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On the improved Kirchhoff equation modelling nonlinear vibrations of beams

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1 Introduction

First, let us briefly discuss the Kirchhoff approximation [1]. Consider the governing equations of nonlinear beam vibration in the following form:

$$\rho F \frac{\partial^2 W}{\partial t^2} + \frac{\partial^2 M}{\partial x^2} - \frac{\partial}{\partial x}(T\theta) = 0, \tag{1}$$

$$\rho F \frac{\partial^2 U}{\partial t^2} - \frac{\partial T}{\partial x} = 0, \tag{2}$$

where $M = EI\kappa$, $\varepsilon = \frac{\partial U}{\partial x} + 0.5\left(\frac{\partial W}{\partial x}\right)^2$, $\kappa = \frac{\partial \theta}{\partial x}$, $T = EF\varepsilon$, E is Young's modulus, F, I are the area and the static moment of transversal beam cross section, respectively, U, W are the longitudinal and normal beam displacements, ρ is the density of beam material, t is the time, and x is the spatial coordinate.

Below, we consider two cases of boundary conditions in the axial direction:

$$U = 0 \quad \text{for} \quad x = 0, L \quad \text{or} \tag{3}$$

$$T = 0 \quad \text{for} \quad x = 0,\ L. \tag{4}$$

Boundary conditions in the direction normal to the beam axis do not essentially influence our further consideration, and we take

$$W = \frac{\partial^2 W}{\partial x^2} = 0 \quad \text{for} \quad x = 0, L. \tag{5}$$

The Kirchhoff hypothesis is that the axial inertial term in Eq. (2) can be neglected. Then one obtains

$$\frac{\partial T}{\partial x} = 0,$$

$$\text{i.e., } \varepsilon = \frac{\partial U}{\partial x} + 0.5\left(\frac{\partial W}{\partial x}\right)^2 = N \equiv \text{const.} \tag{6}$$

Upon integration of relation (6) with boundary conditions (3) we have

$$N = \frac{1}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx,$$

and Eq. (1) is given in the form

$$\rho F\frac{\partial^2 W}{\partial t^2} + EI\frac{\partial^4 W}{\partial x^4} - \frac{EF}{2L}\left(\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx\right)\frac{\partial^2 W}{\partial x^2} = 0. \tag{7}$$

Equation (6) describes the approximate Kirchhoff model.

It is worth noting that in Kirchhoff's original paper [2] Eq. (6) is not presented. Kirchhoff [2, pp. 443–444] in spite of neglecting the longitudinal inertial term in Eq. (2) has also omitted the second term in Eq. (1), and the original 'Kirchhoff equation' has the form

$$\rho F\frac{\partial^2 W}{\partial t^2} - \frac{EF}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 dx\frac{\partial^2 W}{\partial x^2} = 0.$$

For axial boundary conditions (4) one obtains $N = 0$, and Eq. (1) can be linearized

$$\rho F\frac{\partial^2 W}{\partial t^2} + EI\frac{\partial^4 W}{\partial x^4} = 0.$$

Equation (7) is widely applicable in today's nonlinear mechanics. It enables, in particular, a construction of normal forms of nonlinear vibrations of the continuous system for boundary conditions (5) separating space and time variables in the following way [3]:

$$W = T(t)\sin\frac{m\pi x}{L}. \tag{8}$$

For clamped edges, when boundary conditions (4) are substituted by the following ones:

$$W = \frac{\partial W}{\partial x} = 0 \quad \text{for} \quad x = 0, L,$$

normal forms of nonlinear vibrations can be constructed using the method of Bolotin [4]. In addition, Eq. (7) is applied also to solve more complex problems [5]–[8]. However, in all known examples we cannot find any limitations explicitly given and associated with Kirchhoff hypothesis.

2 When the Kirchhoff equation can be used

To estimate the application area of Eq. (6) we transform Eqs. (1) and (2) into the dimensionless form

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial \xi^4} - \frac{\partial}{\partial \xi}\left(\left[\frac{\partial u}{\partial \xi} + 0.5\left(\frac{\partial w}{\partial \xi}\right)^2\right]\frac{\partial w}{\partial \xi}\right) = 0, \tag{9}$$

$$\alpha^2\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \xi^2} = 0.5\frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial \xi}\right)^2, \tag{10}$$

where

$$w = W/h, \quad u = UL/h^2, \quad \xi = x/L, \quad \tau = \sqrt{EI}t\Big/\left(\sqrt{\rho F}L\right), \quad h = (I/F)^{1/2}, \quad \alpha = h/L. \tag{11}$$

Parameter α is small for real structures. Does it mean, however, that the corresponding term in Eq. (10) can always be neglected? No, because A. Gol'denviezer [9] (see also [4], [10], and [11]) shows, that in addition to inclusion of variations of terms occurring in the analyzed equation, one has to take into account also variations of their derivatives. To be more precise, let us analyze the following functions:

$$F_1(\xi) = \sin(\xi), \quad F_2(\xi) = \alpha \sin(\xi/\alpha).$$

For small α (α<<1) one obtains $F_1 \ll F_2$, but $F_{1xx} \gg F_2$. So, in the analysis one has to take into account the so-called indices of the variation of functions u and w [4], [9]–[11], of the form

$$F_\xi \sim \alpha^{-\gamma} F, \quad F_\tau \sim \alpha^{-\delta} F.$$

Let us estimate variations of the functions F_1 and F_2. For F_1 we get $\gamma = 0$, whereas for F_2 we have $\gamma = 1$.

So, the variation of functions u and w with respect to the variables ξ and τ can be estimated using the parameters γ and δ.

Note that the small parameter α in Eq. (10) appears in a dynamical term. A competition with respect to time and space variation occurs due to nonlinearity. If time variation of a solution being sought is remarkably larger than that of space variation, then the first term in Eq. (10) cannot be neglected. Assume that the solution consists of the terms

$$w = A \sin(m\pi\xi) \sin(at) + B \sin((m+1)\pi\xi) \sin(bt), \tag{12}$$

where $m \gg 1$, $a, b \gg 1$.

Since on the right-hand side of Eq. (10) the square term occurs, upon a substitution of relation (12) one obtains

$$C \sin(\pi\xi) \sin((a+b)t).$$

This term exhibits slow variations in space and fast variations in time. For some defined values of a and b, the first term of Eq. (9) will be of the same order as the second term. Therefore, our task is to find variations with respect to ξ and τ, where the first term of Eq. (10) is small in comparison with the second one, and other terms keep the same order. Consequently, our problem is reduced to that of a routine asymptotic analysis [4], [9]–[11].

In Eq. (9) the first and second terms should be of the same order, and hence $\gamma = \delta$. On the right-hand side of Eq. (9) there are terms with variation in time of the order 4γ (due to the square term) and with variation in space equal to 0. Therefore, there are components of the first term of Eq. (9) of order $-4\gamma + 1$, whereas in the second term of Eq. (9) the components of the order -2γ appear. The Kirchhoff approximation can be applied when either $-4\lambda + 1 < -2\gamma$ or $\gamma < 0.5$. If boundary conditions (5) enabling for variable separation (8) are given, the following estimation holds:

$$m\pi < (I/(FL^2))^{1/4}.$$

Consequently, a direct application of Eq. (6) for an arbitrary change of space coordinate is not allowed.

3 Modified Kirchhoff equation

It should be emphasized that Eqs. (1) and (2) are obtained owing to the assumption of smallness of the rotation angle θ. Consider now a general case, when the rotation angle can be arbitrary. The governing equations are as follows:

$$\rho F \frac{\partial^2 W}{\partial t^2} + \frac{\partial}{\partial x}\left(T \sin\theta + \frac{\partial M}{\partial x}\cos\theta\right) = 0, \tag{13}$$

$$\rho F \frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\left(T \cos\theta - \frac{\partial M}{\partial x}\sin\theta\right) = 0, \tag{14}$$

where $M = EI\kappa$, $\varepsilon = \left(1 + 2\frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right)^{1/2} - 1$, $T = EF\varepsilon, \kappa = \frac{\partial\theta}{\partial x}$,

$\theta = \arctan\left(\frac{\partial W}{\partial x} \Big/ \left(1 + \frac{\partial U}{\partial x}\right)\right)$.

Substitution of dimensionless terms (11) to Eqs. (13) and (14) gives

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial}{\partial \xi}\left(\alpha^{-3}\varepsilon \sin\theta + \alpha^{-1}\frac{\partial\kappa}{\partial\xi}\cos\theta\right) = 0, \tag{15}$$

$$\alpha^2 \frac{\partial^2 u}{\partial \tau^2} - \frac{\partial}{\partial \xi}\left(\alpha^{-2}\varepsilon \cos\theta - \alpha\frac{\partial\kappa}{\partial\xi}\sin\theta\right) = 0, \tag{16}$$

$$\theta = \arctan\left(\alpha\frac{\partial w}{\partial\xi} \Big/ \left(1 + \alpha^2\frac{\partial u}{\partial \xi}\right)\right), \tag{17}$$

$$\varepsilon = \left(\left(1 + \alpha^2(2\frac{\partial u}{\partial\xi} + \alpha^2\left(\frac{\partial u}{\partial\xi}\right)^2 + \left(\frac{\partial w}{\partial\xi}\right)^2\right)\right)^{1/2} - 1. \tag{18}$$

Neglecting the terms of α^2 order in comparison to 1 and taking into account that θ has also order α and ε has the order of α^2, Eqs. (16)–(18) give

$$\frac{\partial}{\partial\xi}\varepsilon\cos\theta = 0,$$

i.e., $\varepsilon\cos\theta = N \equiv \text{const.}$, (19)

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial}{\partial\xi}\left(\frac{\partial\kappa_1}{\partial\xi}\cos\theta\right) + \alpha^{-2}N\frac{\partial^2 w}{\partial\xi^2} = 0, \tag{20}$$

where:

$$\theta = \arctan\left(\alpha\frac{\partial w}{\partial\xi}\right), \quad \varepsilon = \left(\left(1 + \alpha^2(2\frac{\partial u}{\partial\xi} + \left(\frac{\partial w}{\partial\xi}\right)^2\right)\right)^{1/2} - 1, \tag{21}$$

$$\kappa_1 = \frac{\partial^2 w}{\partial\xi^2}\left[1 + \alpha^2\left(\frac{\partial w}{\partial\xi}\right)^2\right]^{-1}. \tag{22}$$

Assuming that conditions (3) are satisfied, one may obtain the quantity N of the following form:

$$N = (a/b)\left(-1 + \left(1 + \frac{\alpha^2 b}{a^2}\int_0^1\left(\frac{\partial w}{\partial\xi}\right)^2 d\xi\right)^{1/2}\right), \tag{23}$$

$$a = \int_0^1 \cos^{-1}\theta\, d\xi, b = \int_0^1 \cos^{-2}\theta\, d\xi, \cos\theta = \left[1 + \alpha^2\left(\frac{\partial w}{\partial\xi}\right)^2\right]^{-1/2}. \tag{24}$$

Equation (20) [with conditions (21)–(24)] is called the generalized Kirchhoff equation. Unfortunately, there is no possibility to integrate it exactly. However, a decreasing equation order gives an opportunity for an efficient application of various numerical methods. Note that assuming small values of rotational angles, Eq. (20) is transformed into the classical Kirchhoff equation (7).

4 Conclusions

The Kirchhoff approach is associated with neglecting of some terms in an initial equation. However, it is necessary to estimate an order of the neglected terms to get appropriate results. New terms introduced may have the same order as those neglected within the Kirchhoff approach. The following recipe is recommended: more terms are introduced into a starting equation before applying the Kirchhoff approach, and then a successive asymptotic splitting is carried out. As a result the correct simplified equations are obtained.

Note that Eq. (7) is often considered as a certain mathematical object being isolated from any physical links. It may happen that it is used beyond its domain of application. It is clear that although the obtained results are mathematically correct, they are false from the mechanical point of view.

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