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Journal of Sound and Vibration 288 (2005) 395–398
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Applicability of the Kirchhoff approach to the theory of vibrations of rods

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Received 9 December 2003; received in revised form 6 January 2005; accepted 19 April 2005
Available online 13 June 2005

Abstract

A sufficient condition for applicability of the Kirchhoff approach is proposed.

The Kirchhoff approach, proposed more than 100 years ago [1], still plays a key role in modelling nonlinear transverse vibrations of rods [2–5]. Therefore, study of its range of applicability is important.

First of all, what do we mean by the Kirchhoff approach?

In order to clarify this question, let us consider the partial differential equations governing the nonlinear dynamics of elastic rods

$$\rho F \frac{\partial^2 W}{\partial t^2} + EI \frac{\partial^4 W}{\partial x^4} - EF\varepsilon \frac{\partial^2 W}{\partial x^2} = 0, \tag{1}$$

$$\rho F \frac{\partial^2 U}{\partial t^2} - EF \frac{\partial \varepsilon}{\partial x} = 0, \tag{2}$$

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doi:10.1016/j.jsv.2005.04.002

where $\varepsilon = \partial U/\partial x + 0.5(\partial W/\partial x)^2$; F and I are area and statical moment of the rod cross section, respectively; U and W are axial and normal displacements, respectively; ρ is mass per unit area; x is the axial coordinate; and t is time.

The boundary conditions with respect to U are given as

$$U = 0 \quad \text{for} \quad x = 0, L. \tag{3}$$

The boundary conditions with respect to W do not play a significant role here, and one may apply conditions for a simple support

$$W = \frac{\partial^2 W}{\partial x^2} = 0 \quad \text{for} \quad x = 0, L. \tag{4}$$

Kirchhoff proposed omitting the inertial term in Eq. (2) [1,2] in order to obtain

$$\varepsilon = \frac{\partial U}{\partial x} + 0.5\left(\frac{\partial W}{\partial x}\right) = N \equiv \text{const.} \tag{5}$$

Integration of (5) using the boundary condition (3) yields

$$N = \frac{1}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x,$$

and Eq. (1) is transformed to the following form:

$$\rho F \frac{\partial^2 W}{\partial t^2} + EI\frac{\partial^4 W}{\partial x^4} - \frac{EF}{2L}\int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x \frac{\partial^2 W}{\partial x^2} = 0. \tag{6}$$

Eq. (6) describes the Kirchhoff approximate model for nonlinear vibrations of the rods.

So, the Kirchhoff approach yields a single governing equation instead of a coupled system of partial differential equations describing the longitudinal and transversal vibrations of rods.

In what follows we are going to explain why the Kirchhoff approach is so widely used.

The boundary value problem (4) and (6) can be considered by the method of separation of variables after substitution:

$$W = D(t)\sin\frac{m\pi x}{L}. \tag{7}$$

Hence, we construct normal modes of nonlinear vibrations of the rod [3,4].

For rods with clamped edges

$$W = \frac{\partial W}{\partial x} = 0 \quad \text{for} \quad x = 0, L,$$

the normal modes of nonlinear vibrations can be constructed using the Bolotin method [6,7].

In a general case of nonlinear free or forced vibrations, the Kirchhoff approach also provides important simplifications [5].

In order to estimate the range of applicability of Eq. (6), it is useful to transform Eqs. (1) and (2) into the following non-dimensional form

$$\frac{\partial^2 w}{\partial T^2} + \frac{\partial^4 w}{\partial \xi^4} - \varphi^{-2}\left[\frac{\partial u}{\partial \xi} + 0.5\alpha\left(\frac{\partial w}{\partial \xi}\right)^2\right]\frac{\partial^2 w}{\partial \xi^2} = 0, \tag{8}$$

$$\varphi\frac{\partial^2 u}{\partial T^2}-\frac{\partial^2 u}{\partial \xi^2}=0.5\alpha\frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial \xi}\right)^2, \tag{9}$$

where $w=W/h$; $u=U/h$; $\xi=x/L$; $T=\sqrt{EI}t/(\sqrt{\rho F}L^2)$; $\varphi^2=\sqrt{I}/(\sqrt{F}L^2)$, and $\alpha=h/L$.

It is assumed that φ is a small parameter.

First, let us illustrate the analysis by using the so-called 'naive' or intuitive interpretation. The small parameter φ occurring in Eq. (9) stands for the dynamical term. This term cannot be neglected 'ad hoc' since there is an interaction and competition in both time and spatial variables. Note that if time evolution is essentially larger than spatial variable changes, then first term of Eq. (9) cannot be neglected. Assume that a solution contains the following terms:

$$w=A\sin(m\pi\xi)\sin(at)+B\sin((m+1)\pi\xi)\sin(bt) \tag{10}$$

and the following inequalities hold: $m \gg 1$, $a, b \gg 1$.

Since in the right-hand side of Eq. (9) the squared term appears, substituting expression (10) into it producing the following term:

$$C\sin(\pi\xi)\sin((a+b)t).$$

The given term exhibits slow changes with respect to the spatial coordinate and fast evoultion with respect to time. However, for some values of 'a' and 'b', the first and second terms of Eq. (9) may be of the same order. Our target is to find the appropriate values of 'a' and 'b', i.e. the appropriate changes with respect to ξ and T. This task belongs to a standard procedure of asymptotical analysis [7,8].

For the asymptotic analysis of Eqs. (8) and (9), one has to take into account of changes of the functions w and u with respect to both time T and the coordinate ξ. The indices of variation of the functions u, w [7,8] can be introduced as follows:

$$\frac{\partial}{\partial \xi}\{u;w\}\sim\varphi^{-\delta_1}\{u;w\};\quad \frac{\partial}{\partial T}\{u;w\}\sim\varphi^{-\delta_2}\{u;w\}.$$

In Eq. (8) the first and the second terms must be of the same order, and hence, $\delta_2=2\delta_1$. Note that in the right-hand side of Eq. (9) there are terms with changes in time of order of $4\delta_1$ due to occurrence of squared term, and with changes of order 0 with respect to spatial variable. To conclude, there are components of order $-4\delta_1+1$ in the first term of Eq. (9), whereas the second term of Eq. (9) includes components of order $-2\delta_1$. Therefore, the Kirchhoff approach can be used either if $-4\delta_1+1<-2\delta_1$ or $\delta_1<0.5$. For the input variables this implies that the following inequality should be satisfied:

$$m\pi<(I/(FL^2))^{1/4}.$$

Acknowledgement

The authors are grateful to an anonymous referee, whose valuable suggestions and comments helped to improve the paper.

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