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Continuous models for chain of inertially linked masses

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Abstract

The paper focuses on 1D continuous models derived from a discrete micro-structure. A new continualization procedure that takes into account the nonlocal interaction between variables of the discrete media is proposed. The proposed procedure mainly contains an application of two-point Padé approximations and allows obtaining continuous models suitable to analyze 1D lattice vibrations with arbitrary frequencies.
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1. Introduction

It is well known that difference and difference-differential equations are often used for the numerical solution of partial differential equations. A natural problem is to obtain a difference-differential equation, whose solution approximates a solution of a given partial differential equation. But an analytical study of the difference and difference-differential equations is often more difficult than a study of the corresponding partial differential equation. Therefore the following important problem appears: how one can construct a partial differential equation approximating a given difference-differential equation?

On the other hand, it is well known that a discrete micro-structure plays a crucial role in mechanical behavior of materials. Investigation of this research direction plays an important role in today's micro-mechanics, which is well documented in the existing literature devoted to this subject. In the considered cases a magnitude of excitation is of order of the characteristic micro-structure scale. Micro-structural effects are important in damage and fracture mechanics (Askes and Sluys, 2002; Chang et al., 2002), during modeling of softening effects (Peerlings et al., 2001), molecular dynamics (Blanc et al., 2002), theory of plasticity (Fleck and Hutchinson, 1993), and wave dispersion in granular materials (Lisina et al., 2001). Let us also recall importance of nano-scale effects modeling (Dowell and Tang, 2003). In addition a challenging task in to days physics is that of construction of continuum mechanics from the 'first principles' (Braides and Gelli, 2004; Paroni, 2003).

Although the mentioned effects may be analyzed within the frame of discrete models, a required result is difficult to obtain using high tech computers in an economical way. Therefore, continuous modeling of micro- and nano-effects plays a crucial role in mechanics. Note that applicability of classical continuous models in dynamics is restricted to limited ranges of frequencies (Filimonov, 1991; Filimonov et al., 1991; Kunin, 1982). A construction and development of improved theories yielding better results is reduced to consideration of either higher order partial differential equations (Filimonov, 1998; Filimonov et al., 1991),

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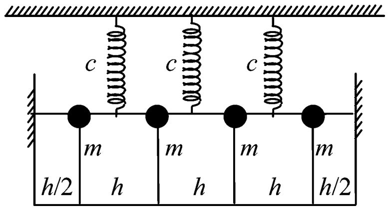

Fig. 1. 1D mass chain.

or to application of one-point Padé approximations (Rosenau, 1986, 1987, 1988, 2003) or to using of composite equations (Obraztsov et al., 1991; Andrianov and Awrejcewicz, 2003), or finally it is associated with some physical ideas (Askes and Metrikine, 2002a, 2002b). In this paper two-point Padé approximations are applied. They have been already used during analysis of analogous lattices (Andrianov, 1991).

The outline of this paper is as follows. In Section 2 the input discrete system is described. In Section 3 construction of continuous system on the basis of Taylor series is studied. In Section 4 one-point Padé approximations is applied. Continuous models are proposed in Section 6 (this is the main result of this paper). In Section 6 some conclusions are drawn and some remarks are given.

2. Discrete model

Consider linear chains composed of the equal masses and springs. The masses are supported by stiff weightless rods linking the point masses (see Fig. 1).

The vibration governing equations of these chains can be derived through the Lagrange functions to yield

$$\frac{m(2\ddot{y}_s - \ddot{y}_{s+1} - \ddot{y}_{s-1})}{4} + cy_s = 0, \quad s = 1, 2, \ldots, n, \tag{1}$$

where: $(\cdots) = \frac{\mathrm{d}}{\mathrm{d}t}(\cdots)$.

System of Eqs. (1) may be rewritten in the following form

$$0.25mD\ddot{y}_s - cy_s = 0,$$

where D is difference operator.

The boundary conditions should be attached to Eq. (1). One can choose, without loss of generality,

$$y_0 = 0, \quad y_{n+1} = 0. \tag{2}$$

An exact solution of problem of natural oscillations of system (1), (2) is written by Landa (2001). The solution to Eqs. (1) is sought in the form of the following normal oscillations: $x_s = a_s \cos\omega t$, where ω one of the normal frequencies is, a_s is the amplitude of s-th mass.

The system frequencies read

$$\omega_k = \sqrt{\frac{c}{m}} \sin^{-1} \frac{k\pi}{2(n+1)}, \quad k = 1, 2, \ldots, n. \tag{3}$$

3. Continuous model: application of Taylor series

For large values of n usually a continuous approximation is used

$$mh^2 \frac{\partial^2 \ddot{y}}{\partial x^2} - 4cy = 0, \tag{4}$$

$$y(0) = y(l) = 0, \tag{5}$$

where: $l = (n+1)h$.

Having a solution of boundary value problem (4), (5), one may easily find the solution of the discrete system due to the formulas

$$y_k(t) = y(kh, t), \quad k = 0, 1, \ldots, n.$$

Both numerical and analytical investigations (Kunin, 1982) proved that for study of lower part of the frequency spectrum of the discrete system a transition to a continuous system is correct. However, investigation of high frequency spectrum and free or forced vibrations requires improved governing equations. Therefore an interesting and challenging research object appears: Is it possible to describe the whole spectrum of vibrations of a discrete system using continuous approximation?

From the mathematical standpoint our problem is focused on accuracy improvement of nonlocal (difference) operator through a local (differential) one.

As a criterion serving for estimation of continuous approximation accuracy, a comparison of n-th frequency of the continuous system with the corresponding frequency of the discrete system (3) is used. It is worth noticing that an estimation of an accuracy of continuous approximation with respect to estimation of the largest frequency of the discrete chain is rather conventional, but the simplest one. For larger values of n formula (3) yields $\omega_n \approx \sqrt{c/m}$, and hence from system (4), (5) one finds $\omega_n \approx \sqrt{c/m}\,(2/\pi)$ (amount of the error is of 36%).

In what follows we consider a possibility of continuous approximation improvement.

Note that system (1) can be reduced to one pseudo-differential equation. For this purpose the translation operator $\exp(\pm\frac{\partial}{\partial n})$ is introduced, and the following formal identity holds: $\exp(\pm\frac{\partial}{\partial n})f(n) = f(n\pm 1)$. On the other hand we may replace a discrete-difference operator D by a high-order differential operator, using the following pseudo-differential operator (Maslov, 1976)

$$D = \sin^2\left(-\frac{\mathrm{i}H}{2}\frac{\partial}{\partial x}\right).$$

Hence, with a help of translation operator system of Eqs. (1) is transformed into the following pseudo-differential equation

$$m\sin^2\left(-\frac{\mathrm{i}h}{2}\frac{\partial}{\partial x}\right)\ddot{y} + cy = 0. \tag{6}$$

Owing to application of the Maclaurin series to the difference operator D one gets

$$D = -0.25h^2\frac{\partial^2}{\partial x^2}\left(1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2} + \frac{h^4}{360}\frac{\partial^4}{\partial x^4} + \cdots\right). \tag{7}$$

Keeping in the series (7) only the first term, continuous approximation (4) is obtained.

Furthermore, approximation accuracy can be increased by keeping more terms in series (6). Remaining three first terms, the following higher order approximation is obtained

$$mh^2\frac{\partial^2}{\partial x^2}\left(1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2} + \frac{h^4}{360}\frac{\partial^4}{\partial x^4}\right)\ddot{y} - 4cy = 0. \tag{8}$$

Observe that a similar like model is called as an intermediate continuous model (Filimonov, 1998).

However, the problem associated with boundary conditions requires more subtle analysis. Namely, it occurs that the problem of equivalence of boundary conditions for Eq. (8) cannot be solved uniquely. We need to know $y_k(t)$ for $k < 0$ and for $k > n+1$. If we choose $y_k(t)$ for $k < 0$ and for $k > n+1$ to satisfy periodicity extension of boundary conditions to keep the translation symmetry ($y_{-1}(t) = -y_1(t)$, etc.), the following boundary conditions associated with Eq. (8) are obtained

$$y = \frac{\partial^2 y}{\partial x^2} = \frac{\partial^4 y}{\partial x^4} = 0 \quad \text{for } x = 0, l. \tag{9}$$

Although continuous approximation (8), (9) yields an amount of error of 5% with respect to estimation of ω_n, but it includes high order differential operator.

4. Continuous model: application of the Padé approximations

A construction of intermediate continual models is based on approximation of the difference operator by the Taylor series. However, more effective way is associated with the Padé approximations (PA). Rosenau (1986, 1987, 1988, 2003) and Wattis (2000) constructed continuous models (a so called quasi-continuum approximation) applying one-point PA.

A brief description of the PA follows. Let the function $F(\varepsilon)$ is represented by the Maclaurin series

$$F(\varepsilon) = \sum_{i=0}^{\infty} a_i\varepsilon^i \quad \text{for } \varepsilon \to 0. \tag{10}$$

The PA $[m/n]$ is defined through the fractional rational function

$$F_{[m/n]} = \frac{\sum_{i=0}^{m}\beta_i\varepsilon^i}{1 + \sum_{i=1}^{n}\gamma_i\varepsilon^i},$$

whose first $m+1$ coefficients of the associated Maclaurin series overlap with the first terms of the series (10).

The PA $[2/2]$ for series (7) has the following form

$$D \approx \frac{\partial^2}{\partial x^2} \Big/ \left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right).$$

The corresponding model of quasi-continuum follows

$$mh^2 \frac{\partial^2 \ddot{y}}{\partial x^2} - 4c\left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)y = 0. \tag{11}$$

The boundary conditions for Eq. (11) have the form (5).

Quasi-continuum approximation (11), (5) yields the error of 14% in estimation of ω_n. Notice that this estimation is better than the standard approximation of (4), (5), but is worse than that yielded by intermediate continuous model (8), (9). On the other hand, an advantage of quasi-continuum model (11), (5) in comparison to intermediate one (8), (9) is mainly exhibited through a lower order of the differential equations.

5. Continuous model: application of two-point PA

Since two-point PA in many cases yields more accurate results than one point ones, therefore it is essentially to apply this approach for construction of continuous approximations of the difference operator (7). Recall briefly the definition of two-point PA. Let

$$F(\varepsilon) = \sum_{i=0}^{\infty} a_i \varepsilon^i \quad \text{for } \varepsilon \to 0, \tag{12}$$

$$F(\varepsilon) = \sum_{i=0}^{\infty} b_i \varepsilon^i \quad \text{for } \varepsilon \to A. \tag{13}$$

Two-point PA F_p is represented by the following fractional function $F_p = \frac{\sum_{i=0}^{m} \beta_i \varepsilon^i}{1+\sum_{i=1}^{n} \gamma_i \varepsilon^i}$, whose first k coefficients of the associated Maclaurin series and $m+n+1-k$ first coefficients of its development into the Taylor series in the neighborhood of the point $x = A$ overlap with the first coefficients of the series (12) and (13).

Taking into account the first term of the series (7) and requiring $\omega_n = \sqrt{c/m}$, the following new approximate differential operator for difference one is obtained using the two-point PA

$$\frac{\partial^2}{\partial x^2} \Big/ \left(1 - \alpha^2 h^2 \frac{\partial^2}{\partial x^2}\right),$$

where: $\alpha^2 = 0.25 - \pi^{-2}$.

In what follows the continuous approximation is governed by the following equation

$$m\frac{\partial^2 \ddot{y}}{\partial x^2} + 4c\left(1 - \alpha^2 h^2 \frac{\partial^2}{\partial x^2}\right)y = 0 \tag{14}$$

and the boundary conditions have the form (5).

The largest error in eigenfrequencies estimation does not exceed 3%. Observe that Eq. (14) is of second order with respect to spatial coordinate, i.e. essential increase of accuracy is not achieved in a way of increasing the order of differential operator.

6. Conclusions

To sum up, using two-point PA gives mathematically justified continuous models of 1D mass chain, valid for spectrum of discrete systems.

One may use the proposed method for constructing of continuous media, when taking into account the main micro-structural effects, for 2D and 3D cases.

Also generalization of obtained results for chains with unequal elements (Landa, 2001) and investigation of nonlinear case posses a challenging problem for further study.

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References

Andrianov, I.V., 1991. Continuous approximation for higher-frequency oscillations of a mass chain. Dokl. Akad. Nauk Ukrain. SSR Ser. A 2, 13–15.

Andrianov, I.V., Awrejcewicz, J., 2003. On the average continuous representation of an elastic discrete medium. J. Sound Vib. 264, 1187–1194.

Askes, H., Metrikine, A.V., 2002a. One-dimensional dynamically consistent gradient elasticity models derived from a discrete microstructure. Part 1: Generic formulation. Eur. J. Mech. A Solids 21, 555–572.

Askes, H., Metrikine, A.V., 2002b. One-dimensional dynamically consistent gradient elasticity models derived from a discrete microstructure. Part 2: Static and dynamic response. Eur. J. Mech. A Solids 21, 573–588.

Askes, H., Sluys, L.J., 2002. Explicit and implicit gradient series in damage mechanics. Eur. J. Mech. A Solids 21, 379–390.

Blanc, X., Le Bris, C., Lions, P.-L., 2002. From molecular models to continuum mechanics. Arch. Rational Mech. Anal. 164, 341–381.

Braides, A., Gelli, M.S., 2004. The passage from discrete to continuous variational problems: a nonlinear homogenisation process. In: Ponte Castaneda, P., Telega, J.J., Gambin, B. (Eds.), Nonlinear Homogenization and its Applications to Composites, Polycrystals and Smart Materials. Kluwer Academic, Dordrecht, pp. 45–64.

Chang, C.S., Askes, H., Sluys, L.J., 2002. Higher-order strain/higher-order stress gradient models derived from a discrete microstructure, with application to fracture. Engrg. Fracture Mech. 69, 1907–1924.

Dowell, E.H., Tang, D., 2003. Multiscale, multiphenomena modelling and simulation at the nanoscale: on constructing reduced-order models for nonlinear dynamical systems with many degrees-of-freedom. ASME J. Appl. Mech. 70, 328–338.

Filimonov, A.M., 1991. Some unexpected results in the classical problem of vibrations of the string with N beads. The case of multiple frequencies. C. R. Acad. Sci. Paris, Ser. I Math. 315 (8), 957–961.

Filimonov, A.M., 1998. Continuous approximations of difference operators. J. Difference Equations and Applications 2 (4), 411–422.

Filimonov, A.M., Kurchanov, P.F., Myshkis, A.D., 1991. Some unexpected results in the classical problem of vibrations of the string with N beads when N is large. C. R. Acad. Sci. Paris, Ser. I Math. 313, 961–965.

Fleck, N.A., Hutchinson, J.W., 1993. A phenomenological theory for gradient effects in plasticity. J. Mech. Phys. Solids 41, 1825–1857.

Kunin, I.A., 1982. Elastic Media with Microstructure. Vol. 1. One-Dimensional Models. Springer-Verlag, Berlin, New York.

Landa, P.S., 2001. Regular and Chaotic Oscillations. Springer-Verlag, Berlin, New York.

Lisina, A.S., Potapov, A.I., Netrebko, V.F., 2001. A nonlinear granular medium with particle rotation: a one-dimensional model. Acoust. Phys. 47 (5), 596–606.

Maslov, V.P., 1976. Operational Methods. Mir, Moscow.

Obraztsov, I.F., Andrianov, I.V., Nerubaylo, B.V., 1991. Continuous approximation for high-frequency oscillations of a chain and composite equations. Soviet Phys. Dokl. 36 (7), 522.

Paroni, R., 2003. From discrete to continuum: a Young measure approach. Z. Angew. Math. Phys. 54 (2), 328–348.

Peerlings, R.H.J., Geers, M.G.D., de Borst, R., Brekelmans, W.A.M., 2001. A critical comparison of nonlocal and gradient-enhanced softening continua. Int. J. Solids Structures 38, 7723–7746.

Rosenau, Ph., 1986. Dynamics of nonlinear mass-spring chains near the continuum limit. Phys. Lett. A 118 (5), 222–227.

Rosenau, Ph., 1987. Dynamics of dense lattices. Phys. Rev. B 36 (11), 5868–5876.

Rosenau, Ph., 1988. Dynamics of dense discrete systems. Progr. Theoret. Phys. 89 (5), 1028–1042.

Rosenau, Ph., 2003. Hamiltonian dynamics of dense chains near and lattices: or how to correct the continuum. Phys. Lett. A 311 (5), 39–52.

Wattis, J.A.D., 2000. Quasi-continuum approximations to lattice equations arising from the discrete nonlinear telegraph equation. J. Phys. A: Math. Gen. 33, 5925–5944.