Homogenization of Quasi-Periodic Structures

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A novel asymptotical approach to integrate differential equations governing quasi-regular structures, particularly suitable on the stage of construction design, is proposed.

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Nowadays, periodic structures are often used in Mechanical Engineering, for example, composite, ribbed, corrugated, perforated, etc., shells. All of the mentioned structures possess regular (periodic) structure, although in many cases quasi-regular structures can be more optimal.

Homogenization approach is widely used for computation of periodically non- homogenenous structures [1–3]. In this paper modification of homogenization procedure for quasi-regular structures is proposed.

The mentioned novel approach will be classified briefly, with the use of a classical problem governed by the following equations [1]

$$\frac{d}{dx}\left[a\left(\frac{x}{\epsilon}\right)\frac{du}{dx}\right]+b\left(\frac{x}{\epsilon}\right)u=q(x) \tag{1}$$

where $a(x/\epsilon)$ is periodic with respect to x, and possesses period $\epsilon(\epsilon\leqslant 1)$.

Now let us suppose $a=a\left(\frac{f(x)}{\epsilon}\right)$, $b=b\left(\frac{f(x)}{\epsilon}\right)$, then the problem (1) can be rewritten into the form

$$\frac{d}{dx}\left\{a\left[\frac{f(x)}{\epsilon}\right]\frac{du}{dx}\right\}+b\left[\frac{f(x)}{\epsilon}\right]u=q(x) \tag{2}$$

where $f(x)$ is a slowly changed function, $f'(x)\sim 1$.

The period of non-homogeneity T is approximately governed by the formula $T\approx\epsilon(f(x))$.

Applying the two-scales approach [1–3], two independent variables $\eta=x$ and $\xi=f(x)/\epsilon$ are introduced, instead of one x. Then, the derivative of x reads

$$\frac{d}{dx}=\frac{\partial}{\partial\eta}+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi} \tag{3}$$

i.e., instead of an ordinary differential equation, a partial differential equation is obtained.

Applying Eq. (3) to Eq. (2) gives

$$\frac{\partial}{\partial\eta}\left(a(\xi)\frac{du}{dx}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\frac{du}{dx}\right)+b(\xi)u=q(\eta) \tag{4}$$

in which the quantity du/dx is given by

$$\frac{du}{dx}=\frac{\partial u}{\partial\eta}+\frac{f'(\eta)}{\epsilon}\frac{\partial u}{\partial\xi} \tag{5}$$

Therefore, Eq. (4) becomes

$$\frac{\partial}{\partial\eta}\left(a(\xi)\left[\frac{\partial u}{\partial\eta}+\frac{f'(\eta)}{\epsilon}\frac{\partial u}{\partial\xi}\right]\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\left[\frac{\partial u}{\partial\eta}+\frac{f'(\eta)}{\epsilon}\frac{\partial u}{\partial\xi}\right]\right)+b(\xi)u=q(\eta) \tag{6}$$

which can be expanded to look like

$$\frac{\partial}{\partial\eta}\left(a(\xi)\frac{\partial u}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(a(\xi)\frac{f'(\eta)}{\epsilon}\frac{\partial u}{\partial\xi}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial u}{\partial\eta}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\left(\frac{f'(\eta)}{\epsilon}\frac{\partial u}{\partial\xi}\right)\right)+b(\xi)u=q(\eta) \tag{7}$$

This can be simplified as follows

$$a(\xi)\frac{\partial^2 u}{\partial\eta^2}+a(\xi)\frac{\partial}{\partial\eta}\left(\frac{f'(\eta)}{\epsilon}\frac{\partial u}{\partial\xi}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial u}{\partial\eta}\right)+\left[\frac{f'(\eta)}{\epsilon}\right]^2\frac{\partial}{\partial\xi}\left(a(\xi)\left(\frac{\partial u}{\partial\xi}\right)\right)+b(\xi)u=q(\eta) \tag{8}$$

A solution to Eq. (8) is sought in the form of the following series

$$u=u_0(\eta,\xi)+\epsilon u_1(\eta,\xi)+\epsilon^2u_2(\eta,\xi)+\ldots \tag{9}$$

where $u_0,u_1,u_2,\ldots$ are periodic functions with respect to ξ with period 1.

Keeping terms of $O(1/\epsilon^2),O(1/\epsilon)$ and $O(1)$ only, we get

$$a(\xi)\frac{\partial^2 u_0}{\partial\eta^2}+a(\xi)\frac{\partial}{\partial\eta}\left(\frac{f'(\eta)}{\epsilon}\frac{\partial(u_0+\epsilon u_1)}{\partial\xi}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial(u_0+\epsilon u_1)}{\partial\eta}\right)+\left[\frac{f'(\eta)}{\epsilon}\right]^2\frac{\partial}{\partial\xi}\left(a(\xi)\times\left(\frac{\partial(u_0+\epsilon u_1+\epsilon^2u_2)}{\partial\xi}\right)\right)+b(\xi)u=q(\eta) \tag{10}$$

This can be expanded as follows

$$a(\xi)\frac{\partial^2 u_0}{\partial\eta^2}+a(\xi)\frac{\partial}{\partial\eta}\left(\frac{f'(\eta)}{\epsilon}\frac{\partial u_0}{\partial\xi}\right)+a(\xi)\frac{\partial}{\partial\eta}\left(\frac{f'(\eta)}{\epsilon}\frac{\epsilon\,\partial u_1}{\partial\xi}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial u_0}{\partial\eta}\right)+\frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\epsilon\,\partial u_1}{\partial\eta}\right)+\left[\frac{f'(\eta)}{\epsilon}\right]^2\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\partial u_0}{\partial\xi}\right)+\left[\frac{f'(\eta)}{\epsilon}\right]^2\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\epsilon\,\partial u_1}{\partial\xi}\right)+\left[\frac{f'(\eta)}{\epsilon}\right]^2\frac{\partial}{\partial\xi}\left(a(\xi)\frac{\epsilon^2\,\partial u_2}{\partial\xi}\right)+b(\xi)u_0=q(\eta). \tag{11}$$

Canceling out all the ϵ's gives

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$$a(\xi)\frac{\partial^2 u_0}{\partial \eta^2} + a(\xi)\frac{\partial}{\partial \eta}\left(\frac{f'(\eta)}{\epsilon}\frac{\partial u_0}{\partial \xi}\right) + a(\xi)\frac{\partial}{\partial \eta}\left(f'(\eta)\frac{\partial u_1}{\partial \xi}\right) + \frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_0}{\partial \eta}\right) + f'(\eta)\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_1}{\partial \eta}\right) + \frac{f'(\eta)^2}{\epsilon^2}\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_0}{\partial \xi}\right) + \frac{f'(\eta)^2}{\epsilon}\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_1}{\partial \xi}\right) + f'(\eta)^2\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_2}{\partial \xi}\right) + b(\xi)u_0 = q(\eta) \quad (12)$$

and collecting terms yields

$$O(1/\epsilon^2): \frac{f'(\eta)^2}{\epsilon^2}\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_0}{\partial \xi}\right) = 0$$

$$O(1/\epsilon): a(\xi)\frac{\partial}{\partial \eta}\left(\frac{f'(\eta)}{\epsilon}\frac{\partial u_0}{\partial \xi}\right) + \frac{f'(\eta)}{\epsilon}\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_0}{\partial \eta}\right) + \frac{f'(\eta)^2}{\epsilon}\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_1}{\partial \xi}\right) = 0$$

$$O(1): a(\xi)\frac{\partial^2 u_0}{\partial \eta^2} + a(\xi)\frac{\partial}{\partial \eta}\left(f'(\eta)\frac{\partial u_1}{\partial \xi}\right) + f'(\eta)\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_1}{\partial \eta}\right) + f'(\eta)^2\frac{\partial}{\partial \xi}\left(a(\xi)\frac{\partial u_2}{\partial \eta}\right) + b(\xi)u_0 = q(\eta) \quad (13)$$

The obtained Eqs. (13) can be cast into the following equivalent set

$$\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_0}{\partial \xi}\right] = 0$$

$$f'(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_0}{\partial \eta}\right] + [f'(\eta)]^2\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_1}{\partial \xi}\right] + a(\xi)\frac{\partial}{\partial \eta}\left[f'(\eta)\frac{\partial u_0}{\partial \xi}\right] = 0$$

$$f'^2(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_2}{\partial \xi}\right] + f'(\eta)\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_1}{\partial \eta}\right] + a(\xi)\frac{\partial}{\partial \eta}\left[f'(\eta)\frac{\partial u_1}{\partial \xi}\right] + a(\xi)\frac{\partial^2 u_0}{\partial \eta^2} + b(\xi)u_0 = q(\eta) \quad (14)$$

The first equation of Eq. (14) yields

$$u_0 = A(\eta)\int a^{-1}(\xi)d\xi + B(\eta)$$

Due periodicity of function u_0 one has $A=0$ [1].

So, $u_0=u_0(\eta)$, and the second equation of Eq. (15) reads

$$\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u_1}{\partial \xi}\right] = -\frac{1}{f'(\eta)}\frac{da(\xi)}{d\xi}\frac{du_0}{d\eta} \quad (15)$$

It gives

$$\frac{\partial u_1}{\partial \xi} = -\frac{1}{f'(\eta)}\frac{du_0}{d\eta} + \frac{c_1(\eta)}{a(\xi)} \quad (16)$$

and the constant $c_1(\eta)$ is defined by the periodicity condition of the function u_1 with respect to ξ, and it reads

$$c_1(\eta) = \frac{a_1}{f'(\eta)}\frac{du_0}{d\eta}, \quad a_1 = \left[\int_0^1 a^{-1}d\xi\right]^{-1}$$

Removing the derivative $\partial u_1/\partial \xi$ from the third equation of Eq. (15), one gets

$$f'^2(\eta)\frac{\partial}{\partial \xi}\left(a\frac{\partial u_2}{\partial \xi}\right) + f'(\eta)\frac{\partial}{\partial \xi}\left(a\frac{\partial u_1}{\partial \xi}\right) + a_1\frac{d^2 u_0}{d\eta^2} + bu_0 = q \quad (17)$$

Let us apply the homogenization procedure to the Eq. (17) with a simultaneous action of the averaging operator $\int_0^1(\ldots)d\xi$ on each of the equation terms. The first two terms are zero due to the periodicity of functions $f'^2(\eta)\frac{\partial}{\partial \xi}(a\frac{\partial u_2}{\partial \xi})$, $f'(\eta)\frac{\partial}{\partial \xi}(a\frac{\partial u_1}{\partial \eta})$ with respect to ξ, and finally one gets

$$a_1\frac{d^2 u_0}{d\eta^2} + b_1 u_0 = q(\eta) \quad (18)$$

where

$$b_1 = \int_0^1 b(\xi)d\xi.$$

The following boundary condition is applied to Eq. (18)

$$u_0 = 0 \quad \text{for} \quad \eta = 0, L \quad (19)$$

Note that the quasiregularity of nonhomogeneity is not exhibited by Eq. (1). Nonregularity is accounted through the changing of variables $\xi=f(x)/\epsilon$.

It is worth noticing that many structures are governed by differential equations with periodically discontinuous coefficients. Here the cylindrical ring reinforced shell with quasi-periodic system ribs is analyzed.

For an axially symmetric case, the equation governing equilibrium state of a shell reads

$$w^{IV} + \left\{\beta + \gamma\sum_{k=1}^{N}\delta[f(x) - \kappa l]\right\}w = P(x) \quad (20)$$

where

$$\beta = \frac{3(1-\nu^2)}{R^2h^2}; \quad P = \frac{q}{D}; \quad \gamma = -\frac{EF}{DR^2}; \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

The following notation is used: w is normal displacement; q is external loading; R, h denote shell radius and thickness; F is the area of rib cross section; E is Young modulus; ν is Poisson coefficient; $\delta(x)$ is Dirac function.

Introducing new variable $\eta=f(x)$(then $x=f^{-1}(\eta)$), Eq. (20) is recast into the form

$$L^4(w) + \left[\beta + \gamma\sum_{k=1}^{N}\delta(\eta - kl)\right]w = Q(\eta)$$

$$L^4(w) = \frac{d}{d\eta}\left\{\frac{1}{\varphi(\eta)}\frac{d}{d\eta}\left[\frac{1}{\varphi(\eta)}\frac{d}{d\eta}\left(\frac{1}{\varphi(\eta)}\frac{dw}{d\eta}\right)\right]\right\} \quad (21)$$

where

$$\varphi(\eta) = [f^{-1}(\eta)]^1_{\eta}, \quad Q = \varphi P[f^{-1}(\eta)]$$

If N is large, then one may apply a homogenization procedure to solve Eq. (21) [2,3]. The equilibrium equation for the shell between successive ribs reads

$$L^4(w) + \beta\varphi w = Q \quad (22)$$

whereas the conditions for junction of shell and rib have the form

$$w^+ = w^-; \quad \frac{dw^+}{d\eta} = \frac{dw^-}{d\eta}; \quad L^2(w)^+ = L^2(w)^- \quad (23)$$

$$L^3(w)^+ - L^3(w)^- = \gamma\varphi w|_{\eta=kl} \quad (24)$$

where: $(w)^+, (w)^-$, etc., are right and left end values calculated in the point $\eta=kl$, respectively.

The boundary conditions at shell ends $\eta=0, l$ are taken in the form

$$w = \frac{dw}{d\eta} = 0 \tag{25}$$

Introducing the small parameter $\epsilon = N^{-1}$, and fast and slow variables $\xi = \eta/\epsilon$ and η, one gets

$$\frac{d}{d\eta} = \frac{\partial}{d\eta} + \epsilon^{-1}\frac{\partial}{\partial\xi} \tag{26}$$

Deflection w, following [2,3] is sought in the form

$$w = w_0(\eta) + \epsilon^4 w_1(\eta, \xi) + \ldots \tag{27}$$

where: w_i $(i=1,2,\ldots)$ are periodic functions of ξ with the period L.

Substituting Eqs. (26), (27) into Eqs. (22), (24) and carrying out the asymptotical splitting with respect to ϵ powers, and according to functions w_i with respect to ξ, the following equations are obtained

$$\frac{1}{\varphi^3}\frac{\partial^4 w_1}{\partial\xi^4} + L^4(w_0) + \beta\varphi w_0 = Q \tag{28}$$

$$\left(w_1, \frac{\partial w_1}{\partial\xi}, \frac{\partial^2 w_1}{\partial\xi^2}\right)_{\xi=0} = \left(w_1, \frac{\partial w_1}{\partial\xi}, \frac{\partial^2 w_1}{\partial\xi^2}\right)_{\xi=L} \tag{29}$$

$$\frac{\partial^3 w_1}{\partial\xi^3}{}_{\xi=L} - \frac{\partial^3 w_1}{\partial\xi^3}{}_{\xi=0} = \varphi^3\gamma N w_0 \tag{30}$$

$$w_{\eta=0,l} = \frac{dw_0}{d\eta}{}_{\eta=0,L} = 0 \tag{31}$$

During the derivation of the Eq. (30), it has been assumed that $\gamma \sim \epsilon$.

Integrating Eq. (28) with respect to δ, one has

$$w_1 = \varphi^3[Q - L^4(w_0) - \beta\varphi w_0]\frac{\xi^4}{24} + C_1(\eta)\xi^3 + C_2(\eta)\xi^2 + C_3(\eta)\xi + C_4(\eta) \tag{32}$$

Taking $C_4=0$ and defining C_1–C_3 from the Condition (30), one gets

$$W_1 = \varphi^3[Q - L^4(W_0) - \beta\varphi W_0]\frac{\xi^2(\xi - L)^2}{24} \tag{33}$$

Substituting Eq. (33) into Eq. (30), the following equation for w_0 is obtained

$$L^4(w_0) + \left(\beta\varphi + \frac{\gamma N}{L}\right)w_0 = Q \tag{34}$$

Coming back to the variable x, the Eq. (34) takes the form

$$w_0^{IV} + \left(\beta + \frac{\gamma N}{L}f'(x)\right)w_0 = P(x) \tag{35}$$

It is worth noticing that the derived Equation (35) describes a beam lying on continuous elastic foundation with changeable stiffness deflection. Therefore, non-regular properties of support are already included in an averaged solution.

Conclusions

We have proposed a novel asymptotical approach to quasi-regular structures. The proposed method is suitable on the stage of optimal design. In addition, the proposed approach is useful for the solution of various optimization problems.

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