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On the normal forms of Hamiltonian systems

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Abstract We propose a novel method to analyze the dynamics of Hamiltonian systems with a periodically modulated Hamiltonian. The method is based on a special parametric form of the canonical transformation $\mathbf{q}, \mathbf{p} \to \mathbf{Q}, \mathbf{P}, H(t, \mathbf{q}, \mathbf{p}) \to \bar{H}(t, \mathbf{Q}, \mathbf{P})$

$$\begin{cases} \mathbf{q} = \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{x}} \\ \mathbf{p} = \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{y}} \end{cases}, \qquad \begin{cases} \mathbf{Q} = \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{x}} \\ \mathbf{P} = \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{y}} \end{cases}$$

$$\bar{H}(t, \mathbf{Q}, \mathbf{P}) = \Psi_t(t, \mathbf{x}, \mathbf{y}) + H(t, \mathbf{Q}, \mathbf{P})$$

using Poincaré generating function $\Psi(t, \mathbf{x}, \mathbf{y})$. As a result, stability problem of a periodic solution is reduced to finding a minimum of the Poincaré function.

The proposed method can be used to find normal forms of Hamiltonians. It should be emphasized that we apply the modified concept of Zhuravlev [Introduction to Theoretical Mechanics. Nauka Fizmatlit, Moscow (1997); Prikladnaya Matematika i Mekhanika **66**(3), (2002) in Russian] to define an invariant normal form, which does not require any partition to either autonomous – non-autonomous, or resonance – non-resonance cases, but it is treated in the frame of one approach. In order to find the corresponding normal form asymptotics, a system of equations is derived analogous to Zhuravlev's chain of equations. Instead of the generator method and guiding Hamiltonian, a parametrized guiding function is used. It enables a direct (without the transformation to an autonomous system as in Zhuravlev's method) computation of the chain of equations for non-autonomous Hamiltonians. For autonomous systems, the methods of computation of normal forms coincide in the first and second approximations.

Using this method we will present solutions of the following problems: nonlinear Duffing oscillator; oscillation of a swinging spring; dynamics of solid particles in the acoustic wave of viscous liquid, and other problems.

Keywords Hamiltonian mechanics · Normal form · Swinging oscillator

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1 Parametric form of canonical transformations

A general result of the Hamiltonian systems parametrisation by canonical transformation is formulated in the frame of the following theorem [1].

Theorem 1. *Let the transformation of variables* $\mathbf{q}, \mathbf{p} \to \mathbf{Q}, \mathbf{P}$ *be given in the following parametric form*

$$\begin{aligned} \mathbf{q} &= \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}} \qquad & \mathbf{Q} &= \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}} \\ \mathbf{p} &= \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}, & \mathbf{P} &= \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}. \end{aligned} \tag{1}$$

Then, for any arbitrary function $\Psi(t, \mathbf{x}, \mathbf{y})$ *the following property holds: Jacobians of two transformations* $\mathbf{q} = \mathbf{q}(t, \mathbf{x}, \mathbf{y})$, $\mathbf{p} = \mathbf{p}(t, \mathbf{x}, \mathbf{y})$ *and* $\mathbf{Q} = \mathbf{Q}(t, \mathbf{x}, \mathbf{y})$, $\mathbf{P} = \mathbf{P}(t, \mathbf{x}, \mathbf{y})$ *are identity ones:*

$$\frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\mathbf{x}, \mathbf{y})} = \frac{\partial(\mathbf{Q}, \mathbf{P})}{\partial(\mathbf{x}, \mathbf{y})} = J(t, \mathbf{x}, \mathbf{y}). \tag{2}$$

In the space $J > 0$ *relation (1) with respect to variables* $\mathbf{q}, \mathbf{p} \to \mathbf{Q}, \mathbf{P}$ *transforms the Hamiltonian system* $H = H(t, \mathbf{q}, \mathbf{p})$ *by the following rule*

$$\Psi_t(t, \mathbf{x}, \mathbf{y}) + H(t, \mathbf{q}, \mathbf{p}) = \tilde{H}(t, \mathbf{Q}, \mathbf{P}), \tag{3}$$

where arguments $\mathbf{q}, \mathbf{p}$ *and* $\mathbf{Q}, \mathbf{P}$ *in the Hamiltonians* H *and* $\tilde{H}$ *are expressed via the parameters* $\mathbf{x}, \mathbf{y}$ *in formulas (1).*

In the next step we investigate for which canonical transformations the parametrization exists. A reader is encouraged to follow a survey of normalisation and averaging for Hamiltonian systems reported in references [2–4].

2 Function derivative

A canonical transformation can be represented by generating functions $S_1(t, \mathbf{q}, \mathbf{P})$ and $S_2(t, \mathbf{p}, \mathbf{Q})$ in the following way

$$\begin{aligned} dS_1 &= \mathbf{p}d\mathbf{q} + \mathbf{Q}d\mathbf{P} + (\tilde{H} - H)dt, \quad & \det S_{1\mathbf{qP}} &\neq 0, \\ dS_2 &= -\mathbf{q}d\mathbf{p} - \mathbf{P}d\mathbf{Q} + (\tilde{H} - H)dt, & \det S_{2\mathbf{pQ}} &\neq 0. \end{aligned}$$

The following new generating function is introduced

$$\Phi = \frac{1}{2}\left[S_1(t, \mathbf{q}, \mathbf{P}) - \mathbf{qP} + S_2(t, \mathbf{Q}, \mathbf{p}) + \mathbf{Qp}\right].$$

Its differential form follows [5, 6]

$$d\Phi = \frac{1}{2}\sum_{i=1}^{n} \begin{vmatrix} Q_i - q_i & P_i - p_i \\ dQ_i + dq_i & dP_i + dp_i \end{vmatrix} + (\tilde{H} - H)dt. \tag{4}$$

For $dt = 0$ the differential form $d\Phi$ was introduced by Poincaré (see [7, 8]). He showed that if $\mathbf{Q}(\mathbf{q}, \mathbf{p})$, $\mathbf{P}(\mathbf{q}, \mathbf{p})$ is the canonical transformation, then $d\Phi$ is the full differential and function $\Phi(\mathbf{q}, \mathbf{p})$ exists.

Solving (1) with respect to $\mathbf{x}$, $\mathbf{y}$ and $\Psi_{\mathbf{y}}$, $\Psi_{\mathbf{x}}$ one gets

$$\begin{aligned} \mathbf{x} &= \frac{1}{2}(\mathbf{q} + \mathbf{Q}), \qquad & \mathbf{y} &= \frac{1}{2}(\mathbf{p} + \mathbf{P}), \\ \Psi_{\mathbf{y}} &= \mathbf{Q} - \mathbf{q}, & \Psi_{\mathbf{x}} &= -\mathbf{P} + \mathbf{p}. \end{aligned} \tag{5}$$

Assuming that the transformation Jacobian (5) differs from zero ($\partial(\mathbf{x}, \mathbf{y})/\partial(\mathbf{q}, \mathbf{p}) \neq 0$), one obtains $d\Phi = d\Psi$, and hence the identity of the function Φ and Ψ follows

$$\Psi(\mathbf{x}, \mathbf{y}) = \Psi\left(\frac{\mathbf{q} + \mathbf{Q}(\mathbf{q}, \mathbf{p})}{2}, \frac{\mathbf{p} + \mathbf{P}(\mathbf{q}, \mathbf{p})}{2}\right) = \Phi(\mathbf{q}, \mathbf{p}).$$

Relations (2) and (5) yield

$$\frac{\partial(\mathbf{x}, \mathbf{y})}{\partial(\mathbf{q}, \mathbf{p})} = 1/J = 2^{-2n}\det(E + A), \qquad A = \frac{\partial(\mathbf{Q}, \mathbf{P})}{\partial(\mathbf{q}, \mathbf{p})},$$

and the condition of non-singularity of transformation (5) is given in the form: $\det(E + A) \neq 0$, where: A – Jacobi matrix; E – unit matrix.

The obtained result is formulated in the form of the following theorem [6].

Theorem 2. *If in the space* $(\mathbf{q}, \mathbf{p}) \in \Omega$ *the transformation* $\mathbf{Q}(\mathbf{q}, \mathbf{p})$, $\mathbf{P}(\mathbf{q}, \mathbf{p})$ *is canonical and none of the eigenvalues of Jacobi matrix* A *is equal to* -1, *then parametrization (1) exists in the space* Ω.

In monograph [8] remarks "with respect to tough" non-invariance of the generating function and concerning a choice of the basis of canonical system coordinates and invariance of the differential Poincaré formula (4) are outlined. It follows that the parametric function $\Psi(\mathbf{x}, \mathbf{y})$ has also an invariant character. If the function $\Psi(\mathbf{x}, \mathbf{y})$ exists for some arbitrary parameters, then it will exist also for any arbitrary transformation of canonical variables.

The condition of existence of parametrization $J \neq 0$ is invariant with respect to the choice of canonical variables, since the equation $\det S_{1\mathbf{qP}} \neq 0$ depends on the choice of canonical variables. The condition on $\det S_{1\mathbf{qP}} \neq 0$ can be violated during a change of canonical variables. Besides, the class of parametrized canonical transformations is essentially wider than the class of canonical transformation through the use of a generating function. For instance, the rotation on amount of $90^\circ : q = -P,\ p = Q$ can not be achieved by generating the function $S(q, P)$, but it can be realized by the parametric function of the form: $\Psi = x^2 + y^2$. These and other advantages of the parametrization in comparison to the method of generating function have been already outlined [1, 9].

Below, we illustrate how an application of equation (3) yields the earlier developed [10, 11] method of invariant normalization of Hamiltonians [5, 12].

3 Invariant normalization of Hamiltonians

The normal form of a Hamilton system is called the normal Birkhoff form [13]. The general compact definition of this form is given in reference [14]. In all cases the generated Hamiltonian is chosen in the form of the simplest quadratic function associated with a linear vibrational system. On the other hand, the definition of the normal form is associated with a choice of the generating Hamiltonian and has non-invariant character [8, 13–16].

In general, there are two widely used methods to construct canonical transfomations leading to the normal form. One of them is based on application of the so called generating functions. This way was chosen by Birkhoff [13]. In the second approach, instead of the generating functions, Lie generators are applied. It seems that the latter approach is more suitable, since it does not need an inverse of the power series required in the case when the generating functions are used.

It is worth noting that Zhuravlev [10, 11] proposed a general criterion of normal Birkhoff form for a perturbed Hamiltonian of the form

$$\bar{H}(t, \mathbf{q}, \mathbf{p}, \epsilon) = H_0(t, \mathbf{q}, \mathbf{p}) + \bar{F}(t, \mathbf{q}, \mathbf{p}, \epsilon),$$
$$\bar{F}(t, \mathbf{q}, \mathbf{p}, \epsilon) = \epsilon \bar{F}_1(t, \mathbf{q}, \mathbf{p}) + \epsilon^2 \bar{F}_2(t, \mathbf{q}, \mathbf{p}) + \cdots$$

Definition. A perturbed Hamiltonian has a normal form if and only if the associated perturbation is the first integral of the non-perturbed form $\frac{\partial \bar{F}}{\partial t} + \{H_0, \bar{F}\} = 0$, where $\{f, g\} = f_{\mathbf{p}} g_{\mathbf{q}} - f_{\mathbf{q}} g_{\mathbf{p}}$ are the Poisson brackets.

There are at least three main advantages of this approach compared to the known ones [8, 13–16].

1°. A solution to the full system of differential Hamilton's equations with the normal form Hamiltonians is obtained through the superposition of the solutions of non-perturbed system and solution of the system with autonomous Hamiltonian equal to $\bar{F}(0, \mathbf{q}, \mathbf{p}, \epsilon)$.

The result has been formulated as a theorem (see [11]).

Theorem (Zhuravlev). *If a system with Hamiltonian $\bar{H}$ satisfies the normal form condition, then the following steps are required to construct a general solution of the corresponding Hamilton's equations*:

(*a*) *find a general solution of the system with Hamiltonian $H_0(t, p, q)$*;
(*b*) *find a general solution of the system defined via the perturbation $\bar{F}(0, p, q, \epsilon)$ under the condition that the explicitly occurred time in this system should be equal to zero.*

Then the general solution of the input non-autonomous system can be presented as matching (in an arbitrary manner) of the obtained solutions (instead of arbitrary constants in solution of the second system, these of the first system are substituted, and vice versa).

2°. An invariant character of this criterion enables normalization without an initial simplification of the non-perturbed part and without splitting into cases of autonomous – non-autonomous, and resonance – non-resonance ones.

3°. Asymptotics of normal form and variables transformations associated with Hamiltonian normalization are found through succesive quadratures of known (on each step) functions.

4 Algorithm of invariant normalization with the help of parametric transformations

Next, we illustrate how equation (3) of Theorem 1 can be transformed to an equivalent one in the Zhuravlev normalization method [5, 12].

Let the Hamiltonian being normalized have the following form:

$$H(t,\mathbf{q},\mathbf{p}) = H_0(t,\mathbf{q},\mathbf{p}) + F(t,\mathbf{q},\mathbf{p},\epsilon),$$
$$F(t,\mathbf{q},\mathbf{p},\epsilon) = \epsilon F_1(t,\mathbf{q},\mathbf{p}) + \epsilon^2 F_2(t,\mathbf{q},\mathbf{p}) + \cdots$$

and let

$$\bar{H}^{(k)}(t,\mathbf{Q},\mathbf{P},\epsilon) = H_0(t,\mathbf{Q},\mathbf{P}) + \bar{F}^{(k)}(t,\mathbf{Q},\mathbf{P},\epsilon)$$

be the k-th order asymptotics of the normal form

$$\bar{F}^{(k)}(t,\mathbf{q},\mathbf{p},\epsilon) = \epsilon \bar{F}_1(t,\mathbf{Q},\mathbf{P}) + \cdots + \epsilon^k \bar{F}_k(t,\mathbf{Q},\mathbf{P})$$

with respect to canonical transformation (1), and let

$$\Psi^{(k)}(t,\mathbf{x},\mathbf{y},\epsilon) = \epsilon\Psi_1(t,\mathbf{x},\mathbf{y}) + \cdots + \epsilon^k\Psi_k(t,\mathbf{x},\mathbf{y})$$

be the k-th order asymptotics of the function $\Psi(t,\mathbf{x},\mathbf{y},\epsilon)$ in relations (1).

Then, it follows from Theorem 1, that the asymptotics $\Psi^{(k)}$ satisfies Equation (3), which can be given in the following form:

$$\begin{aligned}\frac{\partial\Psi^{(k)}}{\partial t} &+ H_0\left(t,\mathbf{x}-\frac{1}{2}\Psi^{(k)}_{\mathbf{y}},\mathbf{y}+\frac{1}{2}\Psi^{(k)}_{\mathbf{x}}\right)\\ &- H_0\left(t,\mathbf{x}+\frac{1}{2}\Psi^{(k)}_{\mathbf{y}},\mathbf{y}-\frac{1}{2}\Psi^{(k)}_{\mathbf{x}}\right)\\ &+ F^{(k)}\left(t,\mathbf{x}-\frac{1}{2}\Psi^{(k)}_{\mathbf{y}},\mathbf{y}+\frac{1}{2}\Psi^{(k)}_{\mathbf{x}}\right)\\ &= \bar{F}^{(k)}\left(t,\mathbf{x}+\frac{1}{2}\Psi_{\mathbf{y}},\mathbf{y}-\frac{1}{2}\Psi_{\mathbf{x}}\right). \end{aligned} \tag{6}$$

The latter result yields a chain of the coefficients of the canonical transformation Ψ_i and the normalized Hamiltonians $\bar{F}_i$ of the form

$$\begin{aligned}&\frac{\partial\Psi_i}{\partial t} + \{H_0,\Psi_i\} + R_i = \bar{F}_i,\\ &\frac{\partial\bar{F}_i}{\partial t} + \{H_0,\bar{F}_i\} = 0; \quad i=1,2,\ldots\end{aligned} \tag{7}$$

Functions R_i are computed successively using the formulas

$$R_1 = F_1, \quad R_2 = F_2 + \frac{1}{2}\{F_1+\bar{F}_1,\Psi_1\},\ldots \tag{8}$$

Observe that if H_0 is the polynom of powers not higher than second order with respect to $\mathbf{q}$ and $\mathbf{p}$, then $R_i,\ i\le k$ are the series coefficients of the function

$$\begin{aligned}&F\left(t,\mathbf{x}-\frac{1}{2}\Psi_{\mathbf{y}},\mathbf{y}+\frac{1}{2}\Psi_{\mathbf{x}}\right)\\ &-\bar{F}^{(k)}\left(t,\mathbf{x}+\frac{1}{2}\Psi_{\mathbf{y}},\mathbf{y}-\frac{1}{2}\Psi_{\mathbf{x}}\right)\\ &+\bar{F}^{(k)}(t,\mathbf{x},\mathbf{y}) = \epsilon R_1 + \epsilon^2 R_2 + \epsilon^2 R_3 + \cdots.\end{aligned} \tag{9}$$

The chain of Equations (7) has been earlier obtained in [10, 11]. The equations are called homologous and are presented in the following form:

$$R_i = \bar{F}_i - \frac{d\Psi_i}{dt}, \quad \frac{d\bar{F}_i}{dt} = 0; \quad i=1,2,\ldots \tag{10}$$

Here full derivatives d/dt are computed using the rule of differentiation of composite functions $\Psi_i(t,\mathbf{x},\mathbf{y})$, $F_i(t,\mathbf{x},\mathbf{y})$, where $\mathbf{x}(t)$, $\mathbf{y}(t)$ (being the functions of time) are defined by a solution of the non-perturbed system of the form

$$\dot{\mathbf{x}} = H_{0\mathbf{y}}, \quad \dot{\mathbf{y}} = -H_{0\mathbf{x}}, \quad \mathbf{x}(t_0)=\mathbf{x}_0, \quad \mathbf{y}(t_0)=\mathbf{y}_0. \tag{11}$$

Instead of $\mathbf{x}$ and $\mathbf{y}$, the solution of (11) is substituted into relations (10), then from the second equation of (10) one may conclude that function $\bar{F}_i$ does not depend on time. Therefore, the integral of the first equation is

$$\begin{aligned}&\int_{t_0}^{t} R_i(t)dt = (t-t_0)\bar{F}_i(t_0,\mathbf{x}_0,\mathbf{y}_0)\\ &+\Psi_i(t_0,\mathbf{x}_0,\mathbf{y}_0) - \Psi_i(t,\mathbf{x},\mathbf{y}).\end{aligned} \tag{12}$$

In fact, this is a key result, since the quadrature (12) defines both normal form and functions Ψ_i through transformation of the variables (1). However, the proposed representation of integral (12) is not always achieved uniquely. The uniqueness is realized if function R_i with the substituted solution (11) is quasi-periodic, i.e. it is the sum of periodic functions with respect to t. In the mentioned case, the integral of R_i is expressed by linear and quasi-periodic functions $f(t)$. One may compute the averaged part $f(t)$ (independent of time), and then match it with the second part of the right hand side of (12). Next, the representation of (12) defines uniquely $\bar{F}_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$, and the function $\Psi_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ with zero time averaged value: $\overline{\Psi_i(t, \mathbf{x}(t), \mathbf{y}(t))} = 0$. The quasi-periodicity condition of R_i yields constraints on the parameters, for which the normal form exists. Below, the obtained result is formally formulated.

Main Theorem. *Asymptotics of the k-th solution of a normal form and the associated transformation of variables exist and are unique, if after a substitution of the solution (11) into the functions R_i, $(i = 1, 2, \ldots, k)$, they are quasi-periodic functions with respect to time. Then, in the right hand side of the integral (12) $\bar{F}_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ is the coefficient of a linear term with respect to t, and $\bar{\Psi}_i(t_0, \mathbf{x}_0, \mathbf{y}_0)$ are terms independent of time.*

Let us address the link between the generating Hamiltonian G and the function Ψ. In the method proposed in [10,11] the transformation $\mathbf{q}, \mathbf{p} \to \mathbf{Q}, \mathbf{P}$ is sought on the phase flow of a Hamiltonian system, i.e.

$$d\mathbf{X}/d\tau = G_Y, \quad d\mathbf{Y}/d\tau = -G_X,$$
$$\mathbf{X}(0) = \mathbf{q}, \ \mathbf{Y}(0) = \mathbf{p}; \quad \mathbf{X}(\epsilon) = \mathbf{Q}, \ \mathbf{Y}(\epsilon) = \mathbf{P}, \quad (13)$$

where: $\tau \in (0 \le \tau \le \epsilon)$ is the helping parameter that plays the role similar to that of time t.

The mentioned transformation is realized in our algorithm using the parametrization. The function governing transformation on the phase flow of Hamiltonian (13) is defined by the equation

$$\Psi_\tau(\tau, \mathbf{x}, \mathbf{y}) = G\left(\mathbf{x} + \frac{1}{2}\Psi_\mathbf{y}, \mathbf{y} - \frac{1}{2}\Psi_\mathbf{x}\right),$$
$$\Psi(0, \mathbf{x}, \mathbf{y}) = 0.$$

Note that $\tau^3 = \epsilon^3$ is computed with the accuracy $\Psi = \epsilon G$. Therefore, asymptotics $\Psi_1 = G_0$, $\Psi_2 = G_1$, of two first methods overlap, and consequently R_1 and R_2 are identical in both approaches. However, one finds differences in next approximations for $R_3, R_4, \ldots$, in spite of the fact that the normal form does not depend on the selection of method.

5 Examples of asymptotical solutions

Highly didactic examples given in references [10, 11] demonstrate essential simplicity of the proposed method in comparison to all existing classical ones. Our method, from the point of view of its simplification, is equivalent to that proposed in [10, 11]. However, it has one more advantage. Namely, the obtained chain of equations for the asymptotics is completely independent of the input Hamiltonian form, i.e. there is no need to distinguish between autonomous and non-autonomous cases. Recall that in the method proposed in reference [10, 11], the non-autonomous system can be reduced to autonomous one by an increase of the system order, and then the chain of asymptotic equations must be derived. Although we are focused on rather simple examples the approach can be also applied to more complex dynamical systems using computer algebra symbolic computation facilities.

We illustrate the introduced method using two examples of solutions to the problems of excited oscillators in resonance case [5]. In order to solve these problems using classical approaches, one has to introduce another normal form definition [14]. This method does not require this additional operation. The normal form is computed directly from the quadrature and then a solution is constructed.

Example 1. Find a solution of excited oscillators of the linear oscillator in resonance.

Example 2. Let the excited oscillations of the non-linear Duffing oscillator be: $\ddot{q} + q = \epsilon(\sin t - q^3 + 2\lambda q)$. Find λ, for which a solution is periodic with the period 2π, and then analyze its stability.

In both examples, the investigated equations are of Hamiltonian type, and they possess the same unperturbed Hamiltonian $H_0 = \frac{1}{2}(q^2 + p^2)$. The associated solution is as follows:

$$q = q_0 \cos(t - t_0) + p_0 \sin(t - t_0),$$
$$p = -q_0 \sin(t - t_0) + p_0 \cos(t - t_0). \quad (14)$$

Solution to Example 1: $R_1 = F_1 = -q \sin t$. Substituting (14), the time periodic function is obtained with the associated integral $\int_{t_0}^{t} R_1(t)dt = -\frac{1}{2}(q_0 \sin t_0 + p_0 \cos t_0)(t - t_0) - \frac{1}{4}(q_0 \cos t_0 + p_0 \sin t_0) + f(t)$. Therefore, one may easily derive normal form $\bar{F}_1$ of coefficients and the function Ψ_1. A solution to the first approximation is obtained in the following way. Since Hamiltonian $\bar{F}(0, Q, P) = -\frac{\epsilon}{2}P$ is associated with the system $\dot{Q} = -\frac{\epsilon}{2}$, $\dot{P} = 0$, its solution is $Q = Q_0 - \frac{1}{2}\epsilon t$, $P = P_0$. Substituting Q and P (instead of q_0 and p_0) into (14), the following solution is obtained $Q = (Q_0 - \epsilon t/2)\cos t + P_0 \sin t$, $P = -(Q_0 - \epsilon t/2)\sin t + P_0 \cos t$ (Zhuravlev's theorem [11]). Then the transformation $Q = q - \frac{1}{2}\epsilon \sin t$, $P = p + \frac{1}{2}\epsilon \cos t$ is carried out with respect to the function Ψ, and the exact solution expressed by input variables is $q = (q_0 - \epsilon t/2)\cos t + (p_0 + \frac{\epsilon}{2})\sin t$. Since the solution of Example 2 is found using the averaging procedure, and it is reported in references [5,10], the normal form method (to compare both approaches) will be further applied.

Solution to Example 2: One finds $R_1 = F_1 = -q \sin t - \lambda p^2 + q^4/4$. Substituting solution (14) into R_1, one finds the coefficient of the normal form series: $\bar{F}_1(0, Q, P) = -\frac{1}{2}P - \frac{\lambda}{2}(Q^2 + P^2) + \frac{3}{32}(Q^2 + P^2)^2$. A periodic solution is represented by a fixed point corresponding to the system

$$\partial \bar{F}_1/\partial Q = Q\left(-\lambda + \frac{3}{8}A^2\right) = 0,$$

$$\partial \bar{F}_1/\partial P = -\frac{1}{2} + P\left(-\lambda + \frac{3}{8}A^2\right) = 0.$$

One finds $Q = 0$, $P = \pm A$ for $\lambda = \frac{3}{8}A^2 \pm \frac{1}{2A}$, where $A = \sqrt{Q^2 + P^2}$ is the amplitude. The dependence $\omega = 1 - \epsilon\lambda$ vs A is called the amplitude–frequency characteristics.

A fixed point is stable, if the corresponding function F_1 achieves its extremum. This yields stability condition for the periodic solution, i.e.

$$\left(\lambda - \frac{3}{8}A^2\right)\left(\lambda - \frac{9}{8}A^2\right) > 0,$$

which fully overlaps with the condition obtained through the averaging procedure.

Let us give one more example [5]. In this case, in order to get a solution, one has to compute higher order approximations, and rather complicated classical procedure approaches are required. Our method yields the solution in a simpler way.

Example 3. Find Poincaré points for time instants $t_n = 2\pi n$ $(n = 0, 1, 2, \dots)$ for the non-linear equation $\ddot{q} = \epsilon^2 \cos t' \cos q$ with accuracy of ϵ^6.

The equation governs various problems of mechanics and physics. One of them is that of vibrational motion of a spherical particle in a fluid, where a flat standing wave occurs [17, 18]. Consider a vertical tube with stiff horizontal roof. In the tube the standing wave is generated and its velocity is governed by the formula $v = A\omega \sin \omega t \cos kz$, where: ω – wave frequency; t – time; k – wave number; z – axis going vertically up; A – amplitude of fluid particle displacement, which is assumed to be small. The frequency and wave number are dependent through sound velocity in the fluid $\omega = kc$. Note that for a particle with radius "a" the following inequality is satisfied $\mu/(\rho k c a^2) \ll 1$, then the Stokes friction and the Bassé force are neglegible in comparison to inertial forces. Then, the equation governing particle dynamics is

$$(\rho + 2\rho_0)\ddot{z}_0 = 3\rho w - 2(\rho_0 - \rho)g,$$

$$w = \partial v + v\partial v/\partial z \approx \partial v/\partial t = A\omega^2 \cos \omega t \cos kz,$$

where ρ and ρ_0 – densities of fluid and solid particle, μ – coefficient of dynamic fluid viscosity.

For the particle of neutral floating $\rho = \rho_0$, the governing equation can be transformed to that of Example 3, in which $q = kz_0$, $t' = \omega t$, $\epsilon = Ak$. In references [17, 18] in order to solve the defined problem the classical averaging technique is applied [19]. The series is developed with respect to the parameter ϵ. In order to solve the problem, three approximations are required. The solution is obtained in the form $q = \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + O(\epsilon^4)$.

In what follows we illustrate how to obtain the solution to the stated problem using our method. The splitting is carried out using the parameter $\delta = \epsilon^2$. Therefore, in order to achieve essentially higher accuracy, i.e. of ϵ^6 order, only two approximations are required. Again they are obtained in a simpler way in comparison to classical approaches.

Solution. The equation of our example is obtained from the system of Hamilton equations with the Hamiltonian $H = \frac{1}{2}p^2 + \delta F_1(t, q, p)$, $F_1 = -\cos t \sin q$.

A solution of the unperturbed system has the form $q = q_0 + p_0(t - t_0)$, $p = p_0$.

The first integration gives $R_1 = F_1$, and the quadrature

$$\int_{t_0}^{t} R_1 dt = -\frac{\cos(t_0 + q_0)}{2 + 2p_0} + \frac{\cos(-t_0 + q_0)}{2 - 2p_0} + f_1(t).$$

Therefore, one gets

$$\bar{F}_1 = 0, \quad \bar{\Psi}_1(t, q, p) = -\frac{\cos(t+q)}{2+2p} + \frac{\cos(-t+q)}{2-2p}. \quad (15)$$

The second integration gives

$$R_2 = -\frac{1}{2}\frac{\partial F_1}{\partial q}\frac{\partial \bar{\Psi}_1}{\partial p} = \frac{1}{4}\cos t \cos q \left(\frac{\cos(t+q)}{(1+p)^2} + \frac{\cos(t-q)}{(1-p)^2}\right)$$

Integral (12) yields the linear part F_2, which is independent of time Ψ_2. The final normal form and the function defining the parametric transformation are as follows:

$$\bar{H} = \frac{1}{2}P^2 + \frac{\delta^2}{16}\left[\frac{1}{(1+P)^2} + \frac{1}{(1-P)^2}\right] + O(\delta^3),$$

$$\Psi\,(0, x, y) = \frac{\delta y}{1-y^2}\cos x - \frac{\delta^2(1-3y^2-2y^4)}{16y(1-y^2)^3}\sin 2x + O(\delta^3).$$

The normal form can not be applied in vicinity of the resonance points: $P = 0$, $P = \pm 1$. Note that even in resonance cases the normal forms can be directly computed using formula (13) [5].

6 Integration of Hamiltonian equations perturbated by damping

Below, we consider dynamics of a unit material mass driven by a potential periodic force $-\varepsilon\partial F(t, q)/\partial q$ embedded in the medium with damping $-\delta\dot{q}$ governed by the equation

$$\ddot{q} = -\epsilon^2 \partial F(t, q)/\partial q - \delta\dot{q}, \quad (16)$$

where ε and δ are the small parameters.

Equation (16) is transformed to the following form

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} + \delta p = -\frac{\partial H}{\partial q}, \qquad H = \epsilon p^2/2 + \epsilon F(t, q). \quad (17)$$

Note that (17) is not Hamiltonian, since damping force is not potential. However, applying the following non-canonical transformation

$$q = \tilde{q}, \quad p = \tilde{p}\exp(-\delta t) \quad (18)$$

the system assumes the following Hamiltonian form

$$\dot{\tilde{q}} = \frac{\partial \tilde{H}}{\partial \tilde{p}}, \quad \dot{\tilde{p}} = -\frac{\partial \tilde{H}}{\partial \tilde{q}}, \qquad \tilde{H} = \epsilon[\exp(-\delta t)\tilde{p}^2/2 + \exp(\delta t)\tilde{F}(t, \tilde{q})]. \quad (19)$$

Our target is to analyze an asymptotic solution of Hamiltonian (19) on the period $0 \le t \le 2\pi$. We remain only the terms ε^3 and $\varepsilon\delta$, in the Hamiltonian $\tilde{H}$, i.e.

$$\tilde{H}(t, x, y, \epsilon) = \epsilon H_1(t, x, y) + \epsilon^2 H_2(t, x, y) + \epsilon^3 H_3(t, x, y) + \epsilon\delta H_4(t, x, y) + \cdots, \quad (20)$$

The system of equations (19) has the standard form and we apply the Poincaré transformation with respect to the period in the following parametric form

$$\begin{cases} \tilde{q}_{n-1} = x - \frac{1}{2}\Psi_y \\ \tilde{p}_{n-1} = y + \frac{1}{2}\Psi_x \end{cases}, \quad \begin{cases} \tilde{q}_n = x + \frac{1}{2}\Psi_y \\ \tilde{p}_n = y - \frac{1}{2}\Psi_x \end{cases}, \quad (21)$$

where function $\Psi(\tau, x, y)$ is found solving a Jacobian type equation.

Points $\tilde{q}_n, \tilde{p}_n$ lie on the trajectory $\tilde{q}(t), \tilde{p}(t)$, defined by the solution of Hamiltonian (19) and they are called Poincaré points. A distance in time between them is equal to period 2π: $\tilde{q}_n = \tilde{q}(2\pi n)$, $\tilde{p}_n = \tilde{p}(2\pi n)$. In what follows recurrent relations (17) may be obtained for periodic and Poincaré points. For this purpose the function Ψ is sought in the following form

$$\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\Psi_3 + \epsilon\delta\Psi_4 + \cdots, \quad (22)$$

Substituting series (22) into Jacobi equation, the following series coefficients (22) are found [6]

$$\begin{aligned}\Psi_1(t,x,y) &= \int_{t_0}^{t} H_1 dt,\\ \Psi_2(t,x,y) &= \int_{t_0}^{t}\left[H_2 - \frac{1}{2}\{H_1,\Psi_1\}\right]dt,\\ \Psi_3(t,x,y) &= \int_{t_0}^{t}\Big[H_3 - \frac{1}{2}(\{H_2,\Psi_1\}+\{H_1,\Psi_2\}) \\ &\quad + \frac{1}{8}(H_{1xx}\Psi_{1y}\Psi_{1y} - 2H_{1xy}\Psi_{1x}\Psi_{1y} \\ &\quad + H_{1yy}\Psi_{1x}\Psi_{1x})\Big]dt,\\ \Psi_4(t,x,y) &= \int_{t_0}^{t} H_4 dt, \ldots,\end{aligned} \tag{23}$$

where $\{f,g\} = f_y g_x - f_x g_y$ denotes the Poisson bracket.

The obtained formulas allow to carry out the full system analysis, which will be illustrated by the following example.

Example. Investigate dynamics of a material point driven by the force $-\epsilon^2 \cos t \cos q$ and damped by $-\delta\dot{q}$.

In this case for the coefficients of the series (20) one gets

$$H_1 = \frac{y^2}{2} + \cos t \sin x, \qquad H_2 = 0, \quad H_3 = 0,$$
$$H_4 = t(-y^2/2 + \cos\tau \sin x).$$

Substituting the above expressions to (23), one gets the following coefficients for the time instant $t = 2\pi$

$$\begin{aligned}&\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\Psi_3 + \epsilon\delta\Psi_4 + \cdots,\\ &\Psi_1 = 2\pi\tfrac{y^2}{2}, \quad \Psi_2 = 0,\\ &\Psi_3 = 2\pi\left(\frac{1}{8}\cos 2x - y^2 \sin x\right), \quad \Psi_4 = -\pi^2 y^2.\end{aligned} \tag{24}$$

Removing from (21) parameters x and y and using (18) the following recurrent relations are obtained for Poincaré points q_n, p_n, $n = 0, 1, 2, \ldots$ of equations (17)

$$\begin{aligned}q_n &= q_{n-1} + 2\pi\epsilon p_{n-1} + O(\epsilon^3),\\ p_n &= p_{n-1}(1 - 2\pi\delta) + \frac{1}{4}\epsilon^3 \sin 2q_n + O(\epsilon^4),\end{aligned} \tag{25}$$

where ϵ is the order of impulse $p_n = O(\epsilon)$ (invariant curves are shown in Fig. 1).

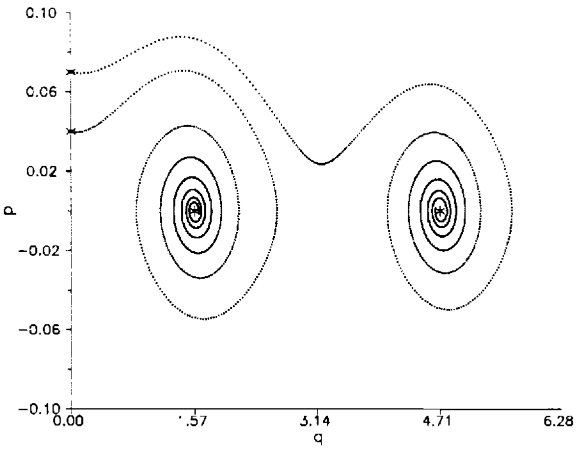

Fig. 1 Invariant curve with damping

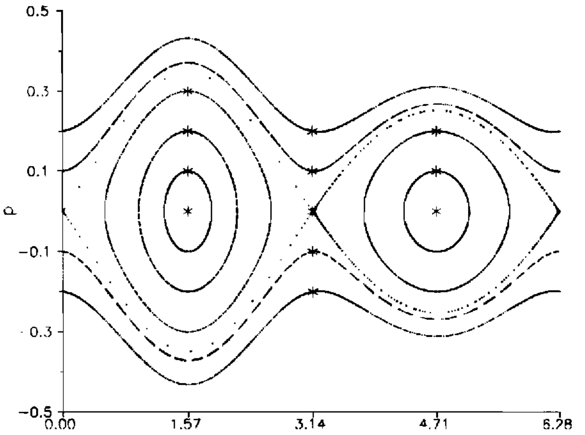

Fig. 2 Invariant curve without damping

For $\delta = 0$ we deal with a Hamiltonian system. It follows from theorem 5 in reference [6] that Poincaré points lie on closed invariant curve $\Psi(q_n, p_n) = const = 2\pi\epsilon^3 C$ with the accuracy of ϵ^5 in the form

$$\frac{p_n^2}{2\epsilon^2} + \frac{1}{8}\cos 2q_n = C. \tag{26}$$

A family of invariant curves is shown in Fig. 2.

For $\delta \neq 0$ our system is non-Hamiltonian and Poincaré points with an increase of n start to lie on curves (26) with a constant $C = c_n$ depended on n. For c_n one may obtain the recurrent relations in the following way. Let a pair $\tilde{q}_{n-1} = q_{n-1}$, $\tilde{p}_{n-1} = p_{n-1}$ and $\tilde{q}_n$, $\tilde{p}_n$ of Hamiltonian (19) lie on the following invariant curve

$$\Psi(\tilde{q}_{n-1}, \tilde{p}_{n-1}) = \Psi(\tilde{q}_n, \tilde{p}_n) = 2\pi\epsilon^3 c_{n-1}. \quad (27)$$

The next pair ($q_n = \tilde{q}_n$, $p_n = \tilde{p}_n(1 - 2\pi\delta)$) and ($\tilde{q}_{n+1} = q_{n+1}$, $\tilde{p}_{n+1}$) lie on the invariant curve with modified constant $2\pi\epsilon^3 c_n$ of the form

$$\Psi(\tilde{q}_n, \tilde{p}_n(1 - 2\pi\delta)) = 2\pi\epsilon^3 c_n. \quad (28)$$

Consequently, taking into account (28) and (27) and applying the Lagrange theorem on gets

$$-2\pi\delta p_n \frac{\partial\Psi}{\partial p_n} = 2\pi\epsilon^3(c_n - c_{n-1}).$$

Substituting the above function Ψ the following recurrent solution is found

$$c_n = c_{n-1} - 2\pi\delta p_n^2/\epsilon^2. \quad (29)$$

For $n = 0$ and having the initial values of q_0, p_0, one finds c_0

$$c_0 = \frac{p_0^2}{2\epsilon^2} + \frac{1}{8}\cos 2q_0,$$

and then constants q_1, p_1, c_1 are found from relations (25) and (26).

It follows from (29) that constant c_n decreases monotonously and tends to minimum of the function.

$$f(x, y) = \frac{y^2}{2\epsilon^2} + \frac{1}{8}\cos 2x.$$

Minimum $x = \pi/2 + \pi n$ $y = 0$ corresponds to stable periodic solution of (16). Poincaré points approach a stable point in a spiral way. Therefore, the found minimum is stable focus (see Fig. 1).

Note that a similar equation is studied in monographs [17, 18]. However, three approximations of averaging KBM method [19] are required in order to get the same result.

7 A swinging oscillator and its normal form

Consider an elastic pendulum with two-degrees-of-freedom, i.e. the heavy point mass swinging in the vertical plane and linked to the massless spring (Fig. 3). The problem is formulated, for instance, in monographs [20–22], where also some investigations with respect to partial results are given.

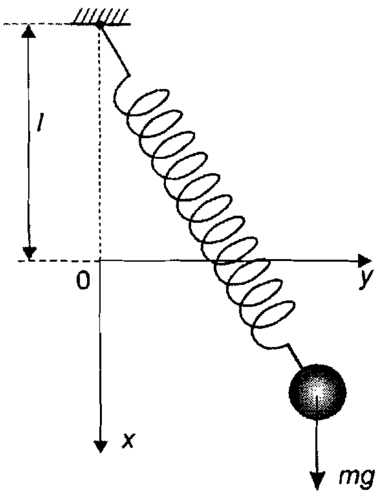

Fig. 3 Scheme of a swinging oscillator

However, high complexity of the used approaches do not allow to carry out a full analysis. Let us apply our method of invariant normalization with the help of parametric variable transformation [23].

The following notation is used: k - spring stiffness; l - its length in the lumped body equilibrium position; m - lumped body mass. In addition, $\omega = \sqrt{g/l}$ (frequency of small vibrations of mathematical pendulum of length l), and

$$\mu = \sqrt{\frac{k}{mg} + 1}.$$

Let us introduce the Cartesian co-ordinates with origin in the lumped system equilibrium position and with the axes along the vertical and horizontal directions. Denote by lx, ly the mass co-ordinates (Fig. 3). The spring length is denoted as lR, where

$$R = \sqrt{(1 + x)^2 + y^2}.$$

The spring tension is $T = k(lR - l_0)/l_0$, where l_0 is the non-stretched spring length. On the other hand, since l is the spring length in the equilibrium position, then $k(l - l_0)/l_0 = mg$. Substituting $l_0 = kl/(k + mg)$ into T, one gets $T = (k + mg)R - k$. Hence, it is clear that $\sqrt{(k+mg)/ml}$ is the vibration mass frequency for vertical spring position, whereas μ is the ratio of this frequency over ω.

The force components acting on the lumped system are: $F_x = mg - T(1+x)/R$, $F_y = -Ty/R$. Newton's equations follow: $ml\ddot{x} = F_x$, $ml\ddot{y} = F_y$.

Potential E_p and kinetic E_c energies of the system have the form

$$E_p = -mglx + \frac{k}{2l_0}(lR - l_0)^2 - \frac{k}{2l_0}(l - l_0)^2,$$

$$E_c = \frac{m}{2}\left[\left(\frac{ldx}{dt'}\right)^2 + \left(\frac{ldy}{dt'}\right)^2\right] = \frac{mgl}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right],$$

where t' is dimensional time and $t = \omega t'$ is undimensional time.

Assuming undimensional impulses $u = \dot{x}$ and $v = \dot{y}$, the following Hamiltonian function $H = (E_c + E_p)/(mgl)$ is found

$$H = \frac{1}{2}(u^2 + v^2) + \frac{\mu^2}{2}(R^2 - 1) - (\mu^2 - 1)(R - 1) - x.$$

The associated constant in H is chosen in the way that $H(0, 0, 0) = 0$.

The system of Hamilton's equations has the form

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial u}, \quad \frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial y}.$$

Next, we are going to investigate the lumped system motion in the neighbourhood of equilibrium (or large time durations t).

Let us change x, y, u, v into $\epsilon x, \epsilon y, \epsilon u, \epsilon v$ and H into $\epsilon^2 H$. Then the system (17) remains unchanged, whereas H takes the following form [22]

$$\begin{aligned} H &= H_0 + \epsilon F_1 + \epsilon^2 F_2, \\ H_0 &= (1/2)(u^2 + v^2 + \mu^2 x^2 + y^2), \\ F_1 &= (1/2)(\mu^2 - 1)xy^2, \\ F_2 &= (1/2)(\mu^2 - 1)(y^4/4 - x^2y^2). \end{aligned}$$

Owing to the illustrated algorithm, the following general solution of the unperturbed system with the Hamilton H_0 is found

$$\begin{aligned} x(t) &= X\cos\mu t + \frac{U}{\mu}\sin\mu t, \\ y(t) &= Y\cos t + V\sin t, \\ u(t) &= U\cos\mu t - \mu X\sin\mu t, \\ v(t) &= V\cos t - Y\sin t \end{aligned} \tag{30}$$

First approximation. The function $R_1 = F_1 = \frac{(\mu^2-1)}{2}xy^2$ is obtained and then a solution of the unperturbed system is substituted to it. In the integral only terms independent of time $(\mu^4 - 4\mu^2)\int_0^t x(t)y^2(t)dt = \mu^2 UY^2 - 2UV^2 - 2UY^2 - 2XYV\mu^2 + \cdots$ remain. In fact, they define the first approximation of the form

$$\Psi_1 = \frac{(\mu^2 - 1)}{2(\mu^4 - 4\mu^2)} \times[\mu^2(UY^2 - 2XYV) - 2UV^2 - 2UY^2]$$

of the parametric transformation.

Since linear (in time) term is equal to zero, the first approximation of the normal form perturbation is equal to zero, i.e. $\bar{F}_1 = 0$.

To conclude, both the normal form and variable transformation associated with the first approximation are computed.

Second approximation. The function $R_2 = F_2 + \frac{1}{2}\{F_1, \Psi_1\}$ is found, where Poisson's bracket is introduced $\{f, g\} = f_u g_x + f_v g_y - f_x g_u - f_y g_v$. We obtain

$$\frac{R_2(x, y, u, v)}{(\mu^2 - 1)} = \frac{1}{2}\left(\frac{y^4}{4} - x^2y^2\right) + \frac{(\mu^2 - 1)}{8(\mu^4 - 4\mu^2)} \times[-y^4\mu^2 + 2y^2v^2 + 2y^4 + 4x^2y^2\mu^2 + 8xyuv].$$

Instead of x, y, u, v the solution of unperturbed system is substituted and integrated again. The multiplier standing by linear (in time) part of this integral defines the normal form of the second approximation

$$\bar{F}_2 = \frac{3\,(\mu^2-1)\,(\mu^2 X^2 + U^2)\,(Y^2+V^2)}{8\,\mu^2\,(\mu^2-4)} - \frac{(\mu^2-1)\,(8+\mu^2)\,(Y^2+V^2)^2}{64\,\mu^2\,(\mu^2-4)}.$$

Note that function Ψ_2 is the part of integral, which does not depend on time, and it is

$$\begin{aligned}\Psi_2 = &\left(\frac{-1}{64\,\mu^2\,(\mu^2-4)}\right)\\ &\times(8\,U\,V^2\,X + 16\,U^2\,V\,Y - 8\,V^3\,Y\\ &+40\,U\,X\,Y^2 - 8\,V\,Y^3 + 16\,U\,V^2\,X\,\mu^2\\ &-40\,U^2\,V\,Y\,\mu^2 + 7\,V^3\,Y\,\mu^2 + 32\,V\,X^2\,Y\,\mu^2\\ &-64\,U\,X\,Y^2\,\mu^2 + V\,Y^3\,\mu^2 + V^3\,Y\,\mu^4\\ &-8\,V\,X^2\,Y\,\mu^4 + 7\,V\,Y^3\,\mu^4)\end{aligned}$$

Let us investigate the system within new coordinates X, Y, U, V and with the Hamilton $H_0(X, Y, U, V) + \epsilon^2 \bar{F}_2(X, Y, U, V)$. Owing to Zhuravlev's theorem, it is sufficient to find integral of the system with Hamiltonian $\epsilon^2 \bar{F}_2(X, Y, U, V)$. The associated system of equations is

$$\begin{aligned}\dot{X} &= \frac{3(\mu^2-1)\epsilon^2}{4\mu^2(\mu^2-4)}(Y^2+V^2)U,\\ \dot{U} &= -\frac{3(\mu^2-1)\epsilon^2}{4(\mu^2-4)}(Y^2+V^2)X,\\ \dot{Y} &= -\frac{3(\mu^2-1)\epsilon^2}{4\mu^2(\mu^2-4)}(\mu^2X^2+U^2)V\\ &\quad+\frac{(\mu^2-1)(\mu^2+8)\epsilon^2}{16\mu^2(\mu^2-4)}(Y^2+V^2)V,\\ \dot{V} &= \frac{3(\mu^2-1)\epsilon^2}{4\mu^2(\mu^2-4)}(\mu^2X^2+U^2)Y\\ &\quad-\frac{(\mu^2-1)(\mu^2+8)\epsilon^2}{16\mu^2(\mu^2-4)}(Y^2+V^2)Y.\end{aligned}$$

The system possesses two integrals: $Y^2 + V^2 = A$, $\mu^2 X^2 + U^2 = B$, and after their account, the linear equations are obtained in the following form

$$\begin{aligned}\dot{X} &= A\frac{3(\mu^2-1)\epsilon^2}{4\mu^2(\mu^2-4)}U,\\ \dot{U} &= -A\frac{3(\mu^2-1)\epsilon^2}{4(\mu^2-4)}X,\\ \dot{Y} &= \frac{(\mu^2-1)\epsilon^2}{\mu^2(\mu^2-4)}\left(-\frac{3}{4}B + \frac{\mu^2+8}{16\mu^2}A\right)V,\\ \dot{V} &= -\frac{(\mu^2-1)\epsilon^2}{\mu^2(\mu^2-4)}\left(-\frac{3}{4}B + \frac{\mu^2+8}{16\mu^2}A\right)V.\end{aligned}$$

Finally, the following linear system is found

$$\begin{aligned}&\dot{X} = \frac{\omega_1}{\mu}U, \quad \dot{U} = -\mu\omega_1 X,\\ &\dot{Y} = \omega_2 V, \quad \dot{V} = -\omega_2 Y,\\ &\omega_1 = \frac{3(\mu^2-1)\epsilon^2}{4\mu(\mu^2-4)}A,\\ &\omega_2 = \frac{(\mu^2-1)\epsilon^2}{\mu^2(\mu^2-4)}\left(\frac{\mu^2+8}{16}A - \frac{3}{4}B\right)\end{aligned}$$

possessing the following solutions

$$\begin{aligned}X &= x_0\cos\omega_1 t + (u_0/\omega_1)\sin\omega_1 t,\\ U &= -\mu x_0\sin\omega_1 t + \mu u_0\cos\omega_1 t,\\ Y &= y_0\cos\omega_2 t + (v_0/\omega_2)\sin\omega_2 t,\\ V &= -y_0\sin\omega_2 t + (v_0/\omega_2)\cos\omega_2 t.\end{aligned}$$

One may substitude the functions into (30) in order to get the full solution.

To sum up, the solution with accuracy of ϵ^4 is successfully constructed.

Let us analyse now solution of the Hamiltonian system (17) as well as the following initial conditions with the Hamiltonian $\bar{H} = H_0 + \epsilon^2 \bar{F}_2$

$$x(0) = x_0, \qquad u(0) = 0, \qquad y(0) = y_0,$$
$$v(0) = 0,$$

and $\varepsilon = 1$.

$$\begin{aligned}
&X = x_0 \cos\omega_1 t, \qquad Y = y_0 \cos\omega_2 t,\\
&U = -\mu x_0 \sin\omega_1 t, \qquad V = -y_0 \sin\omega_2 t,\\
&A = y_0^2, \quad B = \mu^2 x_0^2,\\
&\omega_1 = \frac{3(\mu^2-1)}{4\mu(\mu^2-4)} y_0^2,\\
&\omega_2 = \frac{\mu^2-1}{\mu^2(\mu^2-4)}\left(\frac{\mu^2+8}{16} y_0^2 - \frac{3}{4}\mu^2 x_0^2\right).
\end{aligned}$$

Substituting these expressions into (17), we find

$$\begin{aligned}
x(t) &= x_0[\cos\omega_1 t \cos\mu t - \sin\omega_1 t \sin\mu t]\\
&= x_0 \cos(\omega_1 + \mu)t,\\
y(t) &= y_0[\cos\omega_2 t \cos t - \sin\omega_2 t \sin t]\\
&= y_0 \cos(\omega_2 + 1)t.
\end{aligned}$$

In this way, we find the dependence of the frequency of oscillations on amplitudes x_0 and y_0.

8 Concluding remarks

We have proposed a method to study dynamical systems with periodically modulated Hamiltonians.

In the first part of the paper the parametric form of canonical transformation and the method of normalization have been introduced and illustrated.

Note that the last mentioned method is sometimes called the method of invariant normalization (see [10, 11]). A similar like approach has been applied to carry out averaging and invariant normalization for a giroscope using computer algebra REDUCE program (see [24]).

In the second paper's part dynamics of a swinging oscillator is analysed. In what follows the Hamiltonian of the studied system is defined, and also its normal form is constructed. After integration of the system, the asymptotic solution of the investigated non-linear system is obtained [23].

In our work asymptotical series of transformations close to identity are applied, i.e. similarly to the classical theory case. However, the applied by us approach has better series convergence as well as higher accuracy (with respect to same approximation order) in comparison to classical approach. In the Arnold's monograph a series of advantages of Poincaré versus Jacobi guiding functions are reported. In particular, invariant (non-invariant) properties of the mentioned Poincaré (Jacobi) guiding functions are outlined. Owing to the proved identity of the Poincaré and parametric functions, all advantages hold also for the applied by us parametric form of canonical transformations.

Moser and others applied a property of a normal form to be the first integral of its leading (unperturbed) part, when the leading part is quadratic form of coordinates and impulses. It is remarkable that all authors follow the Birkhoff classical approach, i.e. they apply the mentioned leading part of Hamiltonian. Zhuravlev [10, 11] uses the mentioned property to construct a normal form independently on the form of an unperturbed Hamiltonian. This is the novel point of view of a normal form. Instead of tedious Birkhoff's approach yielding a successive cancellation of resonance terms in the third and fourth approximations, the being sought normal form and canonical transformation are defined by simple quadrature (12). It is clear that there exist cases when a Hamiltonian cannot be reduced to the corresponding normal form. Namely, in this case no one of the existing methods can be applied to yield the normal form. Note that it is easy to define such Hamiltonians using the Zhuravlev's method. One has only to check if the being approximated (by the recurrent formulas) function is a quasi-periodic one.

Note that in our short manuscript it is rather difficult to exhibit all advantages of the applied method. However, this method has been already applied to yield important practical results (see reference [24]).

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