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ITERATIVE PROCESSES AND PADÉ APPROXIMANTS

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Abstract. *In many application oriented researches the initial coefficients characterizng a discontinuity of the governing differential equations (for instance, caused by the dry friction) are often substituted by the continuous functions (approximations). One of the effective methods to use the described methodology is that related to introducing a formal small (perturbation) parameter, and then to carry out the successive iterations due to this parameter. In this paper we show, that a combination of the mentioned iterative procedure with the Padé approximants leads to a drastic decrease of the number of iterations.*

Key words: *iterative processes, Padé approximations.*

1. INTRODUCTION

Kleczka and E. Kreuzer [1, 2] proposed the following iterative procedure dealing with differential equations possessing discontinuous coefficients. As it is known the Heaviside function $H(x)$

$$H(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{1}$$

can be presented in the following form:

$$H(x) = \frac{2}{\pi} \lim_{c \to \infty} \arctan(cx).$$

Increasing the parameter c one may better approximate the function $H(x)$. For enough large values of c one obtains (practically) exact values of the function being approximated. It is clear that it is much easier to work with the smooth coefficients instead of the discontinuous ones, when in addition one of the numerical algorithms is applied. In this case a key problem is focused on decreasing a number of iterations along with the simultaneous guarantee of a high accuracy of the solution being sought.

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In this paper we show that one can apply the Padé transformation in order to realize the mentioned requirements [3-5].

2. Increasing the Convergence of the Iterative Processes Using Padé Approximants

The effectiveness of application of the Padé approximants essentially depends on the occurrence of high order terms of an asymptotical process. Although these principal difficulties can be omitted using the symbolic computations, this problem in general still remains open. Much easier is to apply the Padé approximations to the iterative processes. Let us assume that the following iterative process is applied

$$u \approx u_0 + (u_1 - u_0)\varepsilon + (u_2 - u_1)\varepsilon^2 + ... + (u_n - u_{n-1})\varepsilon^n . \tag{2}$$

For $\varepsilon = 0$ one gets $u \approx u_0$, whereas for $\varepsilon = 1$ one obtains $u \approx u_n$. On the other hand, the series (2) can be presented as the following rational function due to the Padé approximation of the form

$$u_{ml} \approx \frac{u_0 + \sum_{i=1}^{m} a_i \varepsilon^i}{1 + \sum_{j=1}^{l} b_j \varepsilon^j} - [u_0 + (u_1 - u_0)\varepsilon + ... + (u_n - u_{n-1})\varepsilon^n] = 0(\varepsilon^{n+1}), \tag{3}$$

where: $m + l = n$.

Therefore for $\varepsilon = 1$

$$u \approx \frac{u_0 + \sum_{i=1}^{m} a_i}{1 + \sum_{j=1}^{l} b_j} . \tag{4}$$

For $m = l$ the diagonal approximation is obtained. Many examples (see [3-5]) confirm a high accuracy and the efficiency of the described method.

3. Application of the Padé Approximations

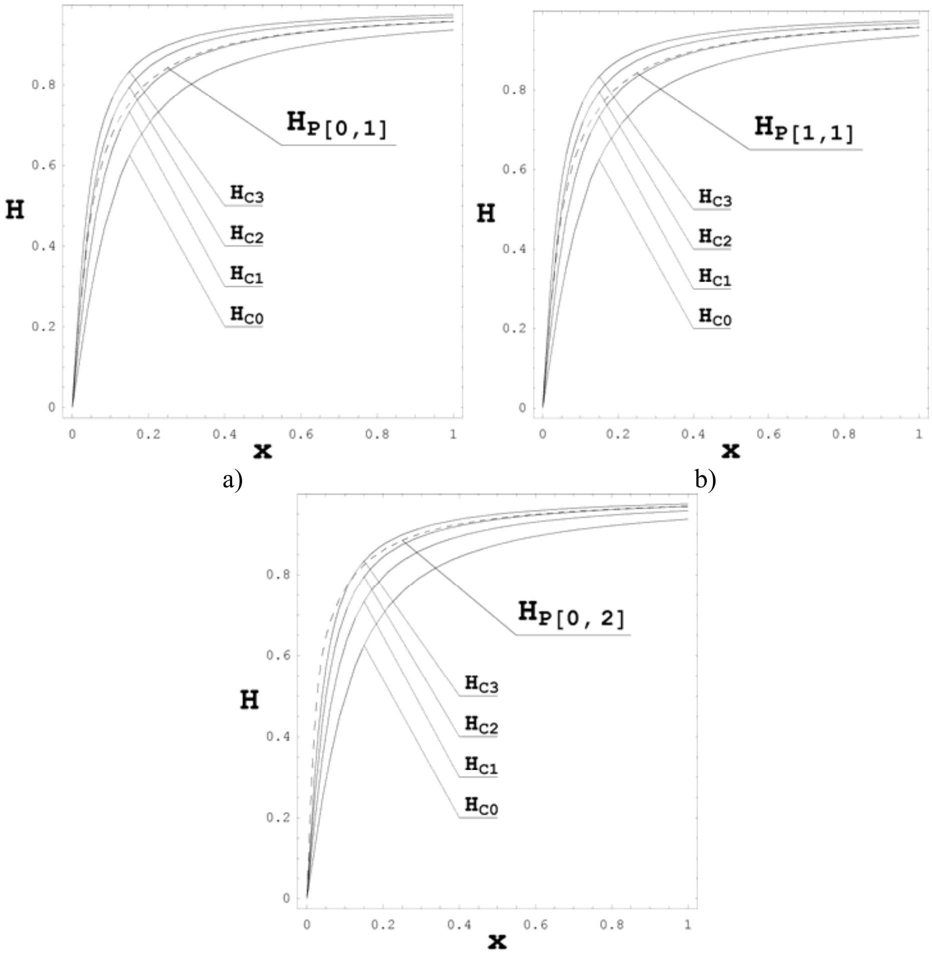
First we consider an application of the Padé approximation to the mentioned function $H(x)$, i.e. let us consider the function

$$H_c(x,c) = \frac{2}{\pi}\arctan(cx),$$

where: $H_c(x,\infty) = H(x)$. Giving values of c as the series of $c_1 < c_2 < c_3 < \ldots$, one gets the iterative series of $H_{c_i} = H(x,c_i)$, and the Padé approximations can be applied.

The appropriate program using the "Mathematica" package and the exemplary results are presented in Appendix A.

In Figure 1 results of the described method are presented. It is clear that Padé scheme allow for essential approximation improvement of a being the sought function.



c)

Fig. 1. Normalized inverse tangent function for different values of c (H_{c0}, H_{c1}, H_{c2}, H_{c3} for $c_0 = 10$, $c_1 = 15$, $c_2 = 20$ and $c_3 = 25$) and Padé approximants $H_{P[m,l]}$ for different m and n values corresponding to (3): (a) $m = 0$, $l = 1$; (b) $m = 1$, $l = 1$; (c) $m = 0$, $l = 2$

We consider now the second example described in the reference [1].

A simple mechanical model of the planar submerged inverted double pendulum is represented in Fig. 2. Applied forces acting at the bars are buoyancy forces, gravity forces, and hydrodynamic forces. The excitation is characterized by the angular amplitude a and the frequency ω, and it is transmitted to the lower bar by means of a torsional spring.

The upright position of this two-degrees-of-freedom rigid-body system is a stable equilibrium position due to the buoyancy forces acting at the bars. For the mathematical description, the absolute angles α_1 and α_2 and the absolute angular velocities $\dot{\alpha}_1$ and $\dot{\alpha}_2$ are chosen as state variables, which are summarized in the state vector $x = [\alpha_1 \alpha_2 \dot{\alpha}_1 \dot{\alpha}_2]^T$.

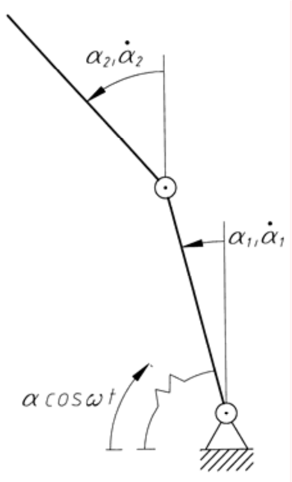


Fig. 2. Mechanical model of a double pendulum

According to Morison's formula, the hydrodynamic damping-force component F acting at an infinitesimal slice dl of the bar takes the form

$$F(x) \sim \begin{cases} u(x)^2 & \text{for} \quad \mathrm{u}(x) \geq 0, \\ -u(x)^2 & \text{for} \quad \mathrm{u}(x) < 0, \end{cases}$$

where $u(x)$ is the normal or tangential velocity component of the surrounding current relative to the bar.

In reference [1] the iterative procedure to trace bifurcation and stability of the periodic solutions of the inverted double pendulum is reported.

Applying the procedure for a stable one-periodic solution of the double pendulum, a symmetry-breaking pitchfork bifurcation with critical eigenvalue $\lambda^* = +1$ and a period-doubling flip bifurcation with critical eigenvalue $\lambda^* = -1$ can be detected. On the basis of the symbolically given map, a very accurate estimation of the bifurcation parameter values a^* of the excitation amplitude a can be obtained. An example of the iteration sequence for the pitchfork bifurcation is given in Table 1.

Table 1. Iteration sequence for pitchfork bifurcation parameter values

Iteration	a^*	λ^*
0	0.1000000	0.48912898 + 0.32400075 j
1	0.1407018	0.84036745
2	0.1605386	0.99358588
3	0.1617132	1.00161352
4	0.1614095	0.99954580
5	0.1614944	1.00012418
10	0.1614761	0.99999996

In spite of the authors' remark about a very good convergence of the numerical iteration sequence it is seen that they have an oscillation-like character. We address two following remarks concerning the results included in the Table 1. First, it is seen that the number of iterations is rather small to get a stationary process. Second, it is not known if in principle this process is expected to be a stationary one. These two remarks indicate a need for the application of the Padé approximations.

In order to apply the Padé approximations the appropriate program using "Mathematica" is presented (together with obtained results) in Appendix B. (Note, that because of the space limitations some of the long analytical expressions have not been shown). A general idea of the algorithm is to take the successive series of the iterations $a_0,a_1,...,a_5$ which corresponds to *pa*[0], and then $a_1,a_2,...,a_5$ which corresponds to *pa*[1], and so on. The same holds for the λ_i series ($i = 0,1,...,5$) (see Table 1). For instance, for *pa*[0] we have 21 possible Padé approximants, and so on (see Appendix B).

It is obvious that for *PA*[3,1], *PA*[3,2] and *PA*[4,1] the results are extremely close to those obtained after the 10^{th} iteration step of the numerical procedure.

In addition, we can obtain a very good approximation even for a few first values of a_i. For instance, for $a_1 - a_5$ we get *PA*[2,1] = 0.16147189, *PA*[2,2] = 0.16147587 and *PA*[3,1] = 0.16147585. For $a_2 - a_5$ we get *PA*[1,1] = 0.16147189, *PA*[1,2] = 0.16147585 and *PA*[2,1] = 0.16147585. For $a_3 - a_5$ we obtain *PA*[1,1] = 0.16147585.

A similar observation holds for λ_i. We would like to point out only the remarkable values of *PA*[3,1] = 0.99996931, *PA*[3,2] = 0.99999786 and *PA*[4,1] = 0.99999776 (for *pa*[0]); *PA*[2,1] = 0.99996931, *PA*[2,2] = 0.99999786 and *PA*[3,1] = 0.99999776 (for *pa*[1]); *PA*[1,1] = 0.99996931, *PA*[1,2] = 0.99999778 and *PA*[2,1] = 0.99999776 (for *pa*[2]); and *PA*[1,1] = 0.99999776 for *pa*[3]. The approximations are valid for both real and the complex values.

Obtained results lead to a conclusion that the advantages of the Padé approximants are very high. Using only a few first iteration values of a_i, we can get practically the same values as those obtained after ten steps of iterations! It means that one integrates numerically a few time less the investigated ODEs in a whole interval of the period of a periodic solution being investigated. It causes a dramatic decrease of the computational time during the integration of the governed differential equations.

Similar approaches have been applied also to the analysis of the complex system of ODEs governing the vocal cords oscillations and during accuracy improvement of the Lorenz homoclinic orbit parameters determination [6, 7].

4. Alternative Approaches

An alternative approach to the Kleczka-Kreuzer method can be realized in the following manner. The discussed discontinuous coefficients can be presented in the form of the Fourier series. Then these coefficients can be presented in the form of the trigonometric Padé approximations, which provide its effective and smooth approximation [8, 9].

The second alternative approach is based on increasing the number of the Fourier terms, and then on applying the Padé approximations to the obtained iterative sequence of the sought functions.

Finally, we described the third approach. Let us assume that a function being sought $u(x,\varepsilon)$ can be approximated by the following series

$$u(x,\varepsilon) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \ldots$$

Let us assume that in addition we have the numerical value of the solution

$$u(x,\varepsilon) \approx u_n(x).$$

Therefore, for given values of ε, i.e. ε_1, ε_2, ε_3 one can write [10] (a generalization for large number of a series can be easily carried out):

$$\begin{cases} u(x_1,\varepsilon_2) - u(x_1,\varepsilon_1) \approx (\varepsilon_2 - \varepsilon_1)u_1(x) + (\varepsilon_2^2 - \varepsilon_1^2)u_2(x), \\ u(x_1,\varepsilon_2) - u(x_1,\varepsilon_3) \approx (\varepsilon_2 - \varepsilon_3)u_1(x) + (\varepsilon_2^2 - \varepsilon_3^2)u_2(x). \end{cases} \tag{5}$$

The sought u_i (x) can be obtained from (5), and then a typical Padé approximation procedure can be applied. In addition, the second order Padé approximation can be applied [4], where the coefficients of the series (3) can be defined by u_{ml}.

5. Concluding Remarks

In this paper we have shown that Padé approximations are very effective tools to improve accuracy of both the analytical und the numerical iterative processes. Among others we have focused on two important aspects of the discussed problem. First, we must emphasize that the Padé approximants for the sequence $m-1(P[m-1])$, $m(p[m])$ and $m+1(P[m+1])$ satisfy the following inequality:

$$P[m-1] \le P[m] \le P[m+1].$$

It means that an interval of changes of a being sought parameter is estimated.

Second, the results of the iterative process and the corresponding Padé approximations lead to a sufficient increase of the heuristic reliability of the results and can serve as a tool of its theoretical background.

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ITERATIVNI PROCESI I PADÉ-OVA APROKSIMACIJA

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Abstrakt na srpskom

U mnogim istraživanjima orjentisanim ka primeni, početni koeficijenti koji karakterišu diferencijalne jednačine (uzrokovani npr. trenjem) često se zamenjuju kontinualnim funkcijama (aproksimacija). Jedan od efektivnih metoda za korišćenje opisane metodologije zasniva se na uvođenju formalno malog (preturbacionog) parametra, a zatim na izvođenju sukcesivnih iteracija po ovom parametru. U ovom radu pokazali smo da kombinacija pomenute iterativne procedure i Padé-ove aproksimacije vodi ka značajnom smanjenju broja iteracija.

Ključne reči: *iterativni procesi, Padé-ova aproksimacija*