ANALYSIS OF NATURAL IN-PLANE VIBRATION OF RECTANGULAR PLATES USING HOMOTOPY PERTURBATION APPROACH

IGOR V. ANDRIANOV, JAN AWREJCEWICZ, AND VLADIMIR CHERNETSKYY

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An analytical solution of the problem of free in-plane vibration of rectangular plates with complicated boundary conditions is proposed.

1. Introduction

We address the important problems of energy transmission by high-frequency excitations [21, 23] and structural noise transmission [25], as well as the analysis of folded [9] and sandwich plates [32]. Although for some boundary conditions even exact solutions are obtained [18, 19], but in general the application of either Rayleigh-Ritz [8, 13, 14, 20, 22, 26, 28–31] or Kantorovich approaches [33], or the method of superposition [15–17, 27] is required.

In this paper, we will use homotopy perturbation approach. Introduction of an artificial small parameter is usually motivated either by the lack of a real physical small parameter or by a rather narrow application zone of the used natural small parameter. In general, the expression "small parameter" can be used in a different manner. Namely, the following key question arises. Is it possible to obtain useful information directly either through a natural small parameter or through an introduction of an artificial one (or by the application of a useful summation procedure)? In this respect, it is worthwhile to speak rather directly on the "methods devoted to development on a parameter" than to speak only on a "small parameter."

From this point of view, there is no difference between a real and an artificial small parameter. However, following the tradition, the phrase an "artificial small parameter" will be further used. It is worth noting that the idea of introducing a small parameter has been proposed with respect to different branches of mathematics. For example, Dorodnitzyn [10] proposed the method of introduction of the parameter ε into the input equations and the boundary conditions in the way that for $\varepsilon = 0$, a simplified problem was obtained,

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whereas for $\varepsilon = 1$ the input problem was governed. In other words, Dorodnitzyn has applied the continuation method widely known in the numerical mathematics. A serious problem appeared due to divergent series occurrence for $\varepsilon = 1$. In order to overcome the difficulties, the so-called methods of analytical continuation have been proposed, but they appeared to be not satisfactory enough.

Some authors used the artificial parameter approach in a special way. Namely, they observed that a transition from $\varepsilon = 0$ to $\varepsilon = 1$ represented a homotopy transformation yielding today's accepted term as the homotopy perturbation technique [10–12, 24]. However, the mentioned technique can be satisfactorily applied only in connection with an effective method of summation.

It has been already shown in [2–5] (see also [1, 6]) that effective results are expected using the Padé approximations matched with the homotopy perturbation techniques.

2. Analysis

We consider free in-plane vibrations of a rectangular plate with hybrid-type boundary conditions in its surface (Figure 2.1).

The governing equations are given in the form

$$
\begin{aligned}
&(1+c)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+c\frac{\partial^2 v}{\partial x\partial y}+\rho\omega^2 u=0,\\
&(1+c)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+c\frac{\partial^2 u}{\partial x\partial y}+\rho\omega^2 v=0,
\end{aligned}
\tag{2.1}
$$

where $c = 1/(1-2\mu^*)$, $\mu^* = \mu/(1+\mu)$, μ is Poisson's coefficient, ω is the frequency of free vibrations, $\rho = \rho_0/E$, ρ_0 is the plate material density, and E is the Young modulus.

The boundary conditions can be formulated with help of the Heaviside function:

$$
H(x)=\begin{cases}0, & x<0,\\ 1, & x>0.\end{cases}
\tag{2.2}
$$

The following formulas hold:

$$
\begin{aligned}
&\text{for } x=\frac{a}{2},\ u=0,\quad H_1 v+(1-H_1)S=0,\\
&\text{for } x=-\frac{a}{2},\ u=0,\quad H_2 v+(1-H_2)S=0,\\
&\text{for } y=\frac{b}{2},\ v=0,\quad H_3 u+(1-H_3)S=0,\\
&\text{for } y=-\frac{b}{2},\ v=0,\quad H_4 u+(1-H_4)S=0,
\end{aligned}
\tag{2.3}
$$

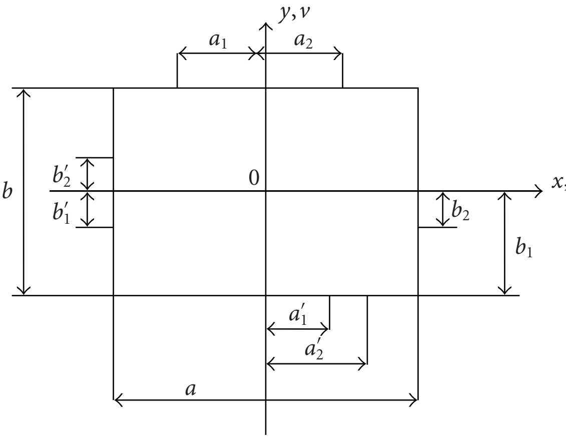

Figure 2.1. Scheme of the investigated plate.

where $S = G(\partial u/\partial y + \partial v/\partial x)$, G is the shear modulus,

$$
\begin{aligned}
H_1 &= H(x, y - a_1) - H(x, -y - a_2),\\
H_2 &= H(x, y - a_1') - H(x, y - a_2'),\\
H_3 &= H(x - b_1, y) - H(x - b_2, y),\\
H_4 &= H(x - b_1', y) - H(x - b_2', y).
\end{aligned}
\tag{2.4}
$$

After introducing small parameter ε, conditions (2.3) take the following form:

$$
\begin{aligned}
&\text{for } x = \frac{a}{2},\ u = 0, \quad \varepsilon\Big[GH_1\frac{v}{a} + (1 - H_1)S\Big] + (1 - \varepsilon)S = 0,\\
&\text{for } x = -\frac{a}{2},\ u = 0, \quad \varepsilon\Big[GH_2\frac{v}{a} + (1 - H_2)S\Big] - (1 - \varepsilon)S = 0,\\
&\text{for } y = \frac{b}{2},\ v = 0, \quad \varepsilon\Big[GH_3\frac{u}{b} + (1 - H_3)S\Big] + (1 - \varepsilon)S = 0,\\
&\text{for } y = -\frac{b}{2},\ v = 0, \quad \varepsilon\Big[GH_4\frac{u}{b} + (1 - H_4)S\Big] - (1 - \varepsilon)S = 0.
\end{aligned}
\tag{2.5}
$$

For $\varepsilon = 0$, the boundary conditions enable separation of the variables. The unknown displacement and frequency are developed into the series with reference to the perturbation parameter ε:

$$
\begin{aligned}
u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots,\\
v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots,\\
\omega^2 &= \omega_0^2 + \varepsilon\omega_1^2 + \varepsilon^2\omega_2^2 + \cdots.
\end{aligned}
\tag{2.6}
$$

Substituting (2.6) into (2.1) and into boundary conditions (2.5) and splitting with respect to ε, we get

$$\begin{aligned}&(1+c)\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}+c\frac{\partial^2 v_0}{\partial x\partial y}+\rho\omega_0^2 u_0=0,\\&(1+c)\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x^2}+c\frac{\partial^2 u_0}{\partial x\partial y}+\rho\omega_0^2 v_0=0,\end{aligned}\tag{2.7}$$

$$\begin{aligned}&\text{for } x=\pm\frac{a}{2},\ u_0=0,\quad \frac{\partial v_0}{\partial x}=0,\\&\text{for } y=\pm\frac{b}{2},\ v_0=0,\quad \frac{\partial u_0}{\partial y}=0.\end{aligned}\tag{2.8}$$

The solution to (2.7), satisfying boundary conditions (2.8), has the following form:

$$\begin{aligned}u_0&=A\sin\frac{2m\pi x}{a}\cos\frac{2n\pi y}{b},\\v_0&=B\sin\frac{2n\pi x}{b}\cos\frac{2m\pi y}{a},\\\rho\omega_{01}^2&=4\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right),\qquad B_1=-\frac{Amb}{na},\\\rho\omega_{02}^2&=4\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)(1+c),\qquad B_2=-\frac{Ana}{mb}.\end{aligned}\tag{2.9}$$

In the next approximation, one finds

$$\begin{aligned}&(1+c)\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}+c\frac{\partial^2 v_1}{\partial x\partial y}+\rho\omega_0^2 u_1=-\rho\omega_0^2 u_0,\\&(1+c)\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial x^2}+c\frac{\partial^2 u_1}{\partial x\partial y}+\rho\omega_0^2 v_1=-\rho\omega_1^2 v_0,\\&\text{for } x=\frac{a}{2},\ u_1=0,\quad \frac{\partial v_1}{\partial x}=-\frac{H_1 v_0}{a},\\&\text{for } x=-\frac{a}{2},\ u_1=0,\quad \frac{\partial v_1}{\partial x}=\frac{H_2 v_0}{a},\\&\text{for } y=\frac{b}{2},\ v_1=0,\quad \frac{\partial u_1}{\partial y}=-\frac{H_3 u_0}{b},\\&\text{for } y=-\frac{b}{2},\ v_1=0,\quad \frac{\partial u_1}{\partial y}=\frac{H_4 u_0}{b}.\end{aligned}\tag{2.10}$$

The values ω_1^2 can be found applying an adjoint problem solution:

$$\rho\omega_1^2=\lfloor A(a_1,a_2)+A(a_1',a_2')+B(b_1,b_2)+B(b_1',b_2')\rfloor,\tag{2.11}$$

where

$$A(a_1,a_2) = \frac{n^2 a}{b^2}\left[0.5(a_2-a_1) - \frac{a}{8\pi m}\left(\sin\left(\frac{4\pi m a_2}{a}\right) - \sin\left(\frac{4\pi m a_1}{a}\right)\right)\right],$$
$$B(b_1,b_2) = \frac{m^2 b}{a^2}\left[0.5(b_2-b_1) - \frac{b}{8\pi n}\left(\sin\left(\frac{4\pi m b_2}{b}\right) - \sin\left(\frac{4\pi m b_1}{b}\right)\right)\right], \tag{2.12}$$
$$N(m,n) = \left(n^2a^2 + m^2b^2\right)^{-1}.$$

A particular solution satisfying the first-order boundary conditions is

$$u_1 = \frac{A}{b^2}(-1)^n\left[H_4\left(\frac{by}{2} - \frac{y^2}{2}\right) - H_3\left(\frac{y^2}{2} + \frac{by}{2}\right)\right]\sin\frac{2m\pi x}{a},$$
$$v_1 = \frac{B}{a^2}(-1)^m\left[H_4\left(\frac{ax}{2} - \frac{x^2}{2}\right) - H_1\left(\frac{x^2}{2} + \frac{ax}{2}\right)\right]\sin\frac{2n\pi y}{b}. \tag{2.13}$$

The next approximation gives

$$(1+c)\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + c\frac{\partial^2 v_2}{\partial x \partial y} + \rho\omega_0^2 u_2 = -\rho\left(\omega_2^2 u_0 + \omega_1^2 u_1\right),$$
$$(1+c)\frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial x^2} + c\frac{\partial^2 u_2}{\partial x \partial y} + \rho\omega_0^2 v_2 = -\rho\left(\omega_2^2 v_0 + \omega_1^2 v_1\right),$$
$$\text{for } x = \frac{a}{2},\ u_2 = 0,\quad \frac{\partial v_2}{\partial x} = -\frac{H_1}{a}(v_0+v_1),$$
$$\text{for } x = -\frac{a}{2},\ u_2 = 0,\quad \frac{\partial v_2}{\partial x} = \frac{H_2}{a}(v_0+v_1), \tag{2.14}$$
$$\text{for } y = \frac{b}{2},\ v_2 = 0,\quad \frac{\partial u_2}{\partial y} = -\frac{H_3}{b}(u_0+u_1),$$
$$\text{for } y = -\frac{b}{2},\ v_2 = 0,\quad \frac{\partial u_2}{\partial y} = \frac{H_4}{b}(u_0+u_1).$$

Again solving the adjoint problem, one gets ω_2^2, and finally the following approximation is found:

$$\omega^2 = \omega_1^2\left\lfloor 2.5 - \pi^{-2} + 2\pi^{-2}N(m,n)\left(aA(a_1,a_2) + aA(a_1',a_2') + bB(b_1,b_2) + bB(b_1',b_2')\right)\right\rfloor. \tag{2.15}$$

The application of Padé approximations [7] enables extension of the function using its finite series number, and this allows us to propose a suitable solution to our problem.

The series part obtained so far,

$$\omega^2 + \omega_0^2 + \varepsilon\omega_1^2 + \varepsilon^2\omega_2^2, \tag{2.16}$$

Table 2.1. First vibrations' frequency square $\rho\omega^2$.

a/b	(2.15)	Error %	(2.17)	Error %	Exact solution
0.5	220.951	15.8	277.65	5.8	262.459
1	82.625	4.7	91.336	4.8	86.726
1.5	58.029	6.6	60.379	2.8	62.152
2	49.716	4.0	50.602	2.2	51.791

is taken and the following Padé approximation is obtained:

$$\omega^2 \approx \frac{\omega_0^2(\omega_1^2-\omega_2^2)+\omega_1^4}{\omega_1^2-\omega_2^2}. \tag{2.17}$$

Note that in the limiting case, when on the plate sides perpendicular to the axis $0y$ there is no clamping, and when on the other two plate sides clamping is applied on the whole plate thickness, one may even find an exact solution. In Table 2.1, the frequencies associated with the first vibration mode, for which an influence of the boundary conditions plays an important role, are reported (for $\mu = 0.3$). One may easily conclude that the applied method of boundary conditions perturbation provides fully reliable results.

The proposed method has advantages in comparison with the known methods of solving the problems related to the mixed boundary conditions, that is, the methods of Bubnov-Galerkin, Ritz, Kantorovich, Trefftz, and so forth. Namely, it does not require an a priori knowledge of the shapes of deformed surfaces. Furthermore, the proposed approach does not lead either to a high-order system of transcendental equations.

3. Conclusions

The proposed asymptotic method enables a solution represented in an analytical form, which is important while applying any optimal design in solution of direct problems. It should be emphasized, however, that the FEM method is universal with respect to a space filled by a plate. It is rather difficult to apply the asymptotic method to complex-form spaces, since they require knowledge of an analytical solution of zero-order approximation. Besides, application of the asymptotic method does not provide an easier way to introduce higher accuracy, since it is rather difficult to construct higher approximations. However, one may require a solution obtained by two methods in order to control reliability of the obtained approximate solution. In the case of complex plate forms, the results obtained by the asymptotic method can serve as tests for FEM, if a transition from complex to simple geometry is possible.

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Igor V. Andrianov: Department of General Mechanics, Rheinisch-Westfälische Technische Hochschule (RWTH) Aachen University, Templergraben 64, Aachen 52056, Germany
E-mail address: igor_andrianov@inbox.ru

Jan Awrejcewicz: Department of Automatics and Biomechanics, Technical University of Łódź, 1/15 Stefanowski Street, Łódź 90-924, Poland
E-mail address: awrejcew@p.lodz.pl

Vladimir Chernetskyy: Department of Civil Engineering, Pridneprovyhe State Academy of Civil Engineering and Architecture, 24a Chernyschevskogo Street, Dnepropetrovsk 49005, Ukraine
E-mail address: alexchs@yandex.ru