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On the solvable operators generated by uniformly correct problems

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Abstract

A first order differential-operator equation with an operator-coefficient generating a C_0 class semi-group is studied. Boundary conditions for which this equation possesses a unique solution dependent continuously on its right-hand side are derived. Two theorems are formulated and proved.

Keywords: Operators; Boundary conditions; Generator of C_0 semi-group

1. Introduction

Coupled problems of plate thermo-mechanics (Timoshenko and Kirchhoff types models) [1] can be reduced in a Hilbert space to the following first order differential-operator equation:

$$y' - Ay = f(t) \tag{1}$$

with the uniformly correct Cauchy problem for the associated homogeneous equation [2–4]. A Cauchy problem for the equation $y' - Ay = 0$ with the initial condition $y(0) = y_0$ is called uniformly correct if for any $y_0 \in D(A)$ there exists one solution; and if $y_n(0) \to 0$, then $y_n(t) \to 0$ is uniform with respect to t on each finite interval $[0, T]$, where $y_n(t)$ are solutions to the mentioned equation. It is important from both the theoretical and application points of view to define

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boundary conditions for Eq. (1) which lead to a unique solution depending continuously on its right-hand side. With this aim in view we define a maximal operator L and a minimal operator L_0 generated by differential expression determined by the left side of (1). In terms of boundary conditions we define the appropriate solvable (in the sense of Gorbachuk and Gorbachuk [5]) operators M, i.e., operators with the following properties: $L_0 \subset M \subset L$ and M^{-1} defined and bounded on the whole space. Notice that many works are devoted to description of various extension and restriction properties of differential operators (many references are cited in monograph [5]). One of the main difficulties arises from the observation that a set of boundary values of some differential operators do not overlap with the input (initial) space. In this paper we construct the space of boundary values suitable for description of solvable operators.

2. Method

Let H be a Hilbert space with the scalar product $(\cdot,\cdot)$ and norm $|\cdot|$; $L_2(H;0,b)$ is a space of measurable H-valued functions with norm square-integrable on $[0,b]$; A is a closed linear operator in H, and A is a generator of a C_0 class semi-group. The latter condition is equivalent to the uniform correctness of the Cauchy problem for the equation $y' - Ay = 0$ ([2, Chapter 1, §2], [3, Chapter 23, §3]).

On the set $D(L')$ composed of strongly differentiable functions $y(t)$ with values in $D(A)$ and satisfying the property $l[y] = y' - Ay \in L_2(H;0,b)$, the operator L' is defined: $L'y = l[y]$. The closure L of the operator L' will be called a maximal operator generated in $L_2(H;0,b)$ by the expression l. A minimal operator L_0 is defined as the closure of the restriction of L to the functions $y(t) \in D(L')$ satisfying the condition $y(0) = y(b) = 0$.

According to [5, Chapter 3, §2], an operator M is said to be solvable, if $L_0 \subset M \subset L$ and the inverse M^{-1} exists as a bounded operator defined on the whole space $L_2(H;0,b)$.

Let $U(t)$ be the semi-group generated by A, and let the following norm on H be introduced:

$$|x|_-^2 = \int_0^b \bigl|U(s)x\bigr|^2\,ds \leqslant \alpha(b)|x|^2, \quad x \in H. \tag{2}$$

We denote by H_- the completion of H by this norm. The inequality

$$\bigl|U(t)x\bigr|_-^2 = \int_0^b \bigl|U(s)U(t)x\bigr|^2\,ds \leqslant c\int_0^b \bigl|U(s)x\bigr|^2\,ds = c|x|_-^2 \quad (c > 0),$$

implies that the semi-group $U(t)$ extends continuously up to the semi-group $\widetilde{U}(t)$ in H_-. Note also that if a sequence $\{x_n\}$ in H converges to $x_0 \in H_-$ in H_-, then the sequence $\{U(t)x_n\}$ is the fundamental one in $L_2(H;0,b)$ and is convergent to the limit $\widetilde{U}(t)x_0$. Hence, the function $y(t) = \widetilde{U}(t)x_0$ belongs to $L_2(H;0,b)$. On the contrary, if the sequence $\{U(t)x_n\}$ $(x_n \in H)$ converges to $y(t)$ in $L_2(H;0,b)$, then there exists an element $x_0 \in H_-$ such that $y(t) = \widetilde{U}(t)x_0$.

Indeed, (2) implies that the sequence $\{x_n\}$ is fundamental in H_-, and one may take as x_0 the element to which $\{x_n\}$ converges. The above considerations yield the conclusion: for any $x_0 \in H_-$ the function $y(t) = \widetilde{U}(t)x_0$ belongs to $\ker L$, i.e., $Ly = 0$.

Lemma. *The domain $D(L)$ of operator L consists of the functions defined by the relation*

$$y(t) = \widetilde{U}(t)x + \int_0^t U(t-s)f(s)\,ds, \tag{3}$$

where $x \in H_-$, $f \in L_2(H; 0, b)$, and $Ly = f$.

Proof. Both the results reported in [2, Chapter 1, §6], [4, Chapter 9, §1] concerning a solution to the non-homogeneous equation (1) and the earlier considerations imply that the functions of the form (3) belong to $D(L)$. Assume now $y \in D(L)$ and $Ly = f$. In what follows one may find a sequence $\{y_n\}$, $y_n \in D(L')$, being converged to y in $L_2(H; 0, b)$ that $f_n = L' y_n$ converges to f. The function y_n can be presented in the following form:

$$y_n(t) = U(t)x_n + \int_0^t U(t-s) f_n(s)\,ds.$$

Since $\{y_n\}, \{f_n\}$ are convergent, $\{U(t)x_n\}$ is convergent too. Applying the last equality to achieve a limit, one gets that y can be presented in form (3). The lemma is proved. □

Remark. The operator $x \to \widetilde{U}(t)x$ is one-to-one continuous mapping of H_- onto $\ker L$. This observation follows from the lemma and our earlier considerations.

In addition, one more norm is introduced via the relation

$$\left(|x|_-^*\right)^2 = \int_0^b \left|U^*(s)x\right|^2 ds = \int_0^b \left(U(s)U^*(s)x, x\right) ds \leqslant \beta(b)|x|^2, \quad x \in H,$$

and the completion of H with respect to this norm is denoted by H_-^*. Observe that the space H_-^* can be treated as the negative one with respect to $H_0 = H$ [6, Chapter 1, §1]. The corresponding positive space is denoted by H_+^*. It follows from [6] that the operator I_b, an extension of the operator $\int_0^b U(s)U^*(s)\,ds$ onto H_-^*, is a one-to-one continuous mapping of H_-^* onto H_+^*.

Proceeding in a similar way, one may prove that $U^*(t)$ extends to $\widetilde{U}^*(t)$ defined on H_-^* and the operator $x \to \widetilde{U}^*(t)x$ maps H_-^* into $L_2(H; 0, b)$. Therefore, the corresponding adjoint operator

$$f \to \int_0^b U(s)f(s)\,ds \tag{4}$$

represents a continuous mapping from $L_2(H; 0, b)$ into H_+^*. Furthermore, defining as f the following function $f(s) = \widetilde{U}^*(s)I_b^{-1}x$, where $x \in H_+^*$, one may conclude that (4) maps $L_2(H; 0, b)$ onto the space H_+^*.

Now using a simple change of variables it is not difficult to establish that the operator

$$f \to \int_0^b U(b-s)f(s)\,ds$$

maps $L_2(H;0,b)$ onto H_+^*. Assuming $f(s)=\widetilde{U}(s)x$ $(x\in H_-)$ and applying the equality $\widetilde{U}(b-s)\widetilde{U}(s)=\widetilde{U}(b)$, the relation $\widetilde{U}(b)x\in H_+^*$ is obtained, and hence the operator $\widetilde{U}(b)$ maps H_- continuously into H_+^*.

The following boundary mappings $\gamma_1 : D(L)\to H_-$, $\gamma_2 : D(L)\to H_+^*$ are introduced for the functions $y\in D(L)$ by the formulas $\gamma_1 y=y(0)$ and $\gamma_2 y=y(b)$, respectively. The operators γ_1,γ_2 have the following properties:

(i) for any arbitrary elements $h_1\in H_-$, $h_2\in H_+^*$, there exists a function $y\in D(L)$, that $\gamma_1 y=h_1$, $\gamma_2 y=h_2$;
(ii) γ_1,γ_2 are continuous on $D(L)$ with the norm of the graph of L;
(iii) the restriction γ_1 on $\ker L$ is a one-to-one mapping onto H_-.

Indeed, in order to prove the property (i) it is sufficient to apply $x=h_1$, $f(s)=\widetilde{U}^*(b-s)\times I_b^{-1}(h_2-\widetilde{U}(b)h_1)$ in formula (3). In what follows we prove the property (ii). Let $y_n, y\in D(L)$, $Ly_n=f_n$, $Ly=f$ and let sequences $\{y_n\},\{f_n\}$ converge in $L_2(H;0,b)$ to y, f, respectively. According to the proved lemma, the functions y_n, y can be presented in the form

$$y(t)=\widetilde{U}(t)y_n(0)+\int_0^t U(t-s)f_n(s)\,ds,\qquad y(t)=\widetilde{U}(t)y(0)+\int_0^t U(t-s)f(s)\,ds,$$

and hence $\{\widetilde{U}(t)y_n(0)\}$ converges to $\widetilde{U}(t)y(0)$. Owing to this result and the considerations outlined before the lemma, $\{y_n(0)\}$ converges to $y(0)$ in H_- and $\{y_n(b)\}$ converges to $y(b)$ in H_+^*.

The property (iii) follows directly from the Remark.

To conclude, the ordered quadruple $(H_-,H_+^*,\gamma_1,\gamma_2)$ is a space of boundary values of the operator L in the sense of the work [7].

Theorem 1. *An operator M is solvable if and only if there exists a bounded operator $N : H_+^*\to H_-$, such that $D(M)$ consists only of the elements $y\in D(L)$, which satisfy the following condition*:

$$y(0)=N\big(y(b)-\widetilde{U}(b)y(0)\big). \tag{5}$$

Proof. Denote by $\widetilde{L}$ a restriction of L to the set of elements $y\in D(L)$, satisfying the conditions $y(0)=0$. It follows from (3) that $\widetilde{L}$ is solvable operator. Any element $y\in D(L)$ may be presented in the following form: $y=u+z$, where $u\in\ker L$, $z\in D(\widetilde{L})$, and elements u,z are defined uniquely with respect to y. Owing to this equality, if the operator M is solvable, then $D(M)$ consists of elements of the following form:

$$y=Kf+\widetilde{L}^{-1}f,\quad f\in L_2(H;0,b), \tag{6}$$

where K is a bounded linear operator from $L_2(H;0,b)$ into $\ker L$ and $Kg=0$ for any g belonging to the space values $R(L_0)$ of the operator L_0. On the contrary, a set consisting of elements of the form (6) for any K with the mentioned properties is a domain of the solvable operator. Expression (6) yields

$$M^{-1}=K+\widetilde{L}^{-1}. \tag{7}$$

Taking into account that $R(L_0)$ is closed in $L_2(H;0,b)$, a factor space $\widetilde{F}=L_2(H;0,b)/R(L_0)$ and a canonical mapping π of $L_2(H;0,b)$ onto $\widetilde{F}$ are introduced. Note that the operator K is associated with the operator $\widehat{K} : \widetilde{F} \to \ker L$, defined through the equality $K = \widehat{K}\pi$. Both operators $\widehat{K}$ and K are simultaneously continuous.

Let $T = \gamma_2 \widetilde{L}^{-1}$. Owing to the properties (i), (ii) of boundary operators, the operator T maps continuously $L_2(H; 0, b)$ onto H_+^*.

Definitions of the operators L_0, $\widetilde{L}$ imply that $\ker T$ coincides with $R(L_0)$. Therefore, the operator $\widehat{T}$ defined through the formula $T = \widehat{T}\pi$, maps continuously and one-to-one $\widetilde{F}$ onto H_+^*.

Let us introduce the operator $N : H_+^* \to H_-$ through the equality $N = \gamma_1 \widehat{K}\widehat{T}^{-1}$. Hence

$$K = \beta N T, \tag{8}$$

where $\beta : H_- \to \ker L$ is the inverse of the restriction of γ_1 to $\ker L$, i.e., $\beta x = \widetilde{U}(t)x$, $x \in H_-$. The operators N and K are simultaneously continuous. It follows from (7), (8) that a domain $D(M)$ of any arbitrary solvable operator M consists of the elements of the form

$$y = \beta N \gamma_2 z + z, \tag{9}$$

where z is an arbitrary element in $D(\widetilde{L})$, $N : H_+^* \to H_-$ is a bounded operator. And vice versa, a set consisting of elements of the form (9) creates a domain of the solvable operator.

It follows from (9) that

$$\gamma_1 y = N\gamma_2 z, \qquad \gamma_2 y = \widetilde{U}(b) N \gamma_2 z + \gamma_2 z, \tag{10}$$

and hence

$$\gamma_2 z = \gamma_2 z - \widetilde{U}(b)\gamma_1 y. \tag{11}$$

Applying the operator N to (11) and taking into account (10), formula (5) is obtained. Consequently, any element of form (9) satisfies condition (5). On the contrary, if $y \in D(L)$ satisfies (5) then the equality (9) holds. Indeed, if $y = u + z$, where $u \in \ker L$, $z \in D(\widetilde{L})$, then

$$\gamma_1 y = \gamma_1 u, \qquad \gamma_2 y = \gamma_2 u + \gamma_2 z = \widetilde{U}(b)\gamma_1 u + \gamma_2 z.$$

The obtained equalities and (5) allow us to conclude that $N\gamma_2 z = \gamma_1 y$. Therefore, y can be presented as in (9). The theorem is proved. □

Remark. Equality (11) and property (i) (see properties in the Remark before Theorem 1) of the operators γ_1, γ_2 imply that, for any solvable operator M, a set of elements of the form $y(b) - \widetilde{U}(b)y(0)$, where $y \in D(M)$, coincides with whole spaces H_+^*. Besides, it follows from the proof of Theorem 1 that the operators M, N uniquely determine each other.

Assume now that M is the restriction of L to a set of functions $y \in D(L)$, satisfying the condition

$$\bigl\{y(0), y(b)\bigr\} \in M_0, \tag{12}$$

where $\{\cdot,\cdot\}$ denotes an ordered pair, and $M_0 \subset H_- \oplus H_+^*$ is a linear relation, i.e., a linear manifold. Note that terminology associated with linear relations is introduced for example in Refs. [5,8].

Theorem 2. *The operator M is solvable if and only if the relation $(M_0 - \widetilde{U}(b))^{-1}$ is bounded with domain equal to the whole space.*

Proof. Let $\{y(0), y(b)\} \in M_0$. This is equivalent to $\{y(0), y(b) - \widetilde{U}(b)y(0)\} \in M_0 - \widetilde{U}(b)$, which yields $\{y(b) - \widetilde{U}(b)y(0), y(0)\} \in (M_0 - \widetilde{U}(b))^{-1}$. Now Theorem 2 follows from Theorem 1, where $N = (M_0 - \widetilde{U}(b))^{-1}$. Theorem 2 is proved. $\square$

It should be emphasized that conditions (12) contain a wide class of linear-type boundary conditions. By introducing a relation M_0 in different ways, one obtains various relations combining boundary values.

Let us consider for example the case, when relation M_0 consists of a set of pairs $\{y(0), y(b)\}$, satisfying the following condition:

$$S_1 y(0) = S_2 y(b), \tag{13}$$

where $S_1 : H_- \to B_0$, $S_2 : H_+^* \to B_0$ are bounded linear operators; B_0 is an arbitrary Banach space.

One may observe that condition (13) is equivalent to the following one:

$$Cy(0) = S_2\big(y(b) - \widetilde{U}(b)y(0)\big),$$

where $C = S_1 - S_2\widetilde{U}(b)$. Therefore the relation $M_0 - \widetilde{U}(b)$ consists of the set of pairs $\{g_1, g_2\}$ satisfying the condition $Cg_1 = S_2 g_2$. Therefore the relation $M_0 - \widetilde{U}(b)$ is closed and $(M_0 - \widetilde{U}(b))^{-1} = C^{-1}S_2$. Theorem 2 implies that the operator M is solvable if and only if the operator C is invertible and operator $C^{-1}S_2$ is everywhere defined.

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