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# Geometrical approach to the swinging pendulum dynamics

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**Abstract**

This paper shows how geometrical tools can describe a two degrees-of-freedom mechanical system (a swinging pendulum). First a brief introduction to the subject of geometrodynamics is given and then the fundamental quantities of the technique are found. Next the Jacobi–Levi–Civita equation is explicitely obtained and transformed to a scalar differential equation that is numerically solved. Qualification of regular and chaotic dynamics of our investigated system is illustrated and discussed.


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*Keywords:* Riemannian geometry; Chaos; Pendulum

## 1. Introduction

Riemannian formulation of dynamics makes use of possibility of studying the instability of system motions through the instability properties of geodesics in a suitable manifold. It is believed that geometrical approach may provide a good explanation of the onset of chaos in Hamiltonian systems as an alternative point of view. This technique has been recently applied to study chaos in Hamiltonian systems [1–4], however other mechanical systems can be also analysed in this manner e.g. systems with velocity-dependent potentials that are described by Finslerian geometry. The most important tool in this approach is the Jacobi–Levi–Civita (JLC) equation which describes how nearby geodesics locally scatter. The instability properties are completely determined by curvature properties of the underlying manifold due to the occurrence of the Riemann tensor in the JLC equation. Analysing the JLC equation we observe that manifolds with negative curvature induce chaos, however a chaotic behaviour may occur in systems with positive curvature manifolds due to curvature fluctuations along geodesics [1,2,5,6]. Since a generic Riemannian space consists of the ambient space and a metric tensor, so in order to apply this technique we need to define these quantities. We have a few choices at our disposal. We can use configuration space endowed with a Jacobi metric, enlarged space-time manifold with an Eisenhart metric or tangent bundle of a configuration manifold with a Sasaki metric. However, no matter the ambient space we choose for geometrization, the geodesics projected onto the configuration space of a system have to correspond the trajectories of an investigated system. In other words, the geodesics equations should give us the equations of motion. It should be emphasised that this technique does not provide a new method of solving differential equations. It provides a qualitative description of the behaviour of systems using formalism of differential geometry (Riemannian geometry in this case). This method is still under development and other types of spaces are sought for a purpose of geometrization. So far, Hamiltonian systems with many degrees-of-freedom have been investigated with the use of this approach [1–3,6]. However, Hamiltonian systems with a few degrees-of-freedom are not investigated so often as the previous ones in this manner except in paper [4] that deals with a two degrees-of-freedom system (Hénon–Heiles' model). Note that Hénon–Heiles' model dynamics is described by a Hamiltonian with

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a coordinate-independent kinetic energy term. In our paper, we analyse a two degrees-of-freedom system (the swinging pendulum) of the coordinate-dependent kinetic energy term.

### 1.1. The Jacobi metric and Jacobi–Levi–Civita equation

Below, we introduce basic tools for geometrization. Let us consider a conservative system with $N$ degrees-of-freedom, which is described by the following Lagrangian:

$$L = T - V = \frac{1}{2} a_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - V(x), \quad \text{where } \dot{x}^\mu \equiv \frac{dx^\mu}{dt}. \tag{1}$$

For this conservative system Hamilton's principle can be cast in Maupertuis' principle:

$$\delta\mathcal{S} = \delta \int \frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^\mu \, dt = 0. \tag{2}$$

It is well-known that equations of motion can be obtained using the rules of variational calculus for the functional $\mathcal{S}$. We have the analogous situation in Riemannian geometry, where geodesics equations are obtained from:

$$\delta \int ds = 0, \tag{3}$$

where $ds$ is arc length. Hence, we can identify these two situations in the following way:

$$\delta \int ds = \delta\mathcal{S} = 0. \tag{4}$$

The kinetic energy $T$ is a homogeneous function of degree two in the velocities, hence:

$$\frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^\mu = 2T. \tag{5}$$

Using the above result and substituting it to (2), we get Maupertuis' principle:

$$\delta\mathcal{S} = \delta \int 2T \, dt = 0. \tag{6}$$

Making use of (4) and the fact that the system is conservative, one can easy verify that:

$$2T \, dt = \sqrt{2(E - V(x)) a_{\mu\nu}(x) \, dx^\mu \, dx^\nu} = \sqrt{g_{\mu\nu}(x) \, dx^\mu \, dx^\nu} = ds, \tag{7}$$

where $E$ is the total energy. Hence, we find the metric tensor which is referred as the Jacobi metric:

$$g_{\mu\nu}(x) = 2(E - V(x)) a_{\mu\nu}(x). \tag{8}$$

Note, that the substitution $E - V = T$ made here is essential. As we said earlier, the geometry used in this approach is a Riemannian one hence a metric tensor should not be velocity-dependent. However, the kinetic energy is velocity-dependent by definition and that is why we must put $E - V(x)$ instead of $T$ in (8).

Now, we present a brief derivation of the main tool of this approach, namely the Jacobi–Levi–Civita equation [1–4]. Let us define a congruence of geodesics as a family of geodesics $\{x_u(s) = x(s,u) : u \in \mathcal{R}\}$. The geodesics are parametrized by a parameter $u$ whereas $s$ is arc length. Let $J(s) = \frac{d}{du}$ denote a tangent vector at $u$ ($s$ is fixed). $J(s)$ is called a Jacobi vector that can locally measure the distance between two nearby geodesics. Since $\frac{d}{ds}$ is tangent to a geodesic, we get:

$$\nabla_s \frac{d}{ds} = 0, \quad \text{where } \nabla_s \equiv \nabla_{\frac{d}{ds}}. \tag{9}$$

It is easy to verify that:

$$\nabla_s \frac{d}{du} = \nabla_u \frac{d}{ds}. \tag{10}$$

Hence, we obtain:

$$\nabla_s^2 \frac{d}{du} = \nabla_s \nabla_u \frac{d}{ds}. \tag{11}$$

Let us introduce the Riemann curvature tensor [4]:

$$R\left(\frac{d}{ds}, \frac{d}{du}\right)\frac{d}{ds} = \nabla_s \nabla_u \frac{d}{ds} - \nabla_u \nabla_s \frac{d}{ds} - \nabla_{\left[\frac{d}{ds}, \frac{d}{du}\right]} \frac{d}{ds}. \tag{12}$$

Making use of (9) and the fact that $\left[\frac{d}{ds}, \frac{d}{du}\right] = 0$ we get:

$$R\left(\frac{d}{ds}, \frac{d}{du}\right)\frac{d}{ds} = \nabla_s \nabla_u \frac{d}{ds}. \tag{13}$$

Substituting the obtained result to (11), we find:

$$\nabla_s^2 \frac{d}{du} + R\left(\frac{d}{du}, \frac{d}{ds}\right)\frac{d}{ds} = 0, \tag{14}$$

where we use the antisymmetry of the Riemann curvature tensor with respect to its first two arguments. The obtained equation is called Jacobi–Levi–Civita equation and it describes the evolution of geodesics separation along geodesic. In order to make any further calculation Eq. (14) will be expressed in local coordinate system later on.

## 2. The analysed system

In this paper we consider a two degrees-of-freedom, conservative mechanical system which consists of elastic swinging pendulum with nonlinear stiffness (see Fig. 1). The nonlinearity of the spring has the form:

$$k(y) = k_1(y + y_{st}) + k_2(y + y_{st})^3, \tag{15}$$

where $y_{st}$ is a position of the mass at rest. The Lagrangian describing the dynamics of our system has the form:

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{k_1}{2}(r - l_0)^2 - \frac{k_2}{4}(r - l_0)^4 + mgr\cos\varphi. \tag{16}$$

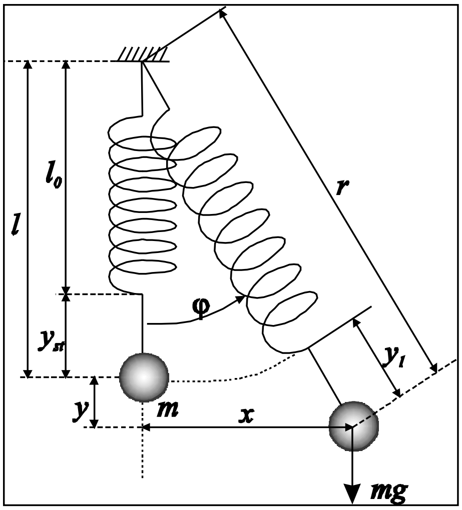


Fig. 1. The investigated system.

From Euler–Lagrange equations one can easily obtain:

$$\begin{cases} \ddot{r} = r\dot{\varphi}^2 - \frac{k_1}{m}(r - l_0) - \frac{k_2}{m}(r - l_0)^3 + g\cos\varphi, \\ \ddot{\varphi} = -\frac{2}{r}\dot{r}\dot{\varphi} - \frac{g}{r}\sin\varphi. \end{cases} \tag{17}$$

In order to obtain dimensionless equations we apply the following substitutions:

$$z = \frac{r - l_0}{l} - z_0, \quad z_0 = \frac{y_{st}}{l}, \quad \alpha = \frac{k_2 l^3}{mg},$$

$$\beta^2 = \frac{k_1 l}{mg}, \quad t \to t\sqrt{\frac{g}{l}}. \tag{18}$$

Hence, we get:

$$\begin{cases} \ddot{z} = (1+z)\dot{\varphi}^2 - \beta^2(z + z_0) - \alpha(z + z_0)^3 + \cos\varphi, \\ \ddot{\varphi} = -\frac{2}{1+z}\dot{z}\dot{\varphi} - \frac{1}{1+z}\sin\varphi, \end{cases} \tag{19}$$

where the corresponding dimensionless Lagrangian reads:

$$\widetilde{\mathscr{L}} = \dot{z}^2 + (1+z)^2\dot{\varphi}^2 - \beta^2(z+z_0)^2 - \frac{\alpha}{2}(z+z_0)^4 + 2(1+z)\cos\varphi. \tag{20}$$

### 3. Geometrization

As the Riemannian space manifold we choose a configuration manifold endowed with a Jacobi metric $g$. In our case the Jacobi metric has the following form:

$$g = 4(\mathscr{E} - \mathscr{V})\begin{pmatrix} 1 & 0 \\ 0 & (1+z)^2 \end{pmatrix}, \tag{21}$$

where $\mathscr{V}$ is dimensionless potential:

$$\mathscr{V} = \beta^2(z+z_0)^2 + \frac{\alpha}{2}(z+z_0)^4 - 2(1+z)\cos\varphi \tag{22}$$

and $\mathscr{E}$ is total energy of the system.

In order to find JLC equation, first we have to find coefficients of the Riemann curvature tensor. It is easy to verify that connection coefficients have the following form:

$$\Gamma^1_{11} = -\frac{1}{2\mathscr{W}}\frac{\partial\mathscr{V}}{\partial z}, \quad \Gamma^1_{12} = -\frac{1}{2\mathscr{W}}\frac{\partial\mathscr{V}}{\partial\varphi}, \tag{23}$$

$$\Gamma^1_{22} = -\frac{1}{2\mathscr{W}}\left(2(1+z)\mathscr{W} - (1+z)^2\frac{\partial\mathscr{V}}{\partial z}\right),$$

$$\Gamma^2_{11} = \frac{1}{2\mathscr{W}(1+z)^2}\frac{\partial\mathscr{V}}{\partial\varphi}, \tag{24}$$

$$\Gamma^2_{12} = \frac{1}{2\mathscr{W}(1+z)^2}\left(2(1+z)\mathscr{W} - (1+z)^2\frac{\partial\mathscr{V}}{\partial z}\right),$$

$$\Gamma^2_{22} = -\frac{1}{2\mathscr{W}}\frac{\partial\mathscr{V}}{\partial\varphi}. \tag{25}$$

where $\mathscr{W} = \mathscr{E} - \mathscr{V}$. Since the Riemannian space manifold is two-dimensional, there is only one nonzero coefficient of the Riemann curvature tensor. It has the following form:

$$R_{2121} = 2\frac{\partial^2\mathscr{V}}{\partial\varphi^2} + \frac{2}{\mathscr{W}}\left(\frac{\partial\mathscr{V}}{\partial\varphi}\right)^2 + 2(1+z)^2\frac{\partial^2\mathscr{V}}{\partial z^2} + 2(1+z)\frac{\partial\mathscr{V}}{\partial z} + 2\frac{(1+z)^2}{\mathscr{W}}\left(\frac{\partial\mathscr{V}}{\partial z}\right)^2. \tag{26}$$

Let us choose special base vectors $\{E_1, E_2\}$, such that $E_1 = \frac{d}{ds}$ and

$$g(E_1, E_2) = 0, \quad g(E_2, E_2) = 1. \tag{27}$$

In other words the base is formed by the orthonormal set of vectors. In local coordinate system these base vectors have the form[1]

$$E_i = Y^j_i e_j, \quad \text{where } i = 1, 2 \tag{28}$$

and

$$\begin{pmatrix} Y^1_1 & Y^2_1 \\ Y^1_2 & Y^2_2 \end{pmatrix} = \begin{pmatrix} \frac{dz}{ds} & \frac{d\varphi}{ds} \\ (1+z)\frac{d\varphi}{ds} & -\frac{1}{1+z}\frac{dz}{ds} \end{pmatrix}, \quad e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial\varphi}. \tag{29}$$

The Jacobi vector in this base reads[2] $J = J^n E_n$. Substituting this into the JLC Eq. (14) we get

$$\nabla^2_s(J^n E_n) + R(J^n E_n, E_1)E_1 = 0. \tag{30}$$

Since $E_1$ is tangent to the geodesics, one gets

$$\frac{d^2 J^n}{ds^2}E_n + J^i R(E_i, E_1)E_1 = 0. \tag{31}$$

Since $R(E_n, E_1)E_1$ is a vector, so it can be decomposed in the orthonormal base as follows:

$$R(E_i, E_1)E_1 = \sum_{n=1}^{2} g(R(E_i, E_1)E_1, E_n)E_n. \tag{32}$$

We put sum explicitely here because $i$ indices are on the same level so we cannot apply Einstein's summation convention. Substituting (32) to (31) one gets

$$\frac{d^2 J^n}{ds^2} + J^i g(R(E_i, E_1)E_1, E_n) = 0, \quad \text{where } n = 1, 2. \tag{33}$$

[1] Einstein's summation convention is used in the paper.
[2] The quantity $J^n$ should not be confused with $n$-power of $J$.

Due to the antisymmetry of the Riemann tensor the following equation is obtained

$$\frac{d^2J^n}{ds^2} + J^2 g(R(E_2,E_1)E_1,E_n) = 0. \tag{34}$$

Making use of (28) one finds

$$g(R(E_2,E_1)E_1,E_n) = R_{klij}Y^k_n Y^l_1 Y^i_2 Y^j_1. \tag{35}$$

Taking into account (29) two uncoupled equations JLC equations are obtained:

$$\begin{cases} \dfrac{d^2J^1}{ds^2} = 0, \\ \dfrac{d^2J^2}{ds^2} + \dfrac{R_{2121}}{\det g}J^2 = 0. \end{cases} \tag{36}$$

Actually, we are only interested in the evolution of $J^2$ along geodesics because this component of Jacobi vector is a coefficient standing by the vector that is orthogonal to the tangent direction of geodesics. Hence, from now on we consider only the following equation:

$$\frac{d^2J^2}{ds^2} + \frac{R_{2121}}{\det g}J^2 = 0. \tag{37}$$

Due to the dimension of the Riemannian space manifold we find

$$\frac{R_{2121}}{\det g} = \frac{1}{2}R, \tag{38}$$

where $R$ is a scalar curvature. In our case, the scalar curvature $R$ takes the form

$$R = \frac{2R_{2121}}{\det g} = \frac{1}{8\mathcal{W}^2(1+z)^2} \times \left(\frac{\partial^2\mathcal{V}}{\partial\varphi^2} + \frac{1}{\mathcal{W}}\left(\frac{\partial\mathcal{V}}{\partial\varphi}\right)^2 + (1+z)^2\frac{\partial^2\mathcal{V}}{\partial z^2} + (1+z)\frac{\partial\mathcal{V}}{\partial z}\right) + \frac{1}{8\mathcal{W}^3}\left(\frac{\partial\mathcal{V}}{\partial z}\right)^2. \tag{39}$$

Let us go back to the JLC equation

$$\frac{d^2J^2}{ds^2} + \frac{1}{2}J^2R(s) = 0. \tag{40}$$

The above equation is a differential equation with respect to the arc length $s$. In order to make further calculation we have to transform this equation into a differential equation with respect to time $t$. The metric used in this paper is the Jacobi metric, so we have

$$ds = 2\mathcal{W}\,dt. \tag{41}$$

Making use of the above identity we get the desired differential equation with respect to time

$$\frac{d^2J^2}{dt^2} - \frac{\dot{\mathcal{W}}}{\mathcal{W}}\frac{dJ^2}{dt} + 2\mathcal{W}^2RJ^2 = 0. \tag{42}$$

To obtain an oscillator equation we apply the substitution [4]:

$$J^2 = Je^{\frac{1}{2}\int\frac{\dot{\mathcal{W}}}{\mathcal{W}}dt} = J\sqrt{\mathcal{W}}. \tag{43}$$

The quantity $\mathcal{W}$ is never negative since its values equal the kinetic energy by definition, so $\sqrt{\mathcal{W}}$ always exists. Hence we find the oscillator equation:

$$\ddot{J} + \frac{1}{2}J\left(\frac{\ddot{\mathcal{W}}}{\mathcal{W}} + 4\mathcal{W}^2R - \frac{3}{2}\left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right)^2\right) = 0. \tag{44}$$

Observe that $\sqrt{\mathcal{W}} \leqslant \mathcal{E}$ so we can examine only $J$ instead of $J^2$. Eq. (44) is an oscillator equation with time-dependent frequency:

$$\Omega = \frac{1}{2}\left(\frac{\ddot{\mathcal{W}}}{\mathcal{W}} + 4\mathcal{W}^2R - \frac{3}{2}\left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right)^2\right). \tag{45}$$

Thus, the analysis of behaviour of the system is transformed to the analysis of solutions of the following equation:

$$\ddot{J} + \Omega J = 0. \tag{46}$$

As we mentioned earlier, the above equation is an oscillator equation with time-dependent frequency $\Omega$. Note that the frequency is not explicitely time-dependent.[3] However, it consists of functions of positions and velocities of the spring mass that are time-dependent. Hence, in order to solve (46) we must know the solutions of dynamics equations[4] (19). In what follows the JLC equation is numerically solved, some numerical results are illustrated and discussed.

## 4. Numerical results

In order to solve numerically Eq. (46) we put the following values of the parameters: ($\alpha = 2, \beta = 1$), ($\alpha = 0.1, \beta = 0.2$) and ($\alpha = 1, \beta = 2$). The results are displayed in four figures for each case (except the last case, where six figures are presented). The first figure shows the projection of the phase trajectories onto $(z,\dot{z})$ subspace. The second one presents a corresponding Poincaré section or another projection onto $(\varphi,\dot{\varphi})$ subspace. Next figures present the evolution of $J$, $\ln|J|$.

Our first case is a quasiperiodic one (see Fig. 2). It is easy to see that $J$ evolves rapidly and fluctuates along the trajectory. The rate of the growth is presented in next figure, where $\ln|J|$ is put on the vertical axis. The second case is a weak-chaotic one (see Fig. 3). Observe that behaviour of $J$ is similar to the previous one (see Fig. 2), i.e. $J$ evolves rapidly as well. Hence, it is rather hard to distinguish between these two cases. In the last case we have a qualitatively different situation (see Fig. 4). One can observe that the evolution of $J$ is bounded and resembles the vibrations with amplitude modulation. Such evolution of $J$ should indicates that a behaviour of the system is regular.

[3] It should be emphasised that $\Omega$ is not periodic in general.

[4] We have a similar situation in computation of Lyapunov exponents where tangent dynamics is involved.

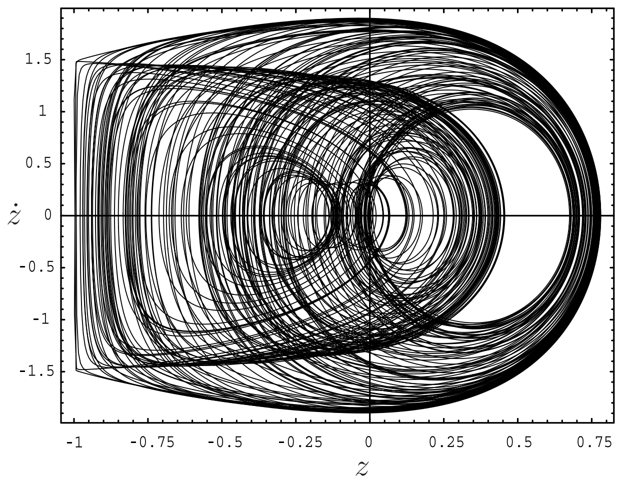


Fig. 2a. Projection $(z, \dot{z})$ for $\alpha = 2$, $\beta = 1$, $z(0) = 0.2$, $\dot{z}(0) = 0.4$, $\varphi(0) = 1.7$, $\dot{\varphi}(0) = 0.8$.

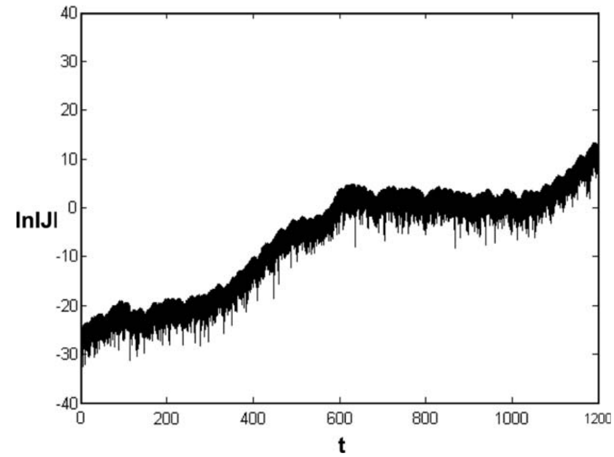


Fig. 2d. Evolution of $\ln|J|$ for $\alpha = 2$, $\beta = 1$, $z(0) = 0.2$, $\dot{z}(0) = 0.4$, $\varphi(0) = 1.7$, $\dot{\varphi}(0) = 0.8$.

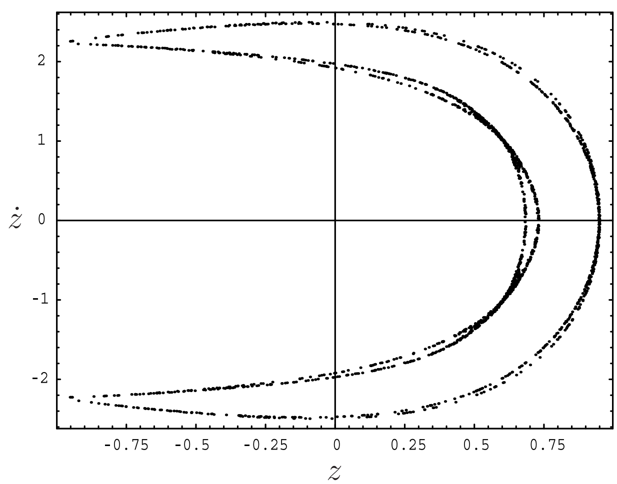


Fig. 2b. Poincaré map $(z, \dot{z})$ for $\alpha = 2$, $\beta = 1$, $z(0) = 0.2$, $\dot{z}(0) = 0.4$, $\varphi(0) = 1.7$, $\dot{\varphi}(0) = 0.8$.

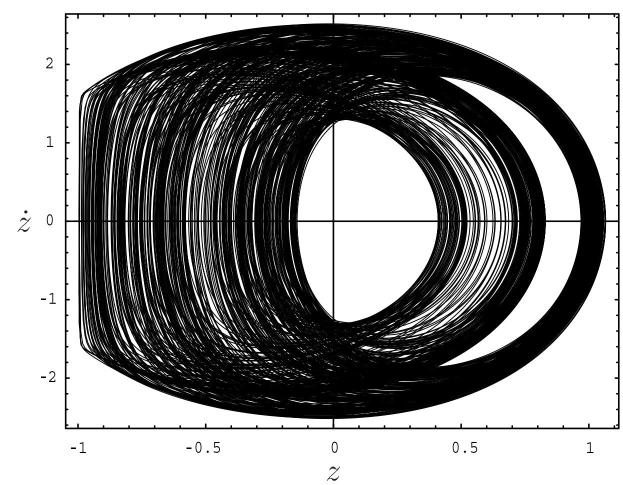


Fig. 3a. Projection $(z, \dot{z})$ for $\alpha = 1$, $\beta = 2$, $z(0) = 0.2$, $\dot{z}(0) = 2.4$, $\varphi(0) = 0.7$, $\dot{\varphi}(0) = 0.4$.

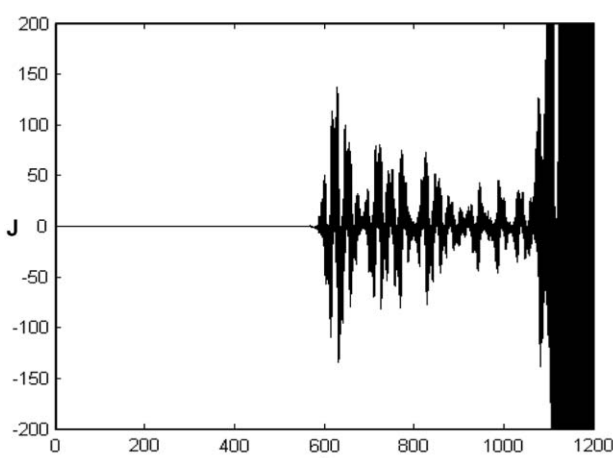


Fig. 2c. Evolution of $J$ for $\alpha = 2$, $\beta = 1$, $z(0) = 0.2$, $\dot{z}(0) = 0.4$, $\varphi(0) = 1.7$, $\dot{\varphi}(0) = 0.8$.

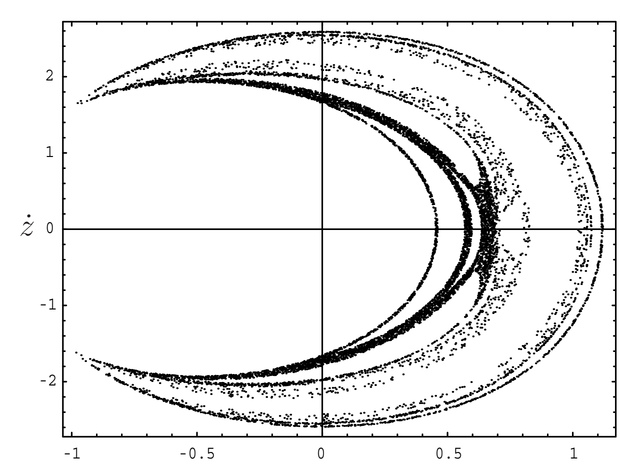


Fig. 3b. Poincaré map $(z, \dot{z})$ for $\alpha = 1$, $\beta = 2$, $z(0) = 0.2$, $\dot{z}(0) = 2.4$, $\varphi(0) = 0.7$, $\dot{\varphi}(0) = 0.4$.

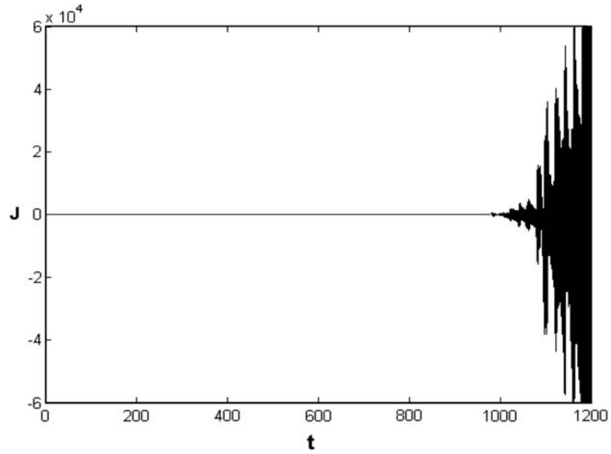


Fig. 3c. Evolution of $J$ for $\alpha = 1$, $\beta = 2$, $z(0) = 0.2$, $\dot{z}(0) = 2.4$, $\varphi(0) = 0.7$, $\dot{\varphi}(0) = 0.4$.

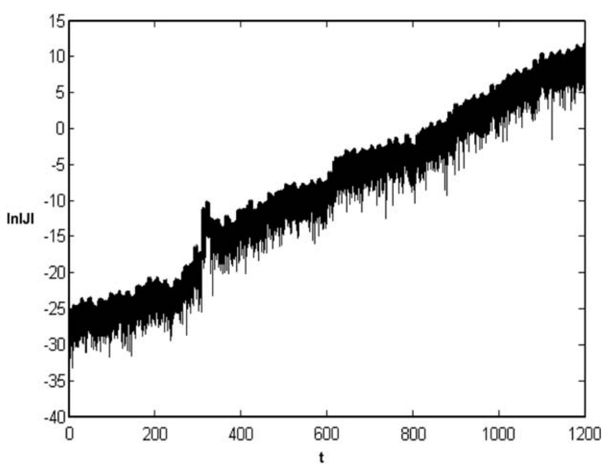


Fig. 3d. Evolution of $\ln|J|$ for $\alpha = 1$, $\beta = 2$, $z(0) = 0.2$, $\dot{z}(0) = 2.4$, $\varphi(0) = 0.7$, $\dot{\varphi}(0) = 0.4$.

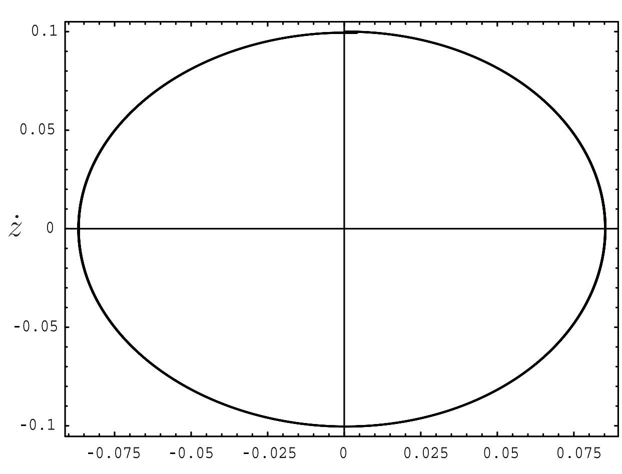


Fig. 4a. Projection $(z, \dot{z})$ for $\alpha = 0.1$, $\beta = 0.2$, $z(0) = 0$, $\dot{z}(0) = 0.1$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0.03$.

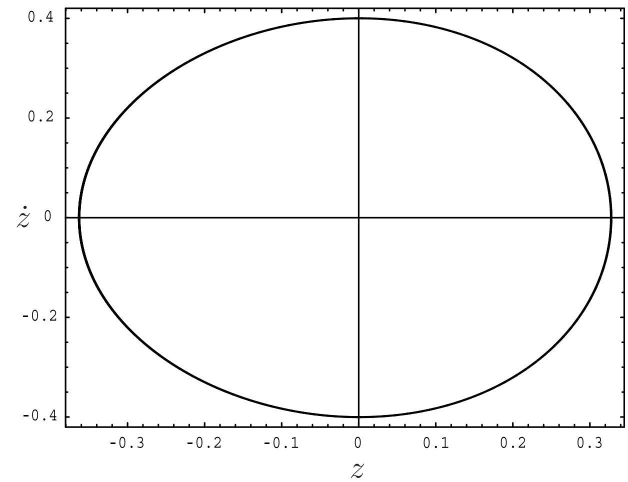


Fig. 4b. Poincaré map $(z, \dot{z})$ for $\alpha = 0.1$, $\beta = 0.2$, $z(0) = 0$, $\dot{z}(0) = 0.1$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0.03$.

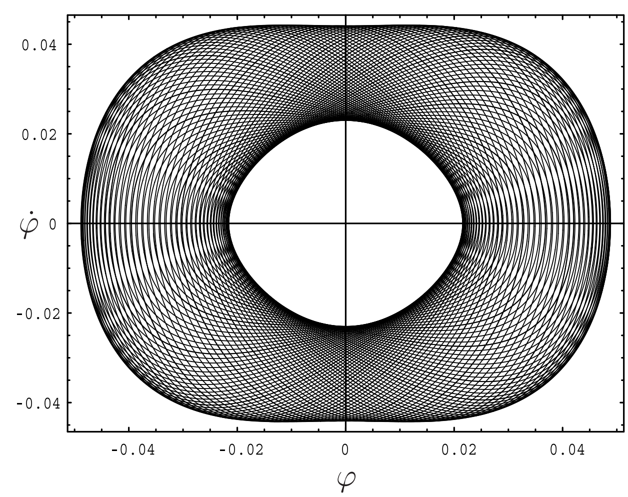


Fig. 4c. Projection $(\varphi, \dot{\varphi})$ for $\alpha = 0.1$, $\beta = 0.2$, $z(0) = 0$, $\dot{z}(0) = 0.1$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0.03$.

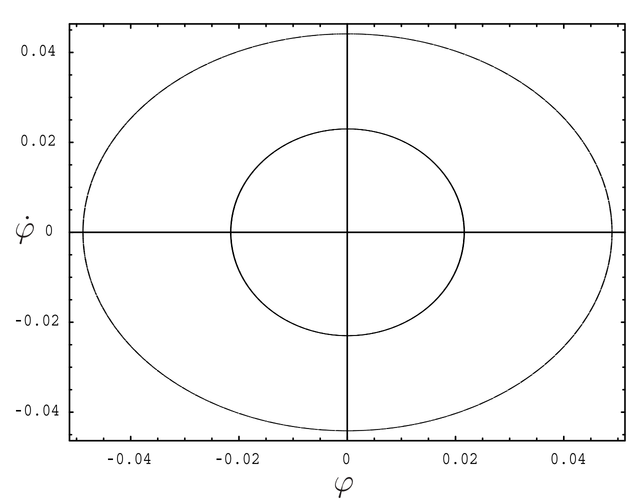


Fig. 4d. Poincaré map $(\varphi, \dot{\varphi})$ for $\alpha = 0.1$, $\beta = 0.2$, $z(0) = 0$, $\dot{z}(0) = 0.1$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0.03$.

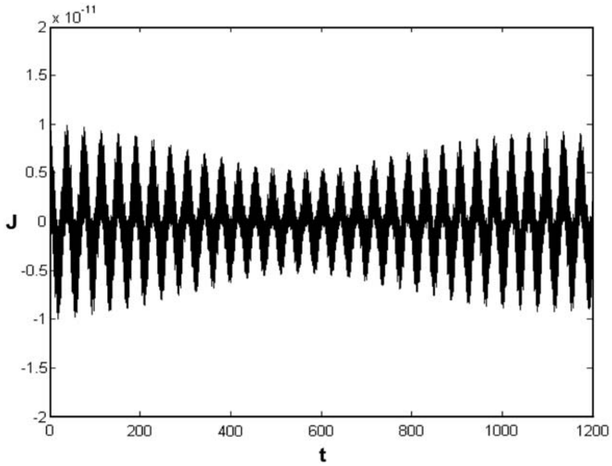


Fig. 4e. Evolution of $J$ for $\alpha = 0.1$, $\beta = 0.2$, $z(0) = 0$, $\dot{z}(0) = 0.1$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0.03$.

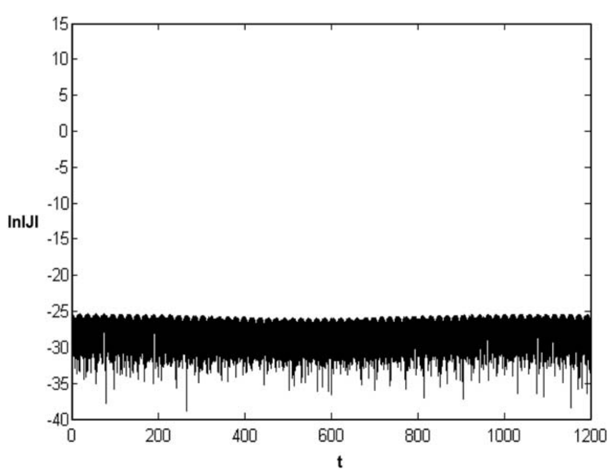


Fig. 4f. Evolution of $\ln|J|$ for $\alpha = 0.1$, $\beta = 0.2$, $z(0) = 0$, $\dot{z}(0) = 0.1$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0.03$.

However, we can observe (see Fig. 4a–d) in phase portraits and Poincaré sections that we have rather quasiperiodic behaviour than periodic one. In order to explain this fact we must take a look at the $(z, \dot{z})$ and $(\varphi, \dot{\varphi})$ Poincaré sections and phase portraits. One can observe that the phase trajectories do not penetrate the whole energy surface. The phase trajectories are bounded to a small region (see the scale in Fig. 4a–d). Hence, there cannot be any rapid growth of $J$ (see Fig. 4e).

## 5. Conclusions

In this paper we have shown how geometrical approach governs behaviour of a two degrees-of-freedom Hamiltonian system (the swinging pendulum) as an alternative way. Our results indicate that in some cases it is rather difficult to distinguish between chaotic and quasiperiodic behaviour of the system. However, it is expected that this situation will improve for systems with many degrees-of-freedom, where some averaging methods can be applied [2]. Notice that some averaging methods can be applied to systems of lower dimensions [7].

The lack of possibility of distinguishing between chaotic and quasiperiodic cases can be caused by the fact that our system possesses two degrees-of-freedom. Hence, the kinetic energy can be close to zero and it can cause the rapid growth of the frequency $\Omega$. Moreover, we showed that quasiperiodic behaviour can give a steady evolution of $J$ as well, that is expected for rather a periodic behaviour. Taking into account these facts, the geometrical approach should be further investigated, especially for systems with coordinate-dependent kinetic energy terms.

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