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On continuous approximation of discontinuous systems☆

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Abstract

By using Tichonov theorem for singularly perturbed differential equations, we study the relationship between dynamics of discontinuous differential equations and their continuous approximations along periodic solutions.

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1. Introduction

There are many physical systems in which the mathematical modeling leads to discontinuous dynamical systems which switch between different states, and the dynamics of each state is given by a different set of differential equations [1–3,5,10]. From the mathematical point of view, several ways exist to handle such discontinuous differential equations.

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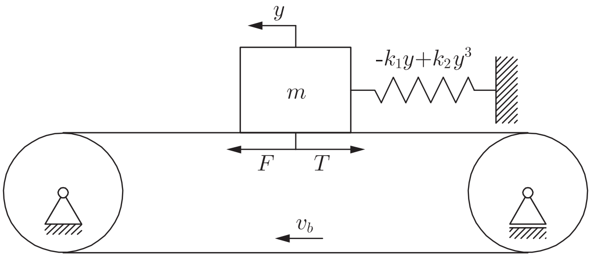

Fig. 1. The considered one-degree of freedom mechanical system with friction.

One way is to use the theory of differential inclusions [7,9]. Another way is a continuous approximation of discontinuities to get smooth differential equations [4].

In this paper we follow the second way. So we consider differential equations with discontinuous nonlinearities. Then we continuously approximate those nonlinearities by using one-parametric families of continuous functions. The parameter is $\varepsilon > 0$ and ε goes to 0. To study the dynamics of the approximated equation, we split it for variables near and far from the discontinuities. We scale the variables near discontinuities to get singular differential equations. Then we use results from the theory of singularly perturbed differential equations, like Tichonov theorem [11]. Finally, we combine the dynamics of singularly perturbed and normal differential equations to get the dynamics of the original approximated differential equations. We use this method for the study of persistence and stability of a periodic solution of the discontinuous systems under continuous approximation. Summarizing, the method is based on the construction of Poincaré maps along periodic solutions of discontinuous systems and their continuous approximations. Then Tichonov theorem is applied to study the relationship between those Poincaré maps. Some transversal assumptions are needed to derive those Poincaré maps.

The plan of this paper is as follows. In Section 2, we present a simple model of one-degree-of-freedom mechanical system to illustrate the main idea of the method used. Then in Section 3, we extend this method for higher-dimensional general discontinuous systems. In both sections, we study the persistence of periodic solutions of discontinuous systems under continuous approximation.

2. An illustrative example

Let us consider a one-degree-of-freedom mechanical system (see Fig. 1) consisting of a mass m oscillating on a belt which moves with constant velocity v_{b}, and which is connected to a nonlinear oscillator with the elastic support characterized by constants k_1 and k_2. Such a model is governed by the following equation of motion:

$$m\ddot{y} - k_1 y + k_2 y^3 - T = 0, \tag{2.1}$$

where friction force T is applied here in the following form:

$$T = -\frac{\mu_0}{1 + |\dot{y} - v_{\mathrm{b}}|}\,\mathrm{sgn}(\dot{y} - v_b).$$

Considering $m = k_1 = k_2 = v_b = 1$ and static friction coefficient $\mu_0 = 0.5$, the following simplified equation governs the dynamics of our dynamical system:

$$\ddot{y} - y + y^3 - \frac{0.5}{1 + |\dot{y} - 1|} \operatorname{sgn}(1 - \dot{y}) = 0. \tag{2.2}$$

By putting $\dot{y} = -w + 1$, we get the system

$$\begin{aligned} \dot{y} &= -w + 1, \\ \dot{w} &= y^3 - y - \frac{0.5}{1 + |w|} \operatorname{sgn} w. \end{aligned} \tag{2.3}$$

Now we approximate (2.3) by the system

$$\begin{aligned} \dot{y} &= -w + 1, \\ \dot{w} &= y^3 - y - f_\varepsilon(w) \end{aligned} \tag{2.4}$$

for $\varepsilon > 0$ small and a function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f_\varepsilon(w) := \begin{cases} \dfrac{0.5}{1 + |w|} \operatorname{sgn} w & \text{for } |w| \geqslant \varepsilon, \\ \dfrac{0.5w}{(1 + \varepsilon)\varepsilon} & \text{for } |w| \leqslant \varepsilon. \end{cases}$$

Then (2.4) for $w \geqslant \varepsilon$ has the form

$$\begin{aligned} \dot{y} &= -w + 1, \\ \dot{w} &= y^3 - y - \frac{0.5}{1 + w}, \end{aligned} \tag{2.5}$$

which is (2.3) for $w > 0$. For $|w| \leqslant \varepsilon$, we take $w = \varepsilon v$, $|v| \leqslant 1$, and (2.4) has the form

$$\begin{aligned} \dot{y} &= -\varepsilon v + 1, \\ \varepsilon \dot{v} &= y^3 - y - \frac{0.5}{1 + \varepsilon} v. \end{aligned} \tag{2.6}$$

If we check the vector field of (2.5) (see Fig. 2) near the line $w = 0$ for $w > 0$, we see that for $y < y_0$ the line $w = 0$ is attracting, and for $y > y_0$ the line $w = 0$ is repelling. Of course, the variable y is increasing. Here $y_0^3 - y_0 = 0.5$, $y_0 = 1.19149$.

Now we can check by the program *Mathematica* that the solution of (2.5) with the initial conditions $y(0) = y_0$, $w(0) = 0$ hits the line $w = 0$ at time $t_0 = 6.73896$ in $y(t_0) := \bar{y}_0 = -0.100068 \in (-y_0, y_0)$. Of course, for the discontinuous system (2.3), we get a periodic solution $p_0(t)$ starting from the point $(y_0, 0)$, which is infinitely stable, i.e. all solutions starting near periodic solution $p_0(t)$ collapse after a finite time to $p_0(t)$. We expect that its approximation (2.4) would also possess a unique periodic solution near $p_0(t)$ with a rapid attractivity. This phenomenon is numerically demonstrated in [4] for a two-degree-of-freedom autonomous system with friction. To show analytically this property for our simple system (2.4), we consider the dynamics of a Poincaré map of (2.4) near periodic orbit $p_0(t)$ of (2.3). For the construction of this Poincaré map, we take the interval

$$I := [y_0 - \delta, y_0 + \delta]$$

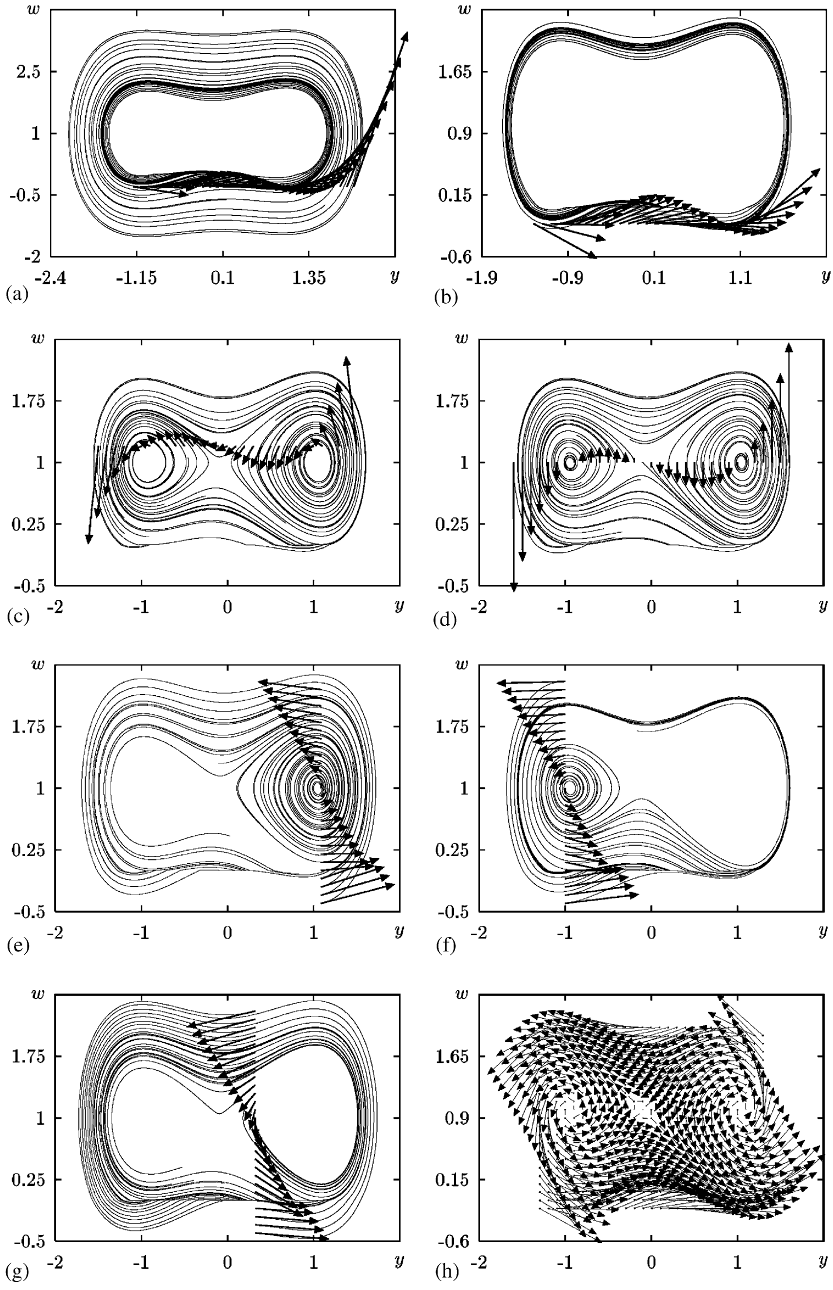

Fig. 2. Vector field of (2.3) for $\mu_0 = 0.2$ and (a) $w_0 = -0.3$, $y_{0i} \in (-1.2, 2)$ for $i = 0 \cdots 32$; (b) $w_0 = -0.2$, $y_{0i} \in (-1.3, 1.3)$ for $i = 0 \cdots 26$; (c) $w_0 = 1.2$, $y_{0i} \in (-1.5, 1.5)$ for $i = 0 \cdots 30$, (d) $w_0 = 1.0$, $y_{0i} \in (-1.6, 1.6)$ for $i = 0 \cdots 32$; (e) $y_0 = 1.08803$, $w_{0i} \in (-0.4, 2.2)$ for $i = 0 \cdots 26$; (f) $y_0 = -1.0$, $w_{0i} \in (-0.4, 2.3)$ for $i = 0 \cdots 27$; (g) $y_0 = 0.321889$, $w_{0i} \in (-0.4, 2.3)$ for $i = 0 \cdots 27$; (h) $w_{0k} \in (-0.2, 2.1)$ for $k = 0 \cdots 23$, $y_{0i} \in (-1.3, 1.3)$ for $i = 0 \cdots 26$.

for a fixed small $\delta > 0$. For any $\bar{y} \in I$, we consider the solution $(y(t), w(t))$ of (2.5) with the initial value conditions $y(0) = \bar{y}$, $w(0) = \varepsilon$. Then for a small $\delta > 0$, there is $\bar{t} \sim t_0$ such that $w(\bar{t}) = \varepsilon$. We put

$$\Phi_\varepsilon(\bar{y}) := y(\bar{t}).$$

We get a mapping $\Phi_\varepsilon : I \to \bar{I}$ for $\bar{I} = [\bar{y}_0 - \bar{\delta}, \bar{y}_0 + \bar{\delta}]$ and $\bar{\delta} = \bar{\delta}(\delta)$ is small. Concerning (2.6), we put $\varepsilon = 0$ and we get

$$V(y) := 2(y^3 - y)$$

as a solution of the equation

$$f(V, y) := y^3 - y - 0.5V = 0.$$

Moreover, we have

$$\frac{\partial f}{\partial V}(V(y), y) < -0.5V < 0.$$

Let us consider the rectangle

$$Q := [-y_0 - 1, y_0 + 1] \times [-1, 1].$$

The graph of $V(y)$ leaves Q only at the points $(-y_0, -1)$ and $(y_0, 1)$. So we can apply Tichonov theorem [11] to the singularly perturbed system (2.6). We take $y(0)=\bar{y} \in \bar{I}$, $v(0)=1$ and the corresponding solution $(y(t), v(t))$ of (2.6) leaves Q near y_0 at time $\tilde{t}$. We put

$$\Psi_\varepsilon(\bar{y}) := y(\tilde{t}).$$

We get a map $\Psi_\varepsilon : \bar{I} \to I$. Finally, we put

$$P_\varepsilon(y) := \Psi_\varepsilon(\Phi_\varepsilon(y))$$

for $y \in I$. Clearly, $P_\varepsilon : I \to I$ and this is the desired Poincaré map of (2.4) near periodic solution $p_0(t)$ of (2.3). The map Φ_ε depends smoothly on ε small and $y \in I$. Similarly, the map Ψ_ε depends smoothly on $\varepsilon > 0$ small and $y \in I$. We need to find the limit of $\Psi_\varepsilon(y)$ as $\varepsilon \to 0_+$. To do this, we apply Tichonov theorem [11]. For system (2.6), we have already verified all assumptions of this theorem except Vth in [11, p. 31]: Namely, we must show that for any $(y, v^0) \in Q$, $y \in \bar{I}$, it holds

$$\begin{aligned} &(v(\tau), y) \in Q \quad \text{for } \tau \geqslant 0, \\ &v(\tau) \to V(y) \quad \text{as } \tau \to \infty, \end{aligned} \tag{2.7}$$

where $v(\tau)$ is the solution of the equation

$$\dot{v}(\tau) = y^3 - y - 0.5v(\tau),$$

$$v(0) = v^0.$$

Clearly, we have

$$v(\tau) = \mathrm{e}^{-0.5\tau}(v^0 - 2(y^3 - y)) + 2(y^3 - y).$$

If $v^0 \geqslant 2(y^3 - y)$ then $2(y^3 - y) \leqslant v(\tau) \leqslant v^0$. Since $|v^0| \leqslant 1$ and $|2(y^3 - y)| \leqslant 1$ then $|v(\tau)| \leqslant 1$, and condition (2.7) holds. Similarly, if $v^0 \leqslant 2(y^3 - y)$ then $2(y^3 - y) \geqslant v(\tau) \geqslant v^0$ and again $|v(\tau)| \leqslant 1$, and condition (2.7) still holds. Summarizing, we can apply Tichonov theorem to (2.6). Consequently, the solution $(y(t), v(t))$ of (2.6) with the initial value conditions $y(0) = \bar{y} \in \bar{I}$ and $v(0) = 1$ has the asymptotic expansion

$$y(t) = \bar{y} + t + O(\varepsilon),$$

$$v(t) = 2((\bar{y} + t)^3 - \bar{y} - t) + \mathrm{e}^{-0.5t/\varepsilon}(1 - 2(y^3 - y)) + O(\varepsilon).$$

Time $\tilde{t}$ is determined from the equation $v(\tilde{t}) = 0$ and we get

$$\bar{y} + \tilde{t} \sim y_0 + O(\varepsilon).$$

Hence,

$$\Psi_\varepsilon(\bar{y}) = y(\tilde{t}) = \bar{y} + \tilde{t} + O(\varepsilon) = y_0 + O(\varepsilon).$$

This gives

$$P_\varepsilon(y) = \Psi_\varepsilon(\Phi_\varepsilon(y)) = y_0 + O(\varepsilon). \tag{2.8}$$

We note that the limit map $P_0(y) = y_0$ in (2.8) is just the Poincaré map along the periodic solution $p_0(t)$ of (2.3). Furthermore, the identity (2.8) holds also in the C^1-topology, i.e. it holds

$$P'_\varepsilon(y) = O(\varepsilon). \tag{2.9}$$

Hence the map $P_\varepsilon : I \to I$ has a unique fixed point $y_\varepsilon \in I$ of the form $y_\varepsilon = y_0 + O(\varepsilon)$, which is according to (2.9) also rapidly attractive. Summarizing, we get the next theorem.

Theorem 1. *The discontinuous system* (2.3) *has the periodic solution* $p_0(t)$ *starting from the point* $(y_0, 0)$, *which is infinitely stable, i.e. all solutions starting near* $p_0(t)$ *collapse after a finite time to* $p_0(t)$. *Its approximation* (2.4) *has also a unique periodic solution* p_ε *starting from the point* $(y_\varepsilon, \varepsilon)$ *which approximates* $p_0(t)$ *and which is rapidly attracting. This coincides with the infinite stability of* $p_0(t)$.

Theorem 1 analytically explains numerical results of [4] concerning stable periodic solutions.

Finally, we note that function f_ε is an approximation of the multivalued mapping

$$\operatorname{Sgn} w := \begin{cases} \operatorname{sgn} w & \text{for } w \neq 0, \\ [-1, 1] & \text{for } w = 0. \end{cases}$$

Hence (2.4) is an approximation of the discontinuous differential inclusion

$$\dot{y} = -w + 1,$$

$$\dot{w} - y^3 + y \in -\frac{0.5}{1 + |w|} \operatorname{Sgn} w,$$

which is a differential inclusion version of (2.3). But, of course, these arguments fit into the general theory of differential inclusions [7,9].

3. Higher dimensional systems

In this section, we consider a general discontinuous system in $\mathbb{R}^n$. For the sake of simplicity, we assume that a system has a discontinuity at level $x=0$ for $\dim x=1$. Hence $z=(x,y)\in\mathbb{R}^n$ and $\dim\ y=n-1$. The system is given by the equation

$$\dot z=\tilde f(z), \tag{3.1}$$

where

$$\tilde f(z)=\begin{cases} f_+(x,y) & \text{for } x>0,\\ f_-(x,y) & \text{for } x<0.\end{cases}$$

The functions $f_\pm:\mathbb{R}^n\to\mathbb{R}^n$ are smooth and $f_+(0,y)\neq f_-(0,y)$ in general. We put

$$f_\pm=(h_\pm(x,y),g_\pm(x,y)),$$

where $h_\pm:\mathbb{R}^n\to\mathbb{R}$ and $g_\pm:\mathbb{R}^n\to\mathbb{R}^{n-1}$. Now we take $\varepsilon>0$ small and consider a continuous approximation of (3.1) given by

$$\dot z=\tilde f(z)\quad \text{for } |x|\geqslant\varepsilon, \tag{3.2}$$

and for $|x|\leqslant\varepsilon$:

$$\begin{aligned}\dot x&=\frac{h_+(\varepsilon,y)-h_-(-\varepsilon,y)}{2\varepsilon}x+\frac{h_+(\varepsilon,y)+h_-(-\varepsilon,y)}{2},\\ \dot y&=\frac{g_+(\varepsilon,y)-g_-(-\varepsilon,y)}{2\varepsilon}x+\frac{g_+(\varepsilon,y)+g_-(-\varepsilon,y)}{2}.\end{aligned} \tag{3.3}$$

We put $x=\varepsilon w$, $|w|\leqslant 1$, in (3.3) to get the system

$$\begin{aligned}\varepsilon\dot w&=\frac{h_+(\varepsilon,y)-h_-(-\varepsilon,y)}{2}w+\frac{h_+(\varepsilon,y)+h_-(-\varepsilon,y)}{2},\\ \dot y&=\frac{g_+(\varepsilon,y)-g_-(-\varepsilon,y)}{2}w+\frac{g_+(\varepsilon,y)+g_-(-\varepsilon,y)}{2}.\end{aligned} \tag{3.4}$$

In order to apply Tichonov theorem, we consider the assumption

$$h_+(0,y)-h_-(0,y)<0. \tag{3.5}$$

If (3.5) fails, then we use (3.2)–(3.3), since the right-hand side of (3.2)–(3.3) belongs to the set

$$\begin{aligned}\tilde F(z)&:=\operatorname{conv}[f_-(x,y),f_+(x,y)]\\ &:=\{\lambda f_-(x,y)+(1-\lambda)f_+(x,y)|\lambda\in[0,1]\},\end{aligned}$$

and

$$\dot z\in\tilde F(z) \tag{3.6}$$

is the differential inclusion corresponding to (3.1).

Now we suppose condition (3.5) along with the following assumption.

(A1): There is a solution $p(t)$ of (3.1) defined on $[0, T]$ such that the x-coordinate $p_1(t)$ of $p(t)$ satisfies $p_1(t) > 0$ for $t \in (0, T)$ and $p_1(0) = p_1(T) = 0$. Moreover, $h_+(p(0)) = 0$, $h_+(p(T)) < 0$, and the gradient $\nabla_y h_+(p(0)) \neq 0$.

The condition $\nabla_y h_+(p(0)) \neq 0$ implies that the set $h_+^{-1}(0)$ is a manifold M on $\mathbb{R}^{n-1}$ near the point $p(0)$. Here we consider the restriction $h_+(0, \cdot) : \mathbb{R}^{n-1} \to \mathbb{R}$.

The reduced system of (3.4) for $\varepsilon = 0$ has the form

$$\dot{y} = H(y) := \frac{g_+(0, y) - g_-(0, y)}{2} V(y) + \frac{g_+(0, y) + g_-(0, y)}{2} \tag{3.7}$$

for

$$V(y) = \frac{h_+(0, y) + h_-(0, y)}{h_-(0, y) - h_+(0, y)}.$$

We suppose the following assumption.

(A2): Let $p_2(t)$ be the y-coordinate of $p(t)$. Then the solution $y_0(t)$ of (3.7) with the initial condition $y_0(0) = p_2(T)$ passes through the point $p_2(0)$ at time T_0, and $h_+(0, y_0(t)) < 0$ for $t \in [0, T_0)$, $h_-(0, y_0(t)) > 0$ for $t \in [0, T_0]$. Moreover, $\dot{y}_0(T_0)$ is transversal to M, i.e. $g_+(p(0))$ is not orthogonal to $\nabla_y h_+(p(0))$.

We note that Eq. (3.7) is related to [10, formula (2.12)]. Condition (A2) implies condition (3.5) along $y = y_0(t)$, $t \in [0, T_0]$, and it also gives a sliding solution $(0, y_0(t))$, $t \in [0, T_0]$, of (3.6). Moreover, we get a periodic solution $p_0(t)$ of (3.6) given by

$$p_0(t) := \begin{cases} p(t) & \text{for } t \in [0, T], \\ (0, y_0(t - T)) & \text{for } t \in [T, T + T_0]. \end{cases}$$

Now we construct a Poincaré map P_ε of (3.2)–(3.3) along $p_0(t)$ as follows. Let $B_\delta(p_2(0))$ be the small ball in $\mathbb{R}^{n-1}$ centered in $p_2(0)$ with the small radius $\delta > 0$. We take a solution $z(t)$ of (3.2) starting from the point (ε, y), $y \in B_\delta(p_2(0))$. This hits the surface $x = \varepsilon$ near $p(T)$ at the point $(\varepsilon, y(\bar{t}))$. We consider the map

$$\Phi_\varepsilon(y) := y(\bar{t}),$$

$$\Phi_\varepsilon : B_\delta(p_2(0)) \to B_{\delta_1}(p_2(T)),$$

for a small $\delta_1 > 0$. Now we consider the solution $(w(t), y(t))$ of (3.4) starting from the point $(1, y)$, $y \in B_{\delta_1}(p_2(T))$. It will hit the surface $w = 1$ at a time $\tilde{t}$ near the point $(1, p_2(0))$. Like in Section 2, Tichonov theorem implies that

$$1 = w(\tilde{t}) = V(y(\tilde{t})) + O(\varepsilon), \tag{3.8}$$

$$y(\tilde{t}) = \bar{y}(\tilde{t}) + O(\varepsilon),$$

where $\bar{y}(t)$ is the solution of (3.7) with the initial condition $\bar{y}(0) = y$. Since y is near to $p_2(T)$, condition (A2) and (3.8) imply that $w(t)$ transversally crosses the line $w = 1$ at $\tilde{t}$. Hence $\tilde{t}$ is locally uniquely defined. We also note that $V(\bar{y}(\tilde{t})) + O(\varepsilon) = 1$. Hence $\bar{y}(\tilde{t})$ is

$O(\varepsilon)$ near to the point $\Theta(y) \in M$ which is the crossing point of M with $\bar{y}(t)$. Thus we can put

$$\Psi_\varepsilon(y) := y(\tilde{t}),$$

$$\Phi_\varepsilon : B_{\delta_1}(p_2(0)) \to \mathbb{R}^{n-1}.$$

Finally, we consider the Poincaré map

$$P_\varepsilon(y) := \Psi_\varepsilon(\Phi_\varepsilon(y)), \quad y \in B_\delta(p_2(0)).$$

We have $\Psi_\varepsilon(y) = \Theta(y) + O(\varepsilon)$ and

$$P_\varepsilon(y) = \Theta(\Phi_0(y)) + O(\varepsilon),$$

$$DP_\varepsilon(y) = D\Theta(\Phi_0(y)) + O(\varepsilon). \tag{3.9}$$

In order to find a periodic solution of (3.2)–(3.3) near $p_0(t)$, we must solve the equation

$$y = P_\varepsilon(y). \tag{3.10}$$

We have $p_0(0) = P_0(p_0(0))$. Now we take a tubular/normal neighborhood $M \times W$ of M in $\mathbb{R}^{n-1}$ near the point $p_0(0)$, where $W \subset \mathbb{R}$ is an open neighborhood of $0 \in \mathbb{R}$. The corresponding projections are as follows: $\Gamma_1 : M \times W \to W$ and $\Gamma_2 : M \times W \to M$. Then Eq. (3.10) is splitted to the system

$$\begin{aligned} y_1 &= \Gamma_1 P_\varepsilon(y_1, y_2), \quad y_1 \in W, \\ y_2 &= \Gamma_2 P_\varepsilon(y_1, y_2), \quad y_2 \in M. \end{aligned} \tag{3.11}$$

Since $\Theta(y) \in M$, (3.9) gives

$$y_1 = \Gamma_1 P_\varepsilon(y_1, y_2) = O(\varepsilon),$$

$$y_2 = \Gamma_2 P_\varepsilon(y_1, y_2) = \Omega(y_2) + O(\varepsilon),$$

where we get a map $\Omega : M \to M$ defined near $p_0(0)$ by

$$\Omega(y_2) := \Theta(\Phi_0(0, y_2)).$$

The map Ω is a Poincaré map of $p_0(t)$ for (3.6) when the dynamics on the level $x = 0$ is given by (3.7) like in [10, p. 111]. Clearly, $\Omega(p_0(0)) = p_0(0)$. Hence, if the linearization $I - D\Omega(p_0(0))$ is nonsingular, then we can solve (3.11) near $p_0(0)$ by using the implicit function theorem. Moreover, if $D\Omega(p_0(0))$ is stable, i.e. all eigenvalues of $D\Omega(p_0(0))$ are inside the unit circle of $\mathbb{C}$, then also the corresponding periodic solution $p_\varepsilon(t)$ of (3.2)–(3.3) is stable. Of course, the stability of $D\Omega(p_0(0))$ gives the stability of periodic solution $p_0(t)$ for (3.6) when again the dynamics on the level $x = 0$ is given by (3.7). Summarizing, we arrive at the following result.

Theorem 2. *Let assumptions* (A1) *and* (A2) *hold. If the linearization* $I - D\Omega(p_0(0))$ *is nonsingular, then the approximate system* (3.2)–(3.3) *possesses a periodic solution* $p_\varepsilon(t)$

near $p_0(0)$. *If* $D\Omega(p_0(0))$ *is stable, then also the corresponding periodic solution* $p_\varepsilon(t)$ *of* (3.2)–(3.3) *is stable.*

Remark 1. Conditions (Al), (A2) and Theorem 2 contain some transversal/generic assumptions, namely that $\nabla_y h_+(p(0)) \neq 0$, $g_+(p(0))$ is transversal to M and that the linearization $I - D\Omega(p_0(0))$ is nonsingular. If one of them fails, then the construction of the Poincaré map P_ε as well as the solvability of Eq. (3.10) becomes problematic. Some bifurcations of periodic solutions are expected.

Remark 2. Theorem 2 states the persistence of a generic periodic solution of discontinuous systems crossing a discontinuity level under a continuous approximation. This persistence could be proved by using the Leray–Schauder degree theory [7], but since we use the implicit function theorem, we get uniqueness and stability of periodic solutions as well. Also, this approach is constructive. Finally, we assume that the discontinuity level has a codimension 1. Higher-codimension problems could also be interesting to study.

We note that $p_0(0) = p(0)$. For application of Theorem 2, we need to find $D\Omega(p(0))$ as a map from $T_{p(0)}M$ to $T_{p(0)}M$, where $T_{p(0)}M$ is the tangent space of manifold M at point $p(0)$ and is given by

$$T_{p(0)}M := \{\eta \in \mathbb{R}^{n-1} | \eta \perp \nabla_y h_+(p(0))\}.$$

From the definition of maps Θ and Φ_0, we can derive after some algebra that for any $\eta \in T_{p(0)}M$, the vector $D\Omega(p(0))\eta \in T_{p(0)}M$ is given by

$$D\Omega(p(0))\eta = w(T_0) - \frac{\langle \nabla_y h_+(p(0)), w(T_0)\rangle}{\langle \nabla_y h_+(p(0)), \dot{y}_0(T_0)\rangle}\, \dot{y}_0(T_0), \tag{3.12}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product and function $w(t)$ (depending on η) is the solution of the initial value problem

$$\begin{aligned} &\dot{w}(t) = DH(y_0(t))w(t), \\ &w(0) = -\frac{\dot{p}_2(T)}{\dot{p}_1(T)}\, x(T) + y(T), \end{aligned} \tag{3.13}$$

where functions $x(t)$, $y(t)$ (depending also on η) are the solutions of the initial value problem

$$\begin{aligned} &\dot{x}(t) = h_{+x}(p(t))x(t) + h_{+y}(p(t))y(t), \\ &\dot{y}(t) = g_{+x}(p(t))x(t) + g_{+y}(p(t))y(t), \\ &x(0) = 0, \quad y(0) = \eta. \end{aligned} \tag{3.14}$$

Formulas (3.12)–(3.14) can be used to compute $D\Omega(p(0))$. For instance, let us consider an extension of system (2.2) to higher dimension given by

$$\begin{aligned} &\ddot{v} - v + v^3 - \frac{0.2}{1 + |\dot{v} - 1|}\operatorname{sgn}(1 - \dot{v}) + zf(z, v) = 0, \\ &\ddot{z} + \delta\dot{z} + zg(z, v) = 0, \end{aligned} \tag{3.15}$$

where $z \in \mathbb{R}$, $\delta > 0$ and f, g are smooth functions. Eq. (3.15) has the form

$$
\begin{aligned}
\dot{x} &= y_1^3 - y_1 - \frac{0.5}{1+|x|}\operatorname{sgn} x + y_2 f(y_2, y_1),\\
\dot{y}_1 &= 1 - x,\\
\dot{y}_2 &= y_3,\\
\dot{y}_3 &= -\delta y_3 - y_2 g(y_2, y_1).
\end{aligned}
\tag{3.16}
$$

Since now

$$
h_{\pm}(x, y) = y_1^3 - y_1 \mp \frac{0.5}{1 \pm x} + y_2 f(y_2, y_1),
$$

for $y = (y_1, y_2, y_3)$, assumption (A1) holds with $T = t_0 = 6.73896$ and with $p(t) = (w(t), y(t), 0, 0)$, where $w(t), y(t)$ solve (2.5) with the initial conditions $y(0) = y_0 = 1.19149$ and $w(0) = 0$. We note that $p(T) = (0, \bar{y}_0, 0, 0)$ for $\bar{y}_0 = -0.100068 \in (-y_0, y_0)$. Furthermore, the reduced system (3.7) now has the form

$$
\begin{aligned}
\dot{y}_1 &= 1,\\
\dot{y}_2 &= y_3,\\
\dot{y}_3 &= -\delta y_3 - y_2 g(y_2, y_1).
\end{aligned}
$$

Hence we have $y_0(t) = (\bar{y}_0 + t, 0, 0)$ with $T_0 = y_0 - \bar{y}_0$ for assumption (A2). Next, since

$$
\dot{y}_0(T_0) = (1, 0, 0), \quad \nabla_y h_+(p(0)) = (3y_0^2 - 1, f(0, y_0), 0),
$$

and $3y_0^2 - 1 = 3.25893 \neq 0$, we see that $\dot{y}_0(T_0)$ is not orthogonal to $\nabla_y h_+(p(0))$. Summarizing, we get that also assumption (A2) holds. So in order to apply Theorem 2 for system (3.16), we need to find the spectrum of $D\Omega(p(0))$. We note that now

$$
\begin{aligned}
T_{p(0)}M &:= \{\eta \in \mathbb{R}^3 | \eta \perp (3y_0^2 - 1, f(0, y_0), 0)\}\\
&= \left\{ \left(-\frac{f(0, y_0)}{3y_0^2 - 1}\eta_2, \eta_2, \eta_3 \right) | \eta_2, \eta_3 \in \mathbb{R} \right\}.
\end{aligned}
$$

According to (3.12), (3.13) and (3.14), we now have

$$
\Omega(p(0))\eta = \left(-\frac{f(0, y_0)}{3y_0^2 - 1} w_2(T_0), w_2(T_0), w_3(t_0) \right),
\tag{3.17}
$$

where $w_1(t)$, $w_2(t)$, $w_3(t)$ solve the following ordinary differential equations:

$$\begin{aligned}
\dot{w}_1 &= 0,\\
\dot{w}_2 &= w_3,\\
\dot{w}_3 &= -\delta y_3 - g(0, \bar{y}_0 + t)w_2,
\end{aligned} \tag{3.18}$$

with the initial value conditions

$$w_1(0) = y_1(T) - \frac{\dot{y}(T)}{\dot{w}(T)}\, x_1(T),$$

$$w_2(0) = y_2(T), \quad w_3(0) = y_3(T),$$

when $x_1(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ solve the following ordinary differential equations:

$$\begin{aligned}
\dot{x}_1 &= (3y(t)^2 - y(t))y_1 + \frac{0.5}{(1+w(t))^2}\, x_1 + f(0, y(t))y_2,\\
\dot{y}_1 &= -x_1,\\
\dot{y}_2 &= y_3,\\
\dot{y}_3 &= -\delta y_3 - g(0, y(t))y_2,
\end{aligned} \tag{3.19}$$

with the initial value conditions

$$x_1(0) = 0, \quad y_1(0) = -\frac{f(0, y_0)}{3y_0^2 - 1}\, \eta_2,$$

$$y_2(0) = \eta_2, \quad w_3(0) = \eta_3,$$

for $\eta_2, \eta_3 \in \mathbb{R}$. We can easily see from (3.17) and from the initial value problems (3.18), (3.19) that the spectrum $\sigma(D\Omega(p(0)))$ is the spectrum of the fundamental matrix solution of the ordinary differential equation

$$\dot{y}_2 = y_3,$$

$$\dot{y}_3 = -\delta y_3 - q(t)y_2,$$

where

$$q(t) = \begin{cases} g(0, y(t)) & \text{for } t \in [0, T],\\ g(0, \bar{y}_0 + t - T) & \text{for } t \in [T, T+T_0]. \end{cases}$$

We note that $q(t)$ is $(T + T_0)$-periodic and $T + T_0 = 8.03052$. By using results of [8, Section 2.5] we obtain that if

$$q(t) > \frac{\delta^2}{4},$$

$$\int_0^{T+T_0} q(t)\,\mathrm{d}t \leqslant \frac{\delta^2}{4}(T + T_0) + \frac{4}{T + T_0}, \tag{3.20}$$

then the spectrum $\sigma(D\Omega(p(0)))$ is inside the unit disk. Summarizing, condition (3.20) implies the stability of periodic solution $p_0(t)$ as well as the existence and stability of periodic solutions $p_\varepsilon(t)$ from Theorem 2 applied to system (3.16).

More general criteria for system (3.16) than condition (3.20) can be derived by using results from [6]. Of course, condition (3.20) trivially holds for a smooth function $g(z, v)$ when $g(0, v) = \delta^2/4 + 2/(T + T_0)^2$. To find a nontrivial example, we use the numerical method. We take $f(y_1, y_2) = y_2$ and $g(y_2, y_1) = y_2 + y_1$. So $g(0, v) = v$. Since $y(t)$ changes the sign over the interval $[0, T]$, so $q(t)$ defined above also changes the sign over the interval $[0, T + T_0]$. Hence, we cannot use criterion (3.20). Instead, we use a criterion by Einaudi from [6] of the form

$$\int_0^{T+T_0} q(t)\,\mathrm{d}t \ \tan \frac{\delta(T+T_0)}{2} \leqslant 2\delta,$$

since now $\int_0^{T+T_0} q(t)\,\mathrm{d}t = 0.174225 > 0$. Consequently, we get the following in equality:

$$\tan(4.01526\delta) \leqslant 11.4794\delta$$

for parameter δ when our theory is applied to get the existence and stability of periodic solutions $p_\varepsilon(t)$ from Theorem 2 to this concrete system (3.16). Finally, we present several numerical solutions to system (3.16) for $f(y_1, y_2) = y_2$ and $g(y_2, y_1) = y_2 - y_1$, since then $g(0, v) = -v$ with $\int_0^{T+T_0} q(t)\,\mathrm{d}t = -0.174225 < 0$ and then the above considerations fail. Taking into account various initial conditions as well as δ parameter, an occurrence of stable (Fig. 3) and unstable (Fig. 4) periodic orbits has been confirmed. On the other hand, we can analytically check by using the result 4.3.7 of [6] that for $\delta \geqslant 2.60768$ this concrete system (3.16) has the unstable periodic solutions $p_\varepsilon(t)$ from Theorem 2.

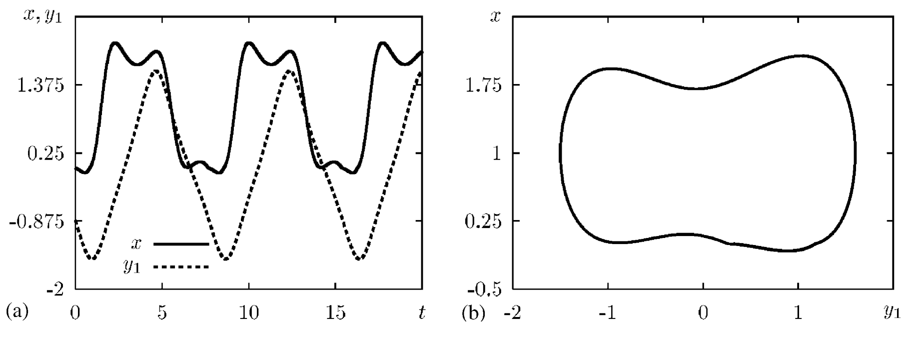

Fig. 3. Stable solution to (3.16) ($\mu_0 = 0.2$), ($x_0 = 0$): (a) $x(t)$, $y_1(t)$ and (b) $x(y_1)$ for $\bar{y}_0 = [0.321889, 0, 0]$, $\delta = 0.1$.

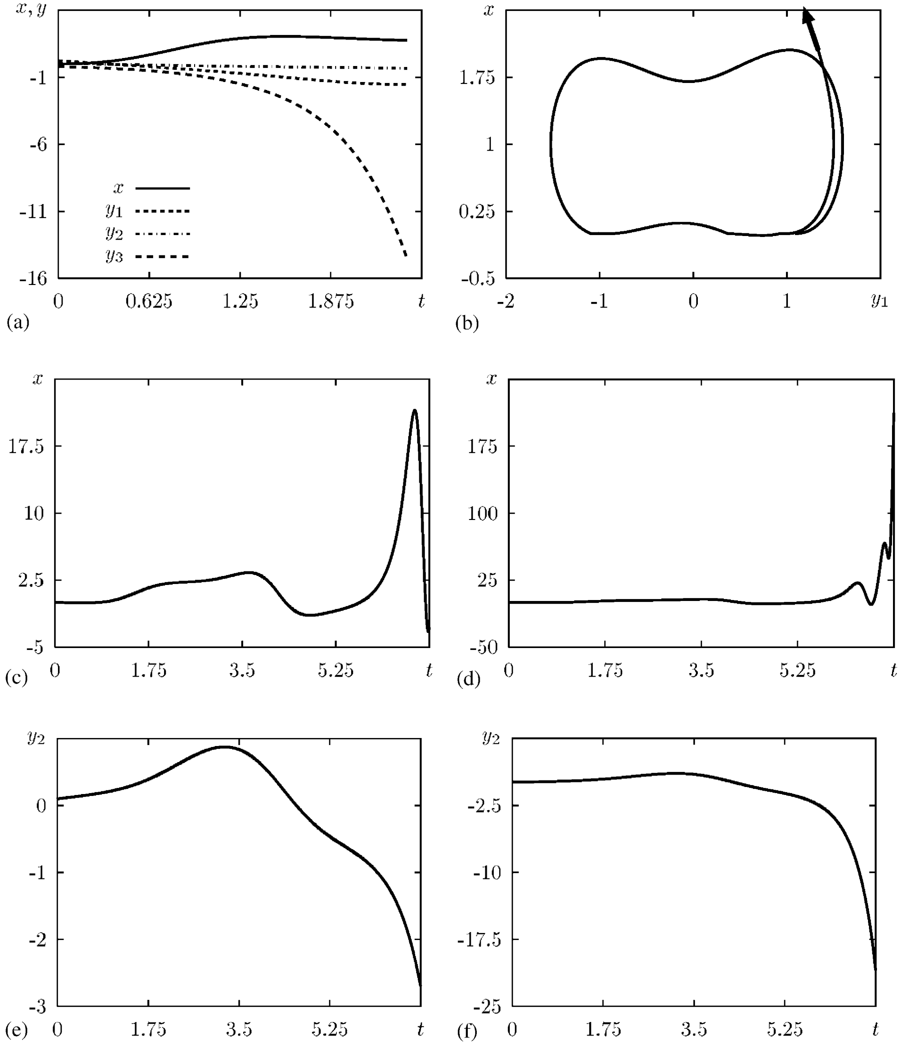

Fig. 4. Unstable solution to (3.16) ($\mu_0 = 0.2$), ($x_0 = 0$): (a) $x(t)$, $\bar{y}(t)$ and (b) $x(y_1)$ for $\bar{y}_0 = [1.08803, 0.01, 0.01]$, $\delta = 0.1$; $x(t)$ for $\bar{y}_0 = [0.321889, 0.1, 0.1]$ if (c) $\delta = 0.2$ and (d) $\delta = 0.07$; $y_2(t)$ for $\bar{y}_0 = [0.321889, 0.1, 0.1]$ if (e) $\delta = 0.5$ and (f) $\delta = 0.2$.

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