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Stability analysis and Lyapunov exponents of a multi-body mechanical system with rigid unilateral constraints

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Abstract

The model of a mechanical system subjected to unilateral constraints, together with the model of stability based on the Aizerman–Gantmakher theory, are presented. Some examples of numerical computation of Lyapunov exponents for the system of triple physical pendulum with the horizontal barrier, using presented model of stability, are reported.

Keywords: Stability; Periodicity; Quasi-periodicity; Chaos; Unilateral constraints; Lyapunov exponents

1. Introduction

It is well known that impact and friction accompanies almost all real behaviour, leading to non-smooth dynamics. The non-smooth dynamical systems can be modelled as the so-called piece-wise smooth systems (PWS) and are also interesting from a theoretical point of view, since they can exhibit certain non-classical phenomena of non-linear dynamics.

One of the important tools of non-linear dynamics is the linear stability theory, useful among others in the analysis of bifurcations of periodic solutions and in the identification of attractors through Lyapunov exponents. These tools are well-developed and known in the case of smooth systems. In what follows, it will be shown how the same tools with small modifications can be used also for the PWS systems. The modifications consist in

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the suitable transformation of the perturbation in the point of discontinuity, accordingly to the so-called Aizerman–Gantmakher theory. Results of this theory, firstly applied to the systems with discontinuous vector field (systems with dry friction, for example), were used for the Lyapunov exponents calculation in systems with impacts (with discontinuous state) in Ref. [4].

In this paper, we present the model of linear stability of trajectory of a multi-degree-of-freedom mechanical system with rigid barriers imposed on its position. In those systems impacts as well as the changes in structure of the system due to the permanent activity of some obstacles, when the number of degrees of freedom of the system actually decreases, are possible. We focus on the use of this model for the numerical calculation of Lyapunov exponents, although the same method can be used in stability analysis of periodic solutions.

Some examples of identification of attractors in the system of triple physical pendulum with the horizontal barrier are given. The used system is the special case of the more general model of triple pendulum investigated in works [1,3].

2. Mathematical model of the system

Let us consider Lagrangian mechanical system of n-degrees-of-freedom with vector of generalized coordinates $\mathbf{q}(t) = [q_1(t), \ldots, q_n(t)]^{\mathrm{T}}$, symmetric $n \times n$ mass matrix $\mathbf{M}(\mathbf{q}, t)$ and $n \times 1$ force vector $\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$. Let us assume that the system is subjected to m rigid unilateral constraints $\mathbf{h}(\mathbf{q}, t) = [h_1(\mathbf{q}, t), \ldots, h_m(\mathbf{q}, t)]^{\mathrm{T}} \geqslant 0$. We define a set $I = \{1, 2, \ldots, m\}$ of indices of all defined unilateral constraints h_i and the set $I_{\mathrm{act}} = \{i_1, i_2, \ldots, i_s\}$ of indices of s constraints permanently active on a certain time interval $[t_i, t_{i+1}]$. Physically, it means that the system slides along obstacles with indices from the set I_{act}.

If the constraints are perfect (frictionless, as we assume in further considerations), the system on time interval $[t_i, t_{i+1}]$ is governed by the following set of differential and algebraic equations (DAEs):

$$\begin{aligned}
&\mathbf{M}(\mathbf{q}, t)\ddot{\mathbf{q}} = \mathbf{f}_{\mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\right)^{\mathrm{T}} \boldsymbol{\lambda}_{\mathrm{act}},\\
&0 = \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t),\\
&0 = \dot{\mathbf{h}}_{\mathrm{act}}(\mathbf{q}, t) = \frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\dot{\mathbf{q}} + \frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial t},
\end{aligned} \tag{1}$$

together with the following conditions:

$$\begin{aligned}
&\boldsymbol{\lambda}_{\mathrm{act}} = [\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_s}]^{\mathrm{T}} > 0,\\
&\mathbf{h}_{\mathrm{inact}}(\mathbf{q}, t) = [h_{j_1}(\mathbf{q}, t), h_{j_2}(\mathbf{q}, t), \ldots, h_{j_{m-s}}(\mathbf{q}, t)]^{\mathrm{T}} > 0,
\end{aligned} \tag{2}$$

where $\mathbf{h}_{\mathrm{act}}(\mathbf{q}, t) = [h_{i_1}(\mathbf{q}, t), h_{i_2}(\mathbf{q}, t), \ldots, h_{i_s}(\mathbf{q}, t)]^{\mathrm{T}}$ is the vector of s constraints permanently active on $[t_i, t_{i+1}]$, $\boldsymbol{\lambda}_{\mathrm{act}}$ is the vector of non-negative Lagrange multipliers and $\mathbf{h}_{\mathrm{inact}}$ is the vector of $m - s$ inactive constraints, i.e. constraints which indices belong to the set $I \backslash I_{\mathrm{act}} = \{j_1, j_2, \ldots, j_{m-s}\}$.

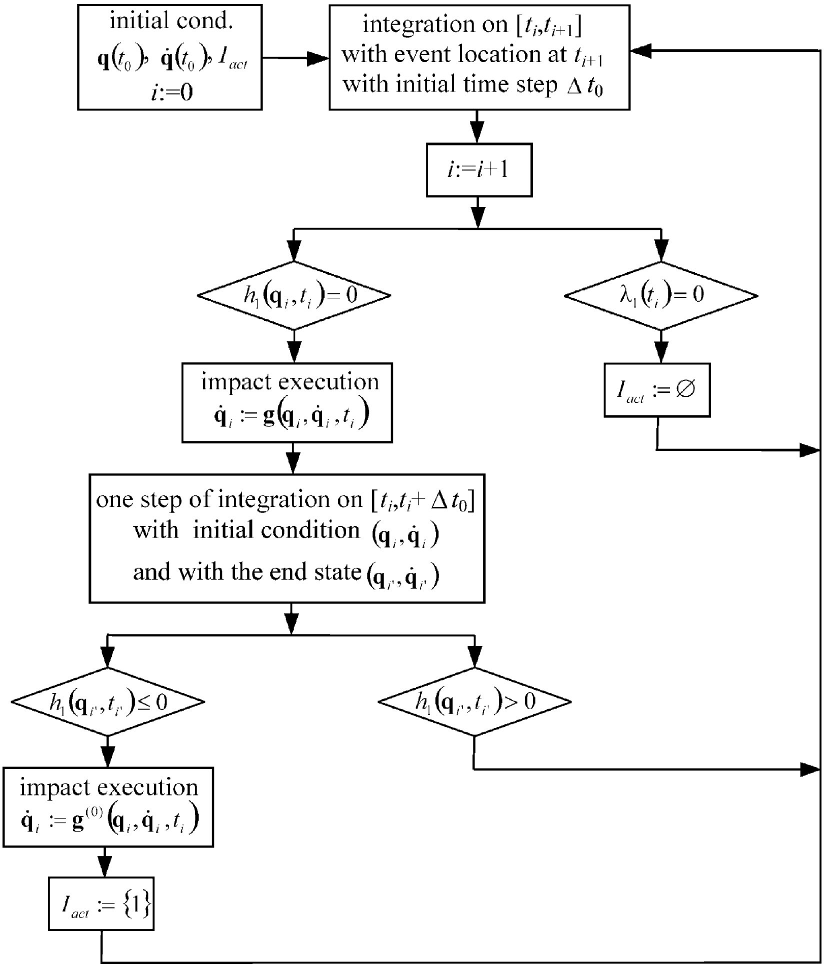

Fig. 1. Scheme for the numerical simulation of the system.

Functions (2) are the event functions and the event t_{i+1} is at a zero of one of the components of $\boldsymbol{\lambda}_{\text{act}}$ or $\mathbf{h}_{\text{inact}}$. At time t_{i+1} the suitable changes in initial conditions (due to the impact) and in the set I_{act} take place and the next piece of solution $[t_{i+1}, t_{i+2}]$ is governed by the new DAEs. In this way the system has been modelled as a PWS DAEs.

The algorithm for the execution of changes in the system state and changes in the set I_{act} at each event time t_j, used in our numerical simulation, is presented in Fig. 1. Because of the limited space, we restrict this scheme to the simplified case, where only one constraint $h_1(\mathbf{q}, t)$ is defined ($I = \{1\}$). In Fig. 1, the following notations are used: $\mathbf{q}_j = \mathbf{q}(t_j)$, $\dot{\mathbf{q}}_j = \dot{\mathbf{q}}(t_j)$, $t_{j'} = t_j + \Delta t_0$ and the function $\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t)$ represents the impact law with the restitution coefficient e while the function $\mathbf{g}^{(0)}(\mathbf{q}, \dot{\mathbf{q}}, t)$ represents impact with the restitution coefficient equal to zero independently from the system parameters.

The applied impact model is the generalized Newton's (restitution coefficient) impact law based on Ref. [2], and has the following final form for the impact with the obstacle defined by $h_i(\mathbf{q},t)=0$:

$$\mathbf{g}(\mathbf{q},\dot{\mathbf{q}},t)=\left[\begin{bmatrix}(\nabla_{\mathbf{q}}h_i(\mathbf{q},t))^{\mathrm{T}}\\ \begin{bmatrix}\mathbf{t}_1^{\mathrm{T}}\\ \cdots\\ \mathbf{t}_{n-1}^{\mathrm{T}}\end{bmatrix}\cdot\mathbf{M}(\mathbf{q},t)\end{bmatrix}\right]^{-1}$$
$$\times\cdot\left(\begin{bmatrix}-e(\nabla_{\mathbf{q}}h_i(\mathbf{q},t))^{\mathrm{T}}\\ \begin{bmatrix}\mathbf{t}_1^{\mathrm{T}}\\ \cdots\\ \mathbf{t}_{n-1}^{\mathrm{T}}\end{bmatrix}\cdot\mathbf{M}(\mathbf{q},t)\end{bmatrix}\dot{\mathbf{q}}+\begin{Bmatrix}-(e+1)\frac{\partial h_i(\mathbf{q},t)}{\partial t}\\ 0\\ \cdots\\ 0\end{Bmatrix}\right),\tag{3}$$

where $\mathbf{t}_j$ are the base vectors of the subspace of the configuration space $\mathbf{q}$, tangent to the impact surface $h_i(\mathbf{q},t)$ at the impact point. For more details on the impact model see works [1,3].

3. Stability model

Let us write the dynamical system in the form

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},t),\tag{4}$$

where $\mathbf{x}=[\mathbf{q}^{\mathrm{T}},\dot{\mathbf{q}}^{\mathrm{T}}]^{\mathrm{T}}$ in the case of the mechanical system considered in the previous section. Then the linear stability theory in the case of smooth system based on the following variational equations:

$$\boldsymbol{\delta}\dot{\mathbf{x}}=\frac{\partial\mathbf{f}(\mathbf{x},t)}{\partial\mathbf{x}^{\mathrm{T}}}\boldsymbol{\delta}\mathbf{x}(t),\tag{5}$$

where we have assumed $\delta t=0$ since the perturbation in time is independent from the perturbation $\boldsymbol{\delta}\mathbf{x}$ ($\delta\dot{t}=0$). Eq. (5) is useful among others in the stability and bifurcation analysis of periodic solutions, as well as in the Lyapunov exponents calculation.

In the case of non-smooth dynamical system, we cannot apply directly the linear stability theory since the Jacobian in (5) is not determined. But in the case of the PWS system the function $\mathbf{f}(\mathbf{x})=\mathbf{f}_i(\mathbf{x})$ is sufficiently smooth on each time interval $[t_i,t_{i+1}]$ between two successive discontinuity points and the linear stability can be applied using variational equations (5) on intervals $[t_i,t_{i+1}]$, and applying at each discontinuity point t_i special transformation rules accordingly to the Aizerman–Gantmakher theory (for $\delta t=0$)

$$\boldsymbol{\delta}\mathbf{x}_i^+=\frac{\partial\mathbf{g}_i(\mathbf{x}_i^-,t_i)}{\partial\mathbf{x}^{\mathrm{T}}}\boldsymbol{\delta}\mathbf{x}_i^-+\left[\frac{\partial\mathbf{g}_i(\mathbf{x}_i^-,t_i)}{\partial\mathbf{x}^{\mathrm{T}}}\mathbf{f}_i(\mathbf{x}_i^-,t_i)+\frac{\partial\mathbf{g}_i(\mathbf{x}_i^-,t_i)}{\partial t}-\mathbf{f}_{i+1}(\mathbf{x}_i^+,t_i)\right]\boldsymbol{\delta}t_e,\tag{6}$$

where

$$\delta t_e = -\frac{[\partial event_i(\mathbf{x}_i^-, t_i)/\partial \mathbf{x}^{\mathrm{T}}]\delta \mathbf{x}_i^-}{(\partial event_i(\mathbf{x}_i^-, t_i)/\partial \mathbf{x}^{\mathrm{T}})\mathbf{f}_i(\mathbf{x}_i^-, t_i) + \partial event_i(\mathbf{x}_i^-, t_i)/\partial t},$$

where $\mathbf{x}_i^- = \lim_{t\to t_i^-} \mathbf{x}(t)$, $\mathbf{x}_i^+ = \lim_{t\to t_i+} \mathbf{x}(t)$, $\delta\mathbf{x}_i^+ = \lim_{t\to t_i^+} \delta\mathbf{x}(t)$, $\delta\mathbf{x}_i^- = \lim_{t\to t_i^-} \delta\mathbf{x}(t)$, $\mathbf{g}_i(\mathbf{x})$ is the function representing jump in the system state $\mathbf{x}_i^+ = \mathbf{g}_i(\mathbf{x}_i^-)$ in the discontinuity point and $event_i(\mathbf{x}, t)$ is the scalar function used for detection of the discontinuity instance at t_i ($event_i(\mathbf{x}_i^-, t_i) = 0$). For details on derivation of Eq. (6) see works [1,3,4].

The linearized (variational) DAEs of system (1) are:

$$\begin{aligned}
\mathbf{M}(\mathbf{q}, t)\delta\ddot{\mathbf{q}} &= \frac{\partial \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\delta\mathbf{q} + \frac{\partial \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}^{\mathrm{T}}}\delta\dot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}^{\mathrm{T}}}\left(\left(\frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q})}{\partial \mathbf{q}^{\mathrm{T}}}\right)^{\mathrm{T}} \boldsymbol{\lambda}_{\mathrm{act}}\right)\delta\mathbf{q} \\
&\quad + \left(\frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q})}{\partial \mathbf{q}^{\mathrm{T}}}\right)^{\mathrm{T}} \delta\boldsymbol{\lambda}_{\mathrm{act}} - \left(\frac{\partial \mathbf{M}(\mathbf{q}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\delta\mathbf{q}\right)\ddot{\mathbf{q}}, \\
0 &= \frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\delta\mathbf{q}, \\
0 &= \dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial^2 \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial \mathbf{q}\partial \mathbf{q}^{\mathrm{T}}}\delta\mathbf{q} + \frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\delta\dot{\mathbf{q}},
\end{aligned} \tag{7}$$

where

$$\ddot{\mathbf{q}} = \mathbf{M}(\mathbf{q}, t)^{-1}\left(\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathbf{h}_{\mathrm{act}}(\mathbf{q}, t)}{\partial \mathbf{q}^{\mathrm{T}}}\right)^{\mathrm{T}} \boldsymbol{\lambda}_{\mathrm{act}}\right)$$

and where we have also assumed $\delta t = 0$.

We have applied Eqs. (7) together with the transformation rules (6) in the Lyapunov exponents calculation for the mechanical system presented in Section 2. Note that Eqs. (6) with the impact law $\mathbf{g}_i(\mathbf{x}, t) = \mathbf{g}_i^{(0)}(\mathbf{x}, t)$ with the restitution coefficient equal to zero applied in the case where the sliding motions starts (see Fig. 1), gives the perturbation ($\delta\mathbf{q}$, $\delta\dot{\mathbf{q}}$) consistent with the algebraic equations in (7) and the perturbation vector $\delta\mathbf{x}^+$ lies in the $(2n-2)$-dimensional subspace (in the case of only one constraint permanently active).

In the well-known algorithm of Lyapunov exponents computation the Gram–Schmidt reorthonormalization procedure is applied after some time of integration of variational equations. After use of this procedure to the vector of perturbations $\delta\mathbf{x}$ fulfilling $2s$ algebraic equations in (7) (in the case of s constraints permanently active), we obtain the new set of perturbation vectors, from which $2n$-$2s$ satisfy the algebraic equations and $2s$ of them do not. Then in our procedure we simply set that $2s$ vectors to zero vectors, obtaining the new "degenerated" set of orthonormal vectors, satisfying algebraic equations.

4. Model of triple physical pendulum

Here we present the special case of the more general model of the externally excited triple physical pendulum with arbitrarily situated barriers, investigated in works [1,3]. Now

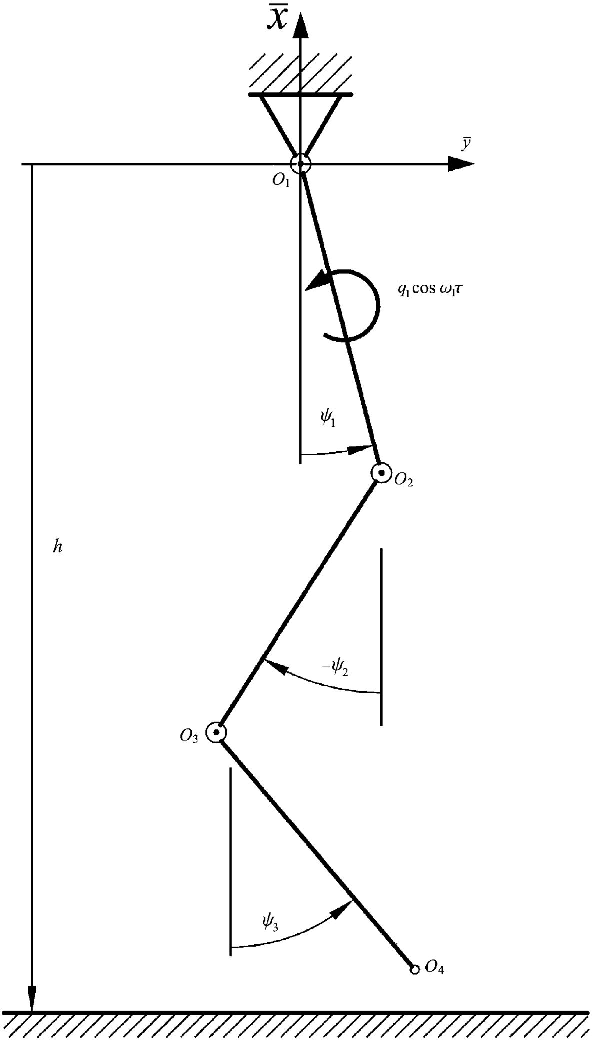

Fig. 2. Harmonically forced triple pendulum with horizontal barrier.

the pendulums, presented in Fig. 2, are reduced to the identical rods with horizontal barrier, coupled in points O_i $(i = 1, 2, 3)$ with viscous damping of coefficients $\bar{c}_i$ and moving on the plane. The system position is defined by three angles ψ_i $(i = 1, 2, 3)$, and the first link is externally forced by $\bar{q}_1 \cos \bar{\omega}_1 \tau$.

A vector of generalized coordinates is the vector of three angles $\mathbf{q} = \boldsymbol{\psi} = [\psi_1, \psi_2, \psi_3]^{\mathrm{T}}$ and the mass matrix and force vector in non-dimensional form are correspondingly

$$\mathbf{M}(\mathbf{q}, t) = \mathbf{M}(\boldsymbol{\psi}) = \begin{bmatrix} 1 & v_{12}\cos(\psi_1 - \psi_2) & v_{13}\cos(\psi_1 - \psi_3) \\ v_{12}\cos(\psi_1 - \psi_2) & \beta_2 & v_{23}\cos(\psi_2 - \psi_3) \\ v_{13}\cos(\psi_1 - \psi_3) & v_{23}\cos(\psi_2 - \psi_3) & \beta_3 \end{bmatrix},$$

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{f}(\boldsymbol{\psi}, \boldsymbol{\psi}, t) = -\mathbf{N}(\boldsymbol{\psi})\boldsymbol{\psi}^{\mathbf{2}} - \mathbf{C}\boldsymbol{\psi} - \mathbf{p}(\boldsymbol{\psi}) + \mathbf{f}_e(\boldsymbol{\psi}, \boldsymbol{\psi}, t), \tag{8}$$

where

$$\mathbf{N}(\boldsymbol{\psi}) = \begin{bmatrix} 0 & v_{12}\sin(\psi_1 - \psi_2) & v_{13}\sin(\psi_1 - \psi_3) \\ -v_{12}\sin(\psi_1 - \psi_2) & 0 & v_{23}\sin(\psi_2 - \psi_3) \\ -v_{13}\sin(\psi_1 - \psi_3) & -v_{23}\sin(\psi_2 - \psi_3) & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix},$$

$$\mathbf{p}(\boldsymbol{\psi}) = \begin{Bmatrix} \sin\,\psi_1 \\ \mu_2 \sin\,\psi_2 \\ \mu_3 \sin\,\psi_3 \end{Bmatrix}, \mathbf{f}_e(\boldsymbol{\psi}, \boldsymbol{\psi}, t) = \begin{Bmatrix} q_1 \cos\,\omega_1 t \\ 0 \\ 0 \end{Bmatrix}$$

and where $\boldsymbol{\psi}^2 = [\dot{\psi}_1^2, \dot{\psi}_2^2, \dot{\psi}_3^2]^{\mathrm{T}}$.

The following relations hold between the non-dimensional quantities and the real ones:

$$\beta_2 = \tfrac{4}{7}, \quad \beta_3 = \tfrac{1}{7}, \quad \mu_2 = \tfrac{3}{5}, \quad \mu_3 = \tfrac{1}{5}, \quad v_{12} = \tfrac{9}{14}, \quad v_{13} = \tfrac{3}{14}, \quad v_{23} = \tfrac{3}{14},$$

$$q_1 = \frac{2\bar{q}_1}{5glm}, \quad c_i = \frac{\sqrt{6}\bar{c}_i}{\sqrt{35gl^3m}},$$

$$t = \alpha_1 \tau,$$

$$\omega = \alpha_1^{-1}\bar{\omega}, \quad \dot{\psi}_j = \alpha_1^{-1}\overset{|}{\psi}_j, \quad \ddot{\psi}_j = \alpha_1^{-2}\overset{||}{\psi}_j, \tag{9}$$

where symbols $\overset{|}{(\ldots)}$ and $\overset{\bullet}{(\ldots)}$ denote derivatives with respect to the real time τ and the non-dimensional time t, respectively, m and l are mass and length of each link, and where

$$\alpha_1 = \sqrt{\frac{15g}{14l}}. \tag{10}$$

The barrier is described by the following unilateral constraint:

$$h_1(\boldsymbol{\psi}) = \eta - (\cos\,\psi_1 + \cos\,\psi_2 + \cos\,\psi_3) \geqslant 0, \tag{11}$$

where

$$\eta = \frac{h}{l}. \tag{12}$$

If the condition $3 > \eta > 2$ is satisfied, then only one inequality (11) is sufficient to describe the horizontal barrier and the obstacle lies in the region reachable by the end of the last link of the pendulum.

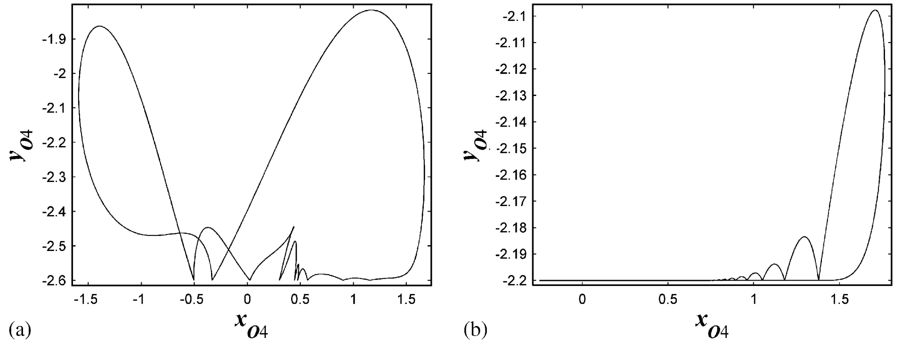

Fig. 3. Projections of the periodic attractors. The parameters are: $c_1 = c_2 = c_3 = 0.1$, $q_1 = 0.7$, $\omega_1 = 1.2$, $\eta = 2.6$, $e = 0.9$ (a); $c_1 = c_2 = c_3 = 0.2$, $q_1 = 0.75$, $\omega_1 = 1$, $\eta = 2.2$, $e = 0.8$ (b).

Table 1
The Lyapunov exponents' spectra of the presented attractors

Fig. name	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	Attractor
3a	**0**	−0.16	−0.18	−0.18	−0.95	−1.41	−2.51	Periodic
3b	**0**	−0.14	−0.14	−1.30	−2.96	−Inf	−Inf	Periodic
4ab	**0.00**	**0**	−0.02	−0.02	−0.56	−0.78	−2.21	Quasi-periodic
4cd	**0.00**	**0**	−0.02	−0.11	−1.63	−Inf	−Inf	Quasi-periodic
4ef	**0.09**	**0**	−0.03	−0.13	−1.31	−Inf	−Inf	Chaotic

5. Numerical results

Here we present some examples of numerical computation of Lyapunov exponents in the mechanical system subjected to the unilateral constraints, described in Sections 2 and 4, by the use of stability model presented in Section 3. Parameters for each example, presented in Figs. 3 and 4, are given in the figure captions. The corresponding Lyapunov exponents' spectra are collected in Table 1. In diagrams the following coordinates have been used:

$$x_{O4} = \frac{\bar{x}_{O4}}{l} = -\sin\psi_1 - \sin\psi_2 - \sin\psi_3,$$

$$y_{O4} = \frac{\bar{y}_{O4}}{l} = \cos\psi_1 + \cos\psi_2 + \cos\psi_3, \tag{13}$$

where $\bar{x}_{O4}$ and $\bar{y}_{O4}$ are the position coordinates of the point O_4 in the pendulum movement plane.

In Fig. 3 we have projection of two periodic trajectories. The first one (Fig. 3a) is the periodic solution with impacts but without the part of trajectory lying on the surface of the barrier and second one (Fig. 3b) has segments of trajectory with permanent contact with

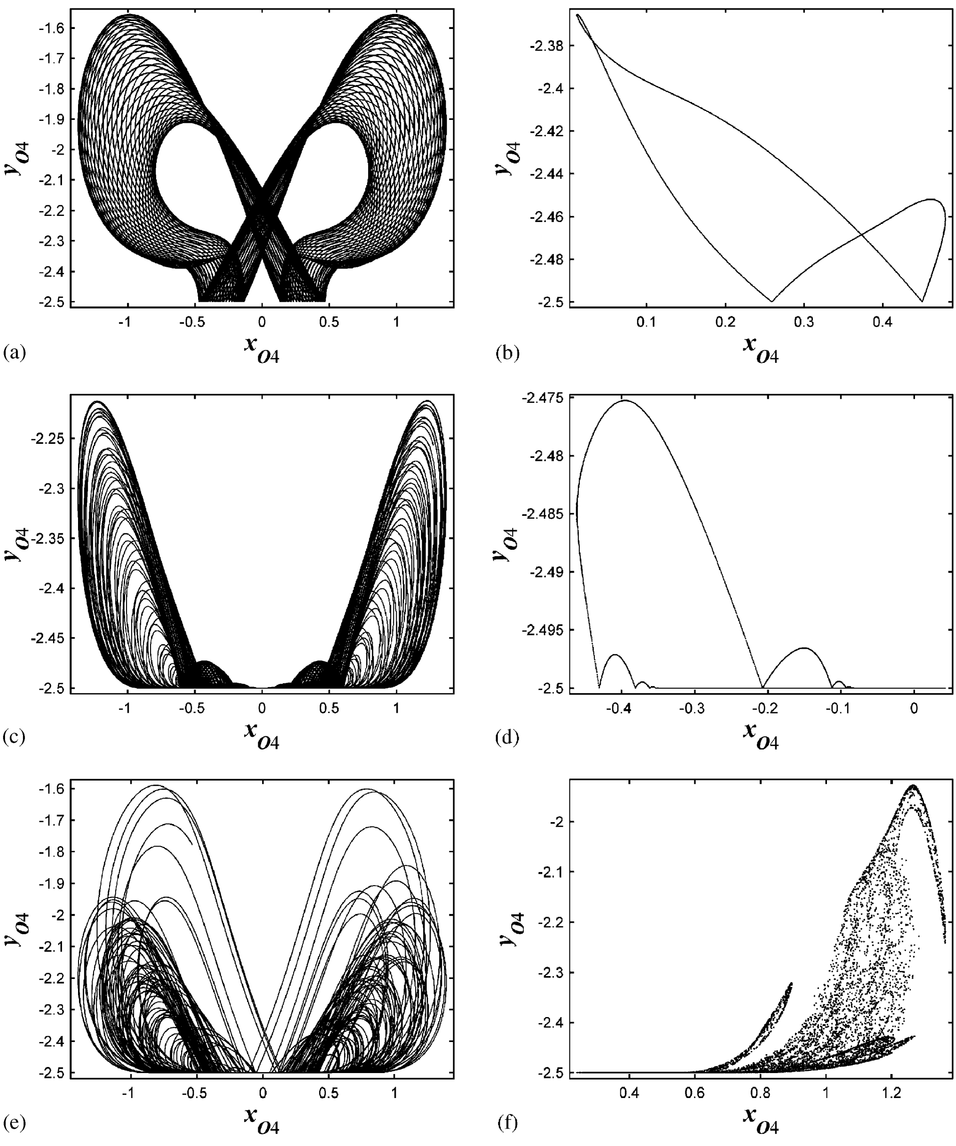

Fig. 4. Projections of two quasi-periodic attractors (a–d) and the chaotic one (e–f). The parameters are: $c_1=c_2=c_3=0.08$, $q_1=0.83$, $\omega_1=1.211$, $\eta=2.5$, $e=0.9$ (a—trajectory; b—Poincaré section); $c_1=c_2=c_3=0.1$, $q_1=0.65$, $\omega_1=1.211$, $\eta=2.5$, $e=0.5$ (c—trajectory; d—Poincaré section); $c_1=c_2=c_3=0.02$, $q_1=0.5$, $\omega_1=1$, $\eta=2.5$, $e=0$ (e—trajectory; f—Poincaré section).

the obstacle in a certain time interval. In the later case there are two Lyapunov exponents in minus infinity (the symbol "Inf" in Matlab language).

In Figs. 4a–d two quasi-periodic attractors, with two zero Lyapunov exponents, are presented. The first one (Figs. 4a and b) is the solution with impacts only and the second one (Figs. 4c and d) is the orbit with parts of trajectory lying on the obstacle and with degenerated spectrum of Lyapunov exponents.

Figs. 4e and f show an example of chaotic attractor with one positive Lyapunov exponent. The solution has two degenerated Lyapunov exponents due to some parts of trajectory with permanent contact with the barrier.

Note also that one of the Lyapunov exponents, related to the perturbation in time, is always strictly zero, and is not computed numerically since the investigated system is non-autonomous.

6. Concluding remarks

In the paper the Aizerman–Gantmakher theory, handling with perturbed solution in points of discontinuity, is used to extend classical method for computing Lyapunov exponents for the multi-degree of freedom mechanical system with rigid barriers imposed on its position. Some examples of identification of attractors in the system of triple pendulum with horizontal barrier are presented, including periodic, quasi-periodic and chaotic attractors with impacts as well as attractors with some segments of trajectory lying on the surface of the obstacle, where the obstacle is permanently active in a certain time interval and the two Lyapunov exponents are degenerated having the value in the minus infinity.

We have focused on the calculation of Lyapunov exponents, although the same method can be used in the stability and bifurcation analysis of periodic solutions.

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