then relaxed. Recall that following results presented in monograph [27] the initiated vertical pendulum oscillations will be unstable regarding arbitrary small horizontal deviation. The described phenomenon is referred to as the breaking process of vertical pendulum oscillations (see [27]). Our proposed method essentially simplifies considerations. Namely, for small horizontal perturbation periodic process of energy pumping from one to another mode occurs, and it can be described via elementary functions in more details and more accurate manner in comparison to those reported in the cited monograph.

Let us consider a solution to the studied problem in the resonance case for initial conditions $x(0) = x_0$, $y(0) = y_0$, $u(0) = v(0) = 0$. In this case a solution to full Hamilton equations with Hamiltonian $H_0 + \bar{F}_1$ is yielded by Zhuravlev's theorem, and it is represented by oscillations with frequencies 2 and 1 and slowly changed amplitudes $X(t)$ and $Y(t)$, and with period T of the form

$$x = X(t)\cos 2t, \qquad y = Y(t)\cos t.$$

Modulations $X(t)$ and $Y(t)$ corresponding to vertical $x(t)$ and horizontal $y(t)$ displacements can be represented by the sum of solutions delayed on the period τ_0 of the form

$$\begin{aligned} X(t) &= x_0\big(|\tanh(\tau - \tau_0/2)| \\ &\quad + |\tanh(\tau - 3\tau_0/2)| - 1\big), \\ Y(t) &= 2x_0\big(\operatorname{sech}(\tau - (1/2)\tau_0) \\ &\quad + \operatorname{sech}(\tau - (3/2)\tau_0)\big), \quad \tau = (3/4)x_0 t. \end{aligned} \tag{38}$$

In Fig. 4, a comparison of both numerical (for $x_0 = 0.1$, $y_0 = 0.01$) and analytical solutions is shown.

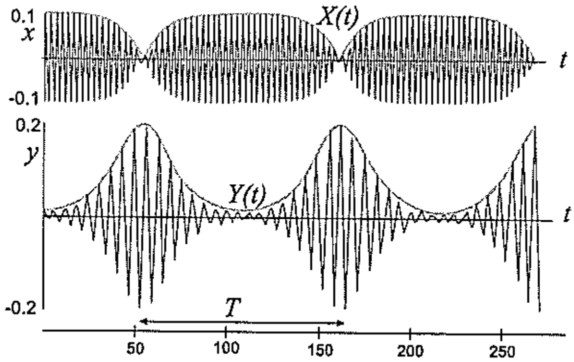

Fig. 4 Analytically (*thick curve*) and numerically (*thin curve*) obtained time histories

Solid thick curves correspond to analytical functions computed via formulas (38). Analytical estimation of modulation (38) and period of energy pumping from one to another mode coincides well with numerical results. Asymptotic formula (34) yields $\tau_0 = 8.075$; $C = 0.20025$; $T = 8\tau_0/(3C) \approx 107.5$. On the other hand, numerical computation of the studied equations presented in Fig. 3 gives the period $T \approx 108$.

The proposed solutions (37) and (38) are not presented in the monograph [27], and hence this is a new result. The same holds for the reported solution in vicinity of the periodic solution (29) and formulas (32), (35), (36) for modulation period.

The obtained solution governs also breaking process of the pendulum horizontal oscillations. They are unstable, and hence they transfer into vertical oscillations.

11 Advantages of the method of invariant normalization

Although this problem has already been discussed in references [11], [14], but it is reconsidered again in the light of the obtained results.

In spite of that, the classical method of normal forms is originated from Poincaré and Birkhhoff works, it is rather rarely applied by non-linear analysts. A reason is that it requires a lot of transformations to achieve construction of a normal form. Algorithms devoted to construction of a normal form given for instance in [28, 29] also require extensive computations to achieve the goal. Besides, classical algorithms essentially differ from resonance and non-resonance, as well as autonomous and non-autonomous cases. Researchers dealing with the investigation of non-linear oscillations either use commercial packages or apply other approaches. This observation holds swinging pendulum with an elastic length [8, 26, 27].

We propose an alternative approach removing the occurring drawbacks. Normal form (20) can be found without a computer or it can be realized via simple programming using either "Maple", or "Mathematica" being applicable simultaneously for resonance-non-resonance, and autonomous-non-autonomous cases.

Furthermore, a normal form oriented analysis has many advantages in comparison to the method of Lyapunov, among the others. First, normal form construction yields two first integrals in the beginning, and the third one is easily found, which allows us to find the full system integral. Second, the obtained integrals allow us to construct the Lyapunov function, and hence the problem of periodic solution stability is solved simultaneously. Third, the method of invariant normalization is interesting from the methodological point of view.

Finally, all of the obtained results have clearly illustrated the mentioned advantages of the method of invariant normalization in comparison to classical normal form approaches.

Concluding remarks

Our main goal has been achieved, i.e. the asymptotic solution regarding small amplitude of oscillations of the spring-type swinging pendulum in the resonance case $\omega_1 : \omega_2 = 1 : 2$ has been constructed. For particular initial conditions, one may find periodic solution (27), where amplitudes of horizontal and vertical modes are constant and frequencies differ from the linear case on amount of first power of oscillation amplitude. Therefore, resonance condition yields strengthening of the Duffing effect. Stability of this solution is yielded by integral (25) and the Lagrange theorem. In other words, for any other initial conditions the oscillations take place with periodically changed in time amplitudes. It has been derived analytical formula (4) governing changes of amplitude T on initial data (35) and (36). Furthermore, elementary functions (38) allow to describe vertical oscillation break during horizontal pendulum coordinate excitation. The comparison of both analytical solution (38) and numerical one exhibits a high accuracy of the presented approach regarding amplitudes variations and the period of these variations. Finally, it is expected that the used novel approach can be also applied to detect analytically an intersection of stable and unstable homoana heteroclinic orbits exhibited by high dimensional dynamical systems, and hence to predict chaotic dynamics. This problem has been recently addressed in the monograph [30], where classical perturbation approaches have been used.

References

1. Poincaré, A.: Collected Works in Three Volumes, vol. II. Nauka, Moscow (1972), in Russian
2. Birkhoff, G.D.: Dynamical Systems. American Mathematical Society, New York (1927)
3. Bruno, A.D.: The Restricted Three Body Three Problem. Walter de Gruyter, Berlin (1994)
4. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Academic Press, Boston (1978)
5. Arnold, V.I.: Geometrical Methods in the Theory of Ordinary Differential Equations. Springer, New York (1983)
6. Arnold, V.I., Kozlov, A.I., Neishtadt, A.I.: Mathematical Aspects of Classical and Celestial Mechanics Dynamical Systems III. In: Encyclopaedia of Mathematical Sciences, vol. 3. Springer, Berlin (1993)
7. Verhulst, F.: Differential Equations and Dynamical Systems, 2nd edn. Springer, Berlin (2006)
8. Nayfeh, A.H.: Perturbation Methods. Wiley, New York (1973)
9. Awrejcewicz, J., Krysko, V.A.: Introduction to Asymptotic Methods. Chapmann and Hall/CRC Press, Boca Raton/London/New York (2006)
10. Zhuravlev, V.F.: Introduction to Theoretical Mechanics. Nauka Fizmatlit, Moscow (1997)
11. Zhuravlev, V.F.: Invariant normalization of non-autonomus Hamiltonian systems. J. Appl. Math. Mech. **66**(3), 347–355 (2002)
12. Petrov, A.G.: A parametric method of construcing Poincaré mapping in hydrodynamic systems. J. Appl. Math. Mech. **66**(6), 905–921 (2002)
13. Petrov, A.G.: Modification of invariant normalization method of Hamiltonians with help parametrization of the canonical transformations. Dokl. Phys. **47**(10), 742–744 (2002)
14. Petrov, A.G.: Invariant normalization of non-autonomus Hamiltonian systems. J. Appl. Math. Mech. **68**(3), 357–367 (2004)
15. Awrejcewicz, J., Petrov, A.G.: On the normal forms of Hamiltonian systems. Nonlinear Dyn. **48**, 185–197 (2007)
16. Zaripov, M.N., Petrov, A.G.: Nonlinear oscillations of a swinging spring. Dokl. Phys. **49**(11), 691–696 (2004)
17. Petrov, A.G.: Nonlinear oscillations of a swinging spring for resonance. Mech. Solids **40**(5) (2006)
18. Bruno, A.D., Petrov, A.G.: On computation of the Hamiltonian normal form. Dokl. Phys. **51**(10), 555–559 (2006)
19. Bruno, A.D.: Analytical form of differential equations. Trans. Mosk. Math. Soc. **26**, 199–239 (1972)

20. Cherry, T.M.: On the solution of Hamiltonian systems of differential equations in the neighbourhood of a singular point. Proc. Lond. Math. Soc. Ser. 2 **27**, 151–170 (1928)
21. Gustavson, F.G.: On constructing formal integrals of a Hamiltonian system near an equilibrium point. Astron. J. **71**, 670–686 (1966)
22. Belitskii, G.R.: Normal Forms, Invariants and Local Mapping. Naukova Dumka, Kiev (1979)
23. Hori, G.I.: Theory of general perturbations with unspecified canonical variables. Astron. Soc. Jpn. **18**(4), 287–296 (1966)
24. Deprit, A.: Canonical transformations depending on a small parameter. Celest. Mech. **1**(1), 12–30 (1969)
25. Vitt, A., Gorelik, G.: Elastic pendulum oscillation as an example of parametrically coupled linear systems. J. Tech. Phys. **3**, 294–307 (1933)
26. Bogaevskii, V.N., Povzner, A.Y.: Algebraic Methods in Nonlinear Theory of Perturbation. Nauka, Moscow (1987), in Russian
27. Starzhinskiy, V.M.: Applied Methods of Nonlinear Oscillations. Nauka, Moscow (1977), in Russian
28. Mersman, W.A.: A new algorithm for the Lie transformation. Celest. Mech. **3**(1), 81–89 (1970)
29. Markeev, A.P., Sokolskiy, A.G.: Algorithms of Hamiltonian Systems Normalization. Institut of Applied Mathematics of AN SSSR, Moscow (1976)
30. Awrejcewicz, J., Holicke, M.M.: Smooth and Nonsmooth High Dimensional Chaos and the Melnikov-Type Methods. World Scientific (2007)