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Active control of two degrees-of-freedom building-ground system

JAN AWREJCEWICZ and PAWEŁ OLEJNIK

The presented idea of active control of buildings is valid for the general concept of stabilization of some constructions subjected to an excitation coming, for instance, from earthquakes. The problem is analyzed in a two case studies describing not excited and externally loaded two degrees-of-freedom dynamical system. 2-DOF linear system is used to model the building-ground interactions. Algorithm that actively controls the system has been implemented and tested as well. The task finally reduces the investigations conducted in the work to estimation of the control force which could guarantee the sufficient minimization of the cost function proposed. On the basis of both analytical derivations and numerical analysis performed a few time histories of convergence of control force, components of the matrix of gain and accelerations of system masses have been illustrated and shortly described.

Key words: active control, constructions stabilization, 2-DOF dynamical systems

1. Introduction

Active control of building structures has been widely studied theoretically, using a variety of control strategies [5]. Control of structural dynamics basically means regulation of the corresponding characteristics for the purpose of providing its controlled response to the effect of the external dynamic loads.

During the last decade, a rapid development has been observed in the field of structural control. Although these achievements in civil engineering are quite recent, it needs to ensure the comfort and safety of occupants and protect the integrity of the structure. There are studied many theoretical and experimental problems of control in order to reduce structural vibrations under any unpredictable conditions [1,2].

The use of active control on a passively base isolated building model is proposed in [4] to counteract vibrations due to a low power excitation. The base isolated building is modeled as a three degrees-of-freedom rigid body. The rotation at the center of this model of building is controlled by means of vertical synchronized actuators. The control methods which are applied to the base isolated model and then compared are as follows: optimal control, eigenvalue assignment using state and output feedbacks.

The Authors are with Technical University of Łódź, Department of Automatics and Biomechanics, 1/15 Stefanowskiego Str., 90-924 Łódź, Poland. E-mails: Jan.Awrejcewicz@p.lodz.pl, Pawel.Olejnik@p.lodz.pl
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A method for design of supplemental dampers in multistory structures is presented in [6]. Active optimal control theory is adapted to design linear passive viscous or viscoelastic devices dependent on their deformation and velocity. The theory using a linear quadratic regulator is used to exemplify the procedure. With the use of Riccati equation the design is aimed at minimizing a performance cost function, which produces a most suitable minimal configuration of devices while minimizing their effect. A progression visible in investigations of earthquake resistant structures from passively controlled base-isolated structures to actively controlled structures has now led to hybrid structures. As it is shown in [7] a hybrid actuator-damper-bracing control system can be composed of viscoelastic dampers and hydraulic actuators in the form of the passive and active controllers, which are installed on the brace and connected to the building floor. The intelligent control strategy is designed to maximally utilize the passive damper and to minimally utilize the active energy. Thus, the passive controller of the hybrid system is designed for small moderate earthquakes and the active controller works for large earthquakes. The hybrid control system is studied under existing earthquake records and the ground motions together with assumption of tectonic movements of seismic plates.

A generalized minimum variance algorithm for the control of civil engineering structures is described in [3]. The algorithm needs the knowledge of the seismic excitation model to drive the autoregressive moving average exogenous model of the structure. The control is designed such that the variance of the generalized cost function is minimized. To demonstrate the effectiveness of this control technique, some simulation tests using a single degree-of-freedom structure were performed.

2. The problem without external loading

The active control of buildings concerns on constructions being analyzed on the base of a general approach. Control of the investigated not subjected to any external loading two degrees-of-freedom continuous dynamical system represents a little particular case of the active control law used in the paper. There is derived a controlling scheme applied to the analyzed 2DOF system, which after an initial placement disturbance x_0 at any initial time t_0 evaluates until the moment of time t_f is reached. The system is not influenced by any external disturbances affecting it from the surrounding environment.

Let the following system of differential equations be given as follows

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = 0, \tag{1}$$

where: A – $(n \times n)$ matrix of structure parameters,
B – $(n \times n)$ matrix of executing (regulatory) elements,
x – n-dimensional state vector of the system.

Our task focuses on searching for the control force $u(t)$ that would satisfactorily minimize the cost function J in time $t = t_f$:

$$J = \frac{1}{2}x(t_f)^T \theta x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & T \\ T^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \tag{2}$$

where: θ, Q, T and R are dependent on time t symmetric weighting coefficients matrices.

2.1. The Riccati equation

Let us assume the existence of such an expression representing the cost function

$$J = \theta(x,t)|_{t_0}^{t_f} + \int_{t_0}^{t_f} \varphi(x,u,t)dt. \tag{3}$$

Hamiltonian of the cost function (3) is defined by the formula

$$H(x,u,\lambda,t) = \varphi - \dot{x}^T \frac{\partial \varphi}{\partial \dot{x}} \equiv \varphi + \dot{x}^T \lambda, \tag{4}$$

where $x(t)$, $\lambda(t)$ are canonical variables.

It results from (3) that φ takes only scalar values, so the summation in (4) can be done if and only if the notation $(dx/dt)^T \lambda$ is a scalar value too. A transposition of scalars does not make any difference in (4), therefore we have

$$H(x,u,\lambda,t) = \varphi + (\dot{x}^T \lambda)^T = \varphi + \lambda^T \dot{x}. \tag{5}$$

Differentiating (4) with respect to x yields

$$\frac{\partial H}{\partial x} = \frac{\partial \varphi}{\partial x} - \frac{\partial \dot{x}^T}{\partial x}\frac{\partial \varphi}{\partial \dot{x}} - \dot{x}^T \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial \dot{x}}. \tag{6}$$

The composite function theorem allows to simplify (6) to the following form

$$\frac{\partial H}{\partial x} = -\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial \dot{x}}. \tag{7}$$

One can guess from (4) that $\lambda = -\partial\varphi/\partial(dx/dt)$. Successively, taking into consideration (7) we get the following Euler-Lagrange equation

$$\frac{\partial \lambda}{\partial t} = -\frac{\partial H}{\partial x}. \tag{8}$$

The Hamiltonian given in (4) satisfying (2) is as follows

$$H(x,u,\lambda,t) = \frac{1}{2}\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & T \\ T^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \lambda^T (Ax(t) + Bu(t)). \tag{9}$$

After differentiating the last equation with respect to λ, x and u one gets

$$\frac{\partial H}{\partial \lambda} = \dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{10}$$

In the background of Euler-Lagrange equation (8) and with application of the properties $\lambda^T A = A^T \lambda$, $T^T = T$ the following relation is obtained

$$\begin{aligned}\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{1}{2}\left(x(t)^T Qx(t) + u(t)^T T^T x(t) + x(t)^T Tu(t) + u(t)^T Ru(t)\right)\right] \\ +A^T\lambda(t) = -Qx(t) - Tu(t) - A^T\lambda(t).\end{aligned} \tag{11}$$

Derivative of(9) with regard to u reads

$$\frac{\partial H}{\partial u} = -\frac{\partial}{\partial u}\left[\frac{1}{2}\left(x(t)^T Qx(t) + u(t)^T T^T x(t) + x(t)^T Tu(t) + u(t)^T Ru(t)\right)\right] + B^T\lambda.$$

If $H = const$ then $\partial H/\partial u = 0$, and hence

$$\frac{\partial H}{\partial u} = T^T x(t) + Ru(t) + B^T\lambda = 0. \tag{12}$$

From the above we find

$$u(t) = -R^{-1}\left(T^T x(t) + B^T\lambda\right), \tag{13}$$

and $R(t)$ is assumed to be reversible in the forthcoming analysis. Substitution of (14) in (1) yields

$$\dot{x}(t) = x(t)\left(A - BR^{-1}T^T\right) - BR^{-1}B^T\lambda. \tag{14}$$

Equations (11) and (14) constitute a system of $2n$ linear ordinary differential equations of the form

$$\begin{gathered}\begin{bmatrix}\dot{x}(t)\\ \dot{\lambda}(t)\end{bmatrix} = \begin{bmatrix} A_n & -BR^{-1}B^T \\ -Q_n & -A_n^T\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}, \\ A_n = A - BR^{-1}T^T, \quad Q_n = Q - TR^{-1}T^T.\end{gathered} \tag{15}$$

Solution of the system (15) can be expressed by means of final (terminal) conditions and the matrix $\varphi(t_f, t)$ decomposed to a four component matrices

$$\begin{bmatrix}x(t_f)\\ \lambda(t_f)\end{bmatrix} = \begin{bmatrix}\varphi_{11}(t) & \varphi_{12}(t)\\ \varphi_{21}(t) & \varphi_{22}(t)\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} \tag{16}$$

where φ_{11}, φ_{12}, φ_{21}, φ_{22} are $n \times n$ component matrices.

Let λ at final time t_f be equal to

$$\lambda(t_f) = \theta(t_f)x(t_f). \tag{17}$$

Taking into account (17) in (16) one obtains

$$x(t_f) = \varphi_{11}(t_f,t)x(t) + \varphi_{12}(t_f,t)\lambda(t). \tag{18}$$

$$\theta(t_f)x(t_f) = \varphi_{21}(t_f,t)x(t) + \varphi_{22}(t_f,t)\lambda(t). \tag{19}$$

Substituting (18) into (19)

$$\theta(t_f)\varphi_{11}(t_f,t)x(t) + \theta(t_f)\varphi_{12}(t_f,t)\lambda(t)) = \varphi_{21}(t_f,t)x(t) + \varphi_{22}(t_f,t)\lambda(t), \tag{20}$$

and consequently

$$\lambda(t) = (\varphi_{22}(t_f,t) - \theta(t_f)\varphi_{12}(t_f,t))^{-1}(\theta(t_f)\varphi_{11}(t_f,t) - \varphi_{21}(t_f,t))x(t). \tag{21}$$

The expressions in braces will be from now denoted by K_x, therefore (21) can be rewritten in the form

$$\lambda(t) = K_x(t)x(t), \tag{22}$$

where K_x is $n \times n$ Riccati matrix.

Putting $\lambda(t)$ expressed by (22) into (13), we get

$$u(t) = -R^{-1}(T^T + B^T K_x)x(t). \tag{23}$$

Observe that the sought control law $u(t)$ is governed by a linear combination of state vectors. It is known that the best method of determination of the K_x matrix and thereby estimation of $u(t)$ is utilization of a proper convergent numerical procedure. By substitution of (22) into (15), we have

$$\dot{x}(t) = (A_n - BR^{-1}B^T K_x)x(t),$$

$$\dot{\lambda}(t) = K_x\dot{x}(t) + \dot{K}_x x(t) = -Q_n x(t) - A_n^T(t)K_x x(t), \tag{24}$$

$$\left(\dot{K}_x + K_x A_n + A_n^T K_x - K_x BR^{-1}B^T K_x + Q_n\right)x(t) = 0. \tag{25}$$

Because validity of (25) covers all $x(t) \neq 0$, then the expression in braces preceding $x(t)$ must equal zero. Therefore, K_x has to satisfy the following Riccati matrix equation

$$\left(\dot{K}_x - K_x A_n - A_n^T K_x + K_x BR^{-1}B^T K_x - Q_n\right)x(t) = 0, \tag{26}$$

also with inclusion of the final condition $K_x(t_f) = \theta(t_f)$, in which $\theta(t_f)$ is the known value. Of course, (26) is numerically integrable in (t_0, t_f). Thus, we are able to determine both the Riccati matrix K_x and the gain matrix F_x for all $t \in (t_0, t_f)$:

$$F_x(t) = -R^{-1}(T^T + B^T K_x). \tag{27}$$

Finally, the following control law can be proposed

$$u(t) = F_x x(t). \tag{28}$$

2.2. Stationary control law

Let A, B, Q and R be some time-independent (constant) matrices. Matrix K_x is then a particular solution to such an equation $K_x(t_f) = \theta(t_f)$. If the solution is stable (in accordance to Lyapunov criterion) for the sufficiently large t_f, then the approximate values of K_x converge to a constant value K_c. Evidently, one can assume that the matrix K_x is also time-independent and is in its final state K_c represented by the solution to the Riccati equation

$$-K_cA_n - A_n^TK_c - Q_n + K_cBR^{-1}B^TK_c = 0. \tag{29}$$

For the purpose of illustration of the principle of Riccati matrix stability the simple mechanical system visible in Fig. 1 of two degrees-of-freedom modeling the dynamics of building-ground connection will be analyzed.

In order to control the system under investigation we additionally dispose of a virtual generator of control force $u(t)$ which is placed between the two masses of the building-ground system. Equations of motion with an auxiliary force $u(t)$ controlling the second material point (building) are written in the following form

$$\begin{aligned} m_1\ddot{x}_1(t) &= -k_1x_1(t) - c_1\dot{x}_1(t) - u(t), \\ m_2\ddot{x}_2(t) &= u(t). \end{aligned} \tag{30}$$

We propose a function of balance, which will precise the starting point as an input in the strategy of searching for the control force characteristics. Our fundamental task is to secure the amplitude of vibrations of the second mass minimized as low as possible in the preset interval of time $t \in (t_0, t_f)$.

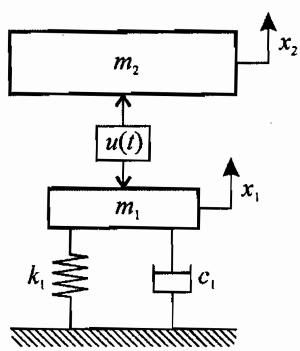

Figure 1. Active building-ground model.

The cost function (2) will be given in a more consisted form. Therefore, let Q, R and T satisfy

$$Q = \begin{bmatrix} q_1+q_2 & -q_2 & 0 & 0 \\ -q_2 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = r, \quad T = 0, \tag{31}$$

where q_1, q_2, r are weighting coefficients.

Additionally, let θ be equal to zero in (t_0, t_f). Matrices A and B are constant and time-independent:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & 0 & \frac{c_1}{m_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ \frac{1}{m_2} \end{bmatrix}. \tag{32}$$

Taking into consideration Q, R, T and θ in (2) (given by (31) and (32)), the following form of cost function can be introduced:

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t) \;\; u^T(t)\right] \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \tag{33}$$

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt, \tag{34}$$

where both elements of the sum are scalars.

Product $x(t)^T Qx(t)$, where $x(t)^T = [x_1, x_2, dx_1/dt, dx_2/dt]$ (see Fig. 1) is equal to

$$\begin{aligned} x^T(t)Qx(t) &= [x_1 \; x_2 \; \dot{x}_1 \; \dot{x}_2] \begin{bmatrix} q_1+q_2 & -q_2 & 0 & 0 \\ -q_2 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &= (x_1(q_1+q_2) - x_2 q_2)x_1 + (-x_1 q_2 + x_2 q_2)x_2, \end{aligned} \tag{35}$$

and $u(t)^T Ru(t) = ru^2$. Thus, we seek a cost function in the form

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(q_1 x_1^2 + q_2(x_2 - x_1)^2 + ru^2\right) dt. \tag{36}$$

2.3. Numerical results for the not externally loaded system

The following set of parameters is assumed: $m_1 = 1500$kg, $m_2 = 11000$kg, $k_1 = 975000$N/m, $c_1 = 10800$Ns/m. Optimal control will be guaranteed for: $q_1 = 5$, $q_2 = 1$, $r = 10^{-10.3}$. Such estimation of control parameters allows for proving of correctness of the method used as well as for making a good observations of influence of the control force on the investigated two degrees-of-freedom building-ground system. Components of the structure matrices A and B in (32) representing a 'control mechanisms' are: $a = k_1/m_1 = 650$, $b = 1/m_1 = 6.66e-4$, $c = 1/m_2 = 9.9091e-5$, $d = c_1/m_1 = 7.2$.

Estimation of $u(t)$ will be preceded by a numerical integration of (26) by means of the standard 4-*th* order Runge-Kutta procedure.

Components of Riccati matrix K_x are:

$$K_x = [\alpha_{ij}], \quad \forall i,j \in (1,\ldots,4). \tag{37}$$

Substitution of the symmetric matrix K_x in (26) implies ten subsequent equations:

$$\begin{aligned}
\dot\alpha_{11} &= -2\alpha_{13}a + q_1 + q_2 + \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)b\alpha_{13} - \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)c\alpha_{14},\\
\dot\alpha_{12} &= -\alpha_{23}a - q_2 + \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)b\alpha_{23} - \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)c\alpha_{24},\\
\dot\alpha_{13} &= \alpha_{11} - \alpha_{13}d - \alpha_{33}a + \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)b\alpha_{33} - \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)c\alpha_{34},\\
\dot\alpha_{14} &= \alpha_{12} - \alpha_{34}a + \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)b\alpha_{34} - \frac{1}{r}(-\alpha_{13}b + \alpha_{14}c)c\alpha_{44},\\
\dot\alpha_{22} &= q_2 + \frac{1}{r}(-\alpha_{23}b + \alpha_{24}c)b\alpha_{23} - \frac{1}{r}(-\alpha_{23}b + \alpha_{24}c)c\alpha_{24},\\
\dot\alpha_{23} &= \alpha_{12} - \alpha_{23}d + \frac{1}{r}(-\alpha_{23}b + \alpha_{24}c)b\alpha_{33} - \frac{1}{r}(-\alpha_{23}b + \alpha_{24}c)c\alpha_{34},\\
\dot\alpha_{24} &= \alpha_{22} + \frac{1}{r}(\alpha_{23}b + \alpha_{24}c)b\alpha_{34} - \frac{1}{r}(-\alpha_{23}b + \alpha_{24}c)c\alpha_{44},\\
\dot\alpha_{33} &= 2\alpha_{13} - 2\alpha_{33}d + \frac{1}{r}(-\alpha_{33}b + \alpha_{34}c)b\alpha_{33} - \frac{1}{r}(-\alpha_{33}b + \alpha_{34}c)c\alpha_{34},\\
\dot\alpha_{34} &= \alpha_{23} + \alpha_{14} - \alpha_{34}d + \frac{1}{r}(-\alpha_{33}b + \alpha_{34}c)b\alpha_{34} - \frac{1}{r}(-\alpha_{33}b + \alpha_{34}c)c\alpha_{44},\\
\dot\alpha_{44} &= 2\alpha_{24} + \frac{1}{r}(-\alpha_{34}b + \alpha_{44}c)b\alpha_{34} - \frac{1}{r}(-\alpha_{34}b + \alpha_{44}c)c\alpha_{44}
\end{aligned} \tag{38}$$

Equations (38) are then integrated numerically since the constant components Riccati matrix K_c results in the following estimation

$$K_c = \lim_{t\to\infty} K_x(t). \tag{39}$$

The knowledge of all components of K_c permits for calculation of the time-dependent matrix of gain, and hence

$$F_x(t) = -R^T B^T K_x = [f_1(t), f_2(t), f_3(t), f_4(t)]. \tag{40}$$

A final form of the resulting time-dependence of control force (control law) is given in the form

$$u(t) = F_x(t)x(t) = f_1x_1 + f_2x_2 + f_3\dot{x}_1 + f_4\dot{x}_2. \tag{41}$$

Note that F_x approaches a constant solution value at F_c for a sufficiently large time t_f.

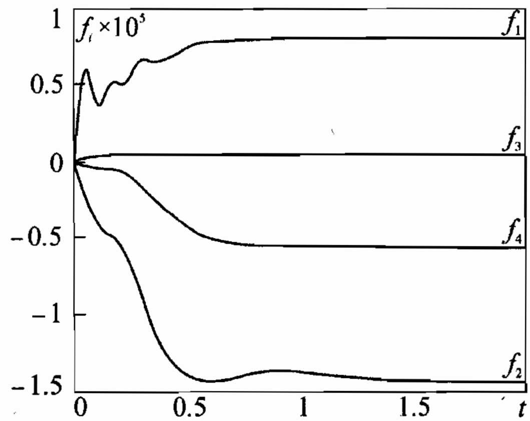

Figure 2. Time history of components f_i of the gain matrix $F_x(t)$.

Fig. 2 presents time histories of components f_i for $i = 1,\ldots,4$ of gain matrix (40). Numerically computed limits of F_x are found:

$$F_c = -R^{-1}B^TK_c = [0.8232\times10^5,\ -0.14126\times10^6,\ 0.5926\times10^4,\ -0.5403\times10^5].$$

Stationary control law is on the basis of the above governed by the general equation

$$u(t) = F_cx(t). \tag{42}$$

Our analysis confirms a strong dependence existing between the control force $u(t)$ and the gain matrix F_x. We mentioned that for any building-ground balance coefficients q_1, q_2, r, the system under investigation should be optimally controlled. With regard to (42) and for the test values $r_1 = 0.5012\times10^{-10}$, $r_2 = 10r_1$ and initial conditions $x_1 = 0.1$, $x_2 = 0$, $dx_1/dt = 0$, $dx_2/dt = 0$ two time histories of the control force $u(t)$ are shown in Fig. 3.

Time histories of acceleration of the second mass m_2 are shown in Fig. 4. Acceleration of that second controlled mass is definitely better damped for the case of the system being under the active control. The obtained results can be corrected (improved) by modification of balance coefficients q_i or r. To do this, one of them is assumed to be constant while the second one is searched for the more accurate value.

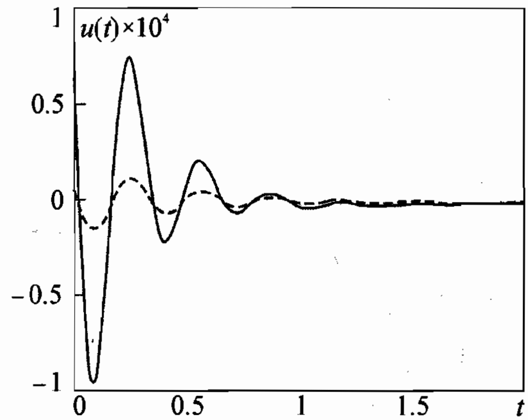

Figure 3. Variations of control force for the coefficients r_1 (solid line) and r_2 (dash line).

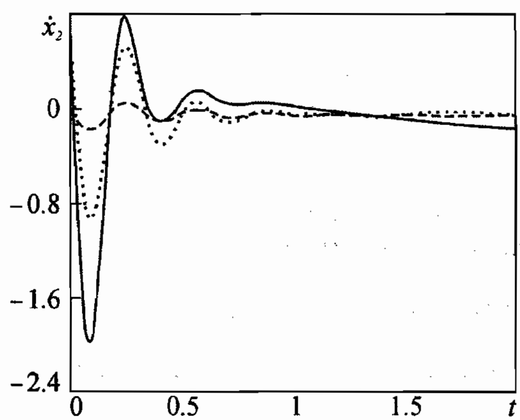

Figure 4. Time histories of acceleration d^2x_2/dt^2 for the passive (solid line) and active control of the second mass for $r_1 = 0.5012 \times 10^{-10}$ (dot line) and $r_2 = 10r_1$ (dash line).

3. The 2-DOF system with an external loading

Let us invoke the system of equations (1) and let it be supplemented with an auxiliary vector of external loading $z(t)$ of dimension n. A general form of the system of motion is given below

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)z(t), \tag{43}$$

where this new matrix $D(t)$ indicates a point of application of the external loading.

If a seismic excitation is assumed to be of a deterministic type the law of the optimal control for the analyzed dynamic construction can be found after minimization of the second order function of effectiveness

$$J(x,z,u) = \frac{1}{2}x^T(t_f)\theta(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) & T(t) \\ S^T(t) & N(t) & L(t) \\ T^T(t) & L^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix} dt, \tag{44}$$

where θ, Q, S, T, N, L and R are the symmetric weighting matrices.

3.1. The Riccati modified equation

Using (4) the Hamiltonian satisfying the cost function J written in (44) is as follows

$$H(x,u,\lambda,t) = \frac{1}{2} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) & T(t) \\ S^T(t) & N(t) & L(t) \\ T^T(t) & L^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix} + \lambda^T \dot{x}. \tag{45}$$

where $\dot{x}$ is expressed with (43).

Differentiation of (45) with respect to λ produces

$$\frac{\partial H}{\partial \lambda} = \dot{x}(t). \tag{46}$$

Similarly to the previous derivations the Euler-Lagrange equation (8) takes the form

$$\frac{\partial \lambda}{\partial t} = -\frac{\partial H}{\partial x} = -Q(t)x(t) - A^T(t)\lambda(t) - S(t)z(t) - T(t)u(t). \tag{47}$$

The partial derivative $\partial H/\partial u$ of (45) together with assumption of $H = const$ is

$$\frac{\partial H}{\partial u} = R(t)u(t) + B^T(t)\lambda(t) + T^T(t)x(t) + L^T(t)z(t) = 0. \tag{48}$$

By calculating $u(t)$ from (48) and applying it successively to (46) and (47) a system of $2n$ linear differential equations is found to be

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A_n & -BR^{-1}B^T \\ -Q_n & -A_n^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} D_n \\ -S_n \end{bmatrix} z, \tag{49}$$

where: $A_n = A - BR^{-1}T^T$, $Q_n = Q - TR^{-1}T^T$, $D_n = D - BR^{-1}L^T$, $S_n = S - TR^{-1}L^T$. Solution to the (49) in which all state variables and matrices are time-dependent is equivalent to the solution of (16) with an additional particular solution

$$\begin{bmatrix} x(t_f) \\ \lambda(t_f) \end{bmatrix} = \phi(t_f,t) \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \tag{50}$$

where ϕ is a transition matrix. Taking the final condition $\lambda(t_f) = \theta(t_f)x(t_f)$ and then substituting it to (50), one gets

$$\lambda(t) = (\phi_{22}(t_f,t) - \theta\phi_{12}(t_f,t))^{-1}((\theta\phi_{11}(t_f,t) - \phi_{21}(t_f,t)x(t) + r_2(t) - \theta r_1(t))). \quad (51)$$

Finally,

$$\lambda(t) = K_x(t)x(t) + r(t), \quad (52)$$

where K_x is the $(n \times n)$ Riccati matrix, $r(t)$ represents a filter of the external excitation $z(t)$.

Substituting $\lambda(t)$ in (48), one finds

$$u = -R^{-1}((B^T K_x + T^T)x + B^T r + L^T z). \quad (53)$$

To estimate K_x and r a numerical procedure has to be applied. Finding the time derivative of (52), using $\dot{x}$ expressed by first equation of system (49) and after that comparing the result for $\dot{\lambda}$ with the second equation of system (49), one gets

$$(\dot{K}_x + K_x A_n + A_n^T K_x - K_x B R^{-1} B^T K_x)x = -\dot{r} + (K_x B R^{-1} B^T - A_n^T)r - (K_x D_x + S_n)z. \quad (54)$$

Vectors x and z are arbitrarily chosen so (54) is true if and only if K_x and r are satisfied by the following

$$\begin{aligned} \dot{K}_x &= -K_x A_n - A_n^T K_x + K_x B R^{-1} B^T K_x - Q_n, \qquad K_x(t_f) = \theta(t_f), \\ \dot{r} &= (K_x B R^{-1} B^T - A_n^T)r - (K_x D_x + S_n)z, \qquad r(t_f) = 0. \end{aligned} \quad (55)$$

Assumption of a deterministic form of the applied force reveals a possibility of replacing r in (52) with $K_z z(t)$. The Lagrange multiplier given in (52) takes the form

$$\lambda(t) = K_x(t)x(t) + K_z(t)z(t), \quad (56)$$

with a final condition $\lambda(t_f) = \theta(t_f)x(t_f)$. System of equations (55) is now rewritten

$$\begin{aligned} K_x A_n + A_n^T K_x - K_x B R^{-1} B^T K_x + Q_n &= -\dot{K}_x, \qquad K_x(t_f) = \theta(t_f), \\ K_x D_n + K_z A_z - K_x B R^{-1} B^T K_z - A_n^T K_z + S_n &= -\dot{K}_z, \qquad K_z(t_f) = 0, \end{aligned} \quad (57)$$

where matrix K_x is the solution to the Riccati equation, and K_z is the Lyapunov solution. If it is still valid our assumption for $r = K_z z(t)$ then the control law $u(t)$ that has been introduced in (53) takes the final estimation

$$\begin{aligned} u(t) &= -R^T(t)((B^T(t)K_x(t) + T^T(t))x(t) + (B^T(t)K_z(t) + L^T(t)z(t)) \\ &= F_x(t)x(t) + F_z(t)z(t). \end{aligned} \quad (58)$$

3.2. The second stationary control law

Our considerations are founded on the observation that if $t_f \to \infty$, the state matrices A, B and D as well as those weighting matrices θ, Q, S, T, N, L and R are some constant value matrices. In a consequence, matrices $K_x(t)$ and $K_z(t)$ approach the direction of constant values $\underline{K}_x$ and $\underline{K}_z$ stationary algebraic solutions to the Riccati and Lyapunov equations

$$\begin{aligned} &\overline{K}_x A_n + A_n^T \overline{K}_x - \overline{K}_x B R^{-1} B^T \overline{K}_x + Q_n = 0, \\ &\overline{K}_x D_n + \overline{K}_z A_z - \overline{K}_x B R^{-1} B^T \overline{K}_z + A_n^T \overline{K}_z + S_n = 0. \end{aligned} \tag{59}$$

In order to illustrate the stationary feature of (59) that investigated in Section 2 building-ground system shown in Fig. 1. will be analyzed once again but with a distinction that the ground is being under action of force loading $f(t)$. For the purpose of control force $u(t)$ generation an actuator has been placed between the two existing masses of the 2-DOF dynamical system and will have a determined influence on their displacements. Equations of motion are derived from the Newton law and are supplemented by the additional $f(t)$ term

$$\begin{aligned} &m_1\ddot{x}_1(t) = -u(t) + k_1(f(t) - x_1(t)) + c_1(\dot{f}(t) - \dot{x}_1(t)) \\ &m_2\ddot{x}_2(t) = u(t). \end{aligned} \tag{60}$$

Let system matrices be defined as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & 0 & -\frac{c_1}{m_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ \frac{1}{m_2} \end{bmatrix}, \quad Q = \begin{bmatrix} q_1+q_2 & -q_2 & 0 & 0 \\ -q_2 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k_1}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 \end{bmatrix}, \quad S^T = \begin{bmatrix} -q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{bmatrix}, \quad N = \begin{bmatrix} q_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{61}$$

$$A_z = \begin{bmatrix} 0 & 1 \\ z_1 & z_2 \end{bmatrix}, \quad T = [0\,0\,0\,0]^T, \quad L = [0\,0], \quad \theta = 0, \quad R = r.$$

Substitution of (61) in (44) provides us with the cost function

$$J(x,z,u) = \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix} x \\ z \\ u \end{bmatrix}^T \begin{bmatrix} Q & S & 0 \\ S^T & N & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix} dt$$

$$= \frac{1}{2}\int_{t_0}^{t_f} (x^T Qx + z^T S^T x + x^T Sz + z^T Nz + u^T Ru)dt. \tag{62}$$

Multiplying the appropriate matrices given in (61) by $x^T = [x_1, x_2, \dot{x}_1, \dot{x}_2]$, one finds the final form of cost function

$$J = \frac{1}{2}\int_{t_0}^{t_f} (ru^2 + q_2(x_1 - x_2)^2 + q_1(x_1 - f)^2)dt. \tag{63}$$

3.3. Numerical results for the externally loaded system

Let our system of two degrees-of-freedom be subjected to the loading of a deterministic force $f_0 exp(\lambda t/2)\cos\omega t$. The following set of parameters is assumed: $m_1 =$ 1500kg, $m_2 =$ 11000kg, $k_1 =$ 975000N/m, $c_1 =$ 10800Ns/m. Optimal control will be guaranteed for: $q_1 = 10$, $q_2 = 1$, $r = 10^{-10.3}$, $z_1 = -\omega^2$, $z_2 = -\lambda$, $\omega = 1\mathrm{s}^{-1}$, $\lambda = 0.5\mathrm{s}^{-1}$, $f_0 =$ 1N. Estimation of $u(t)$ will be preceded by a numerical integration of (57) by means of the standard 4-th order Runge-Kutta procedure. This time we have to integrate 18 equations. Ten of them can be repeated as in (38) but the next eight can be found by substitution of K_x from (37) and $K_z = [\zeta_{ij}]$, $\forall\{i \in \{1,\ldots,4\},\ j \in \{1,2\}\}$ to (57).

Components of the structure matrices A and B in (32) representing a 'control mechanisms' are: $a = k_1/m_1, b = 1/m_1, c = 1/m_2, d = c_1/m_1$. For the known values of matrices K_x and K_z the sum of two gain matrices F_x and F_z is

$$F_x + F_z = -R^{-1}B^T(K_x(t) + K_z(t)) = [f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t)], \tag{64}$$

and subsequently

$$u(t) = f_1 x_1(t) + f_2 x_2(t) + f_3\dot{x}_1(t) + f_4\dot{x}_2(t) + f_5 z(t) + f_6\dot{z}(t). \tag{65}$$

Analogously to the time history in Fig. 2 some six components time dependency of the gain sum of matrices (64) have been presented in Fig. 5.

For a sufficiently large t_f the stationary values f_i are:

$$\begin{aligned} \overline{F}_x + \overline{F}_z = [&1.4250764,\ 1.14125345,\ 0.09274669,\ 0.55781137,\\ &-0.13076894,\ 0.40341027] \times 10^5, \end{aligned} \tag{66}$$

and the stationary control law is given in the following manner

$$u(t) = \overline{F}_x(t) + \overline{F}_z(t). \tag{67}$$

With the use of (67) and assumption of $r_1 = 0.50119 \times 10^{-10}$ and $r_2 = 0.50119 \times 10^{-9}$ the time history of control force fluctuations is shown in Fig. 6.

It is seen in Fig. 6 that these two characteristics are the fast stabilizing ones being similar to theirs not loaded case counterparts presented in Fig. 3.

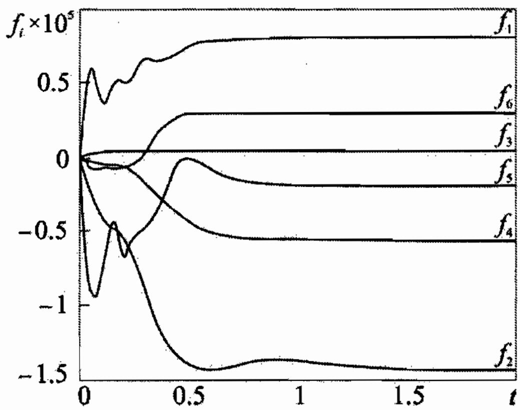

Figure 5. Time histories of components f_i of the gain $F_x(t)+F_z(t)$.

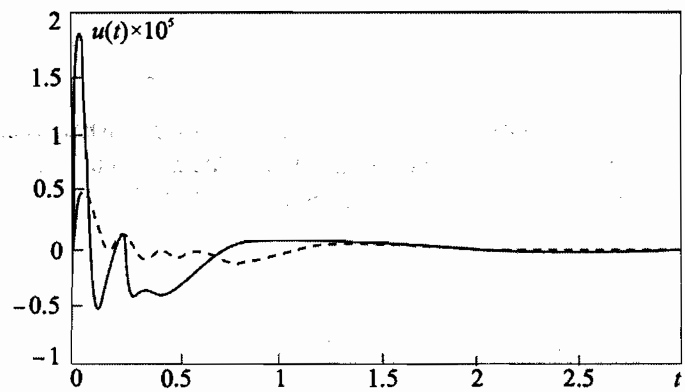

Figure 6. Variations of control force for the coefficients r_1 (solid line) and r_2 (dash line) in the externally loaded case.

4. Conclusions

The presented idea of active control of buildings is valid for the general concept of stabilization of some constructions being not under any external excitation. A two degrees-of- freedom mechanical system was used to model the building-ground interactions, and an algorithm of its active control has been implemented and tested as well. Finally, the problem has reduced the investigations conducted in the work to an estimation of the control force which could guarantee the sufficient minimization of the cost function proposed. On the basis of both analytical derivations and numerical computations performed a few time histories of control force convergence, components of the gain matrix and accelerations of one of the system masses have been illustrated and shortly described. One can observe, that system responses stabilize quite quickly (in about 2 sec.), and the shapes of control force time dependencies (see Figs. 3 and 6) are very perspective, quite fast stabilizing themselves as well as confirm the proper numerical application of the theory of active control used.

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