ASYMPTOTIC-GROUP ANALYSIS OF ALGEBRAIC EQUATIONS

A. D. SHAMROVSKII, I. V. ANDRIANOV, AND J. AWREJCEWICZ

Received 15 June 2004

Both the method of asymptotic analysis and the theory of extension group are applied to study the Descates equation. The proposed algorithm allows to obtain various variants of simplification and can be easily generalized to their algebraic and differential equations.

1. Introduction

First published in 1680, though applied for the first time eleven years earlier, the approach that exploits logarithms of initial values instead of the values themselves is due to Newton (the reader is encouraged to follow the history of the problem described in [7]). The approach is still of crucial importance as it enables transformation of a previously defined nonlinear problem using the initial coordinates to a linear problem applying the logarithmic coordinates. Further development of the idea following this direction is called power geometry (cf. [6, 7, 8]).

Recently, a very active development of two new and theoretically coupled branches of asymptotical analysis has been observed: power geometry [6, 7, 8] and idempotent analysis [9, 10]. Power geometry focuses on investigations of nonlinear problems using logarithmic coordinates instead of classical ones, owing to which many of the original nonlinear properties become linear in new logarithmic coordinates. Power geometry algorithms that implement those linear relations lead to simplification of equations and isolation of their singularities, and to determination of the first approximations of the singularities. Application of power geometry methods makes it possible not only to find solutions that may take the form of their asymptotics but also to use other approaches to arrive at these asymptotics.

The idempotent analysis is oriented mainly towards introduction of a new summation [9, 10] of the form

$$u \oplus_h v = h \ln \left[\exp\left(\frac{u}{h}\right) + \exp\left(\frac{v}{h}\right) \right], \tag{1.1}$$

$u \otimes v = u + v$ and $u \oplus_h v = \max\{u, v\}$ for $h \to 0$; moreover, set A defined by the map $x \to u = h \ln x$ of a halving of positive numbers; R_+ also becomes a halving in relation to the introduced operations with zero $0 = -\infty$ and unity $l = 0$. The semiring is an identical

Mathematical Problems in Engineering 2004:5 (2004) 411–451
2000 Mathematics Subject Classification: 65H05, 52B70, 20G20, 41A10
URL: http://dx.doi.org/10.1155/S1024123X04406036

one, that is, $u \oplus u = u$ for all its elements, which provides a wide applicability spectrum for the idempotent analysis. On the other hand, a simplification similar to that of the initial equations is realized within a frame of the asymptotic-group analysis [11, 12]. The ideas discussed in this paper are clarified via a simple example described in Sections 2–10 of the paper. The example of the Folium of Descartes, whose analysis was already attempted in [7] but unfortunately not brought to completion there, has been chosen as a modeling problem for the purpose of the present paper and this time has been fully analyzed. Besides, in Section 11 of the paper, a possibility of application of Padé approximations is considered [1].

Real behavior modeling via computers is frequently realized through either algebraic or differential equations. Both analytical and numerical approaches or either one can be applied. It is worth noticing that from the mathematical point of view both methods mentioned here supplement each other yielding a full picture of an investigated system. A special role, however, is played by asymptotical methods, targeted at investigation of algebraic and differential equations, not only because they are the most powerful among all known analytical approaches but also because in many cases they are matched with various numerical approaches [11, 12].

We now discuss briefly some problems often met in engineering and associated with either algebraic or algebraic/differential equations.

All problems oriented on investigations of equilibria of nonlinear dynamical systems governed by ODEs are reduced to a study of algebraic equations.

There are many problems in mechanics when a considered dynamical object is subject to unilateral constraints. In this case, the dynamics of the system under consideration is governed by ODEs, whereas potential rigid barriers serve as motion limitations and their influence on motion behavior is represented by algebraic equations (inequalities). A reader is encouraged to follow, for instance, some recent works devoted to dynamics of a triple pendulum with rigid barriers [3, 4, 5].

There is also a vast field of problems related to analysis of continuous mechanical systems, like rods, beams, plates, and shells. The corresponding nonlinear PDEs are reduced through an application of finite difference methods to a set of ODEs and algebraic equations. It has been shown that this approximation to the original problems possesses many advantages in comparison to classical approaches (see, e.g., monograph [2] and the references therein).

However, in spite of the challenging development of asymptotic methods, they still exhibit certain disadvantages, the most significant of which lies in their strong dependence on researchers' intuition.

The aim of the study presented here is to remove this drawback by means of a synthesis of classical, asymptotical, and group theory methods and to arrive at an explicit and intuition-independent algorithm for investigation of complex systems.

2. Extension group associated with an algebraic equation

Consider the following Descartes equation:

$$ax^3 + by^3 - cxy = 0, \tag{2.1}$$

where x and y are rectangular coordinates, and a, b, c are arbitrary coefficients. We focus the attention on the relative magnitude of equation terms, and, in particular, on the role played by coefficients a, b, and c. Note that the latter is not explicitly clear, as, for example, the fact that a coefficient, say a, is large does not imply that the associated term is also large, since it includes x and/or y.

In order to solve the problem, the transformations (extensions) that allow (2.1) to remain in the same form should be established. For this purpose, an arbitrary quantity $\delta \neq 1$ is introduced and the following transformations are defined:

$$x = \delta^{\beta_1} x^*, \qquad y = \delta^{\beta_2} y^*, \qquad a = \delta^{\beta_3} a^*, \qquad b = \delta^{\beta_4} b^*, \qquad c = \delta^{\beta_5} c^*. \tag{2.2}$$

On substituting (2.2) into (2.1), the following equation is obtained:

$$\delta^{3\beta_1+\beta_3} a^* (x^*)^3 + \delta^{3\beta_2+\beta_4} b^* (y^*)^3 - \delta^{\beta_1+\beta_2+\beta_5} x^* y^* c^* = 0. \tag{2.3}$$

Equation (2.1) either can be transformed with respect to (2.2) or it is invariant with respect to these transformations when the exponents standing by all coefficients are the same, that is,

$$3\beta_1 + \beta_3 = 3\beta_2 + \beta_4 + \beta_1 + \beta_2 + \beta_5. \tag{2.4}$$

Note that there are two equations and five unknowns. We describe β_3 and β_4 by other quantities:

$$\beta_3 = \beta_5 - 2\beta_1 + \beta_2, \qquad \beta_4 = \beta_5 + \beta_1 - 2\beta_2. \tag{2.5}$$

Quantities β_1, β_2, and β_5 are arbitrary and may take arbitrary values, whereas β_3 and β_4 are expressed through them. There exist three fundamental solutions to (2.4):

(1)

$$\beta_1 = \gamma_1 \neq 0, \qquad \beta_2 = 0, \qquad \beta_5 = 0 \Longrightarrow \beta_3 = -2\gamma_1, \qquad \beta_4 = \gamma_1, \tag{2.6}$$

which is associated with the transformation

$$x = \delta^{\gamma_1} x^*, \qquad y = y^*, \qquad a = \delta^{-2\gamma_1} a^*, \qquad b = \delta^{\gamma_1} b^*, \qquad c = c^*; \tag{2.7}$$

(2)

$$\beta_2 = \gamma_2, . \qquad \beta_1 = 0, \qquad \beta_5 = 0 \Longrightarrow \beta_3 = \gamma_2, \qquad \beta_4 = -2\gamma_2, \tag{2.8}$$

whose associated transformation reads

$$x = x^*, \qquad y = \delta^{\gamma_2} y^*, \qquad a = \delta^{\gamma_2} a^*, \qquad b = \delta^{-2\gamma_2} b^*, \qquad c = c^*; \tag{2.9}$$

(3)

$$\beta_5 = \gamma_3, \qquad \beta_1 = 0, \qquad \beta_2 = 0 \Longrightarrow \beta_3 = \gamma 3, \qquad \beta_4 = \gamma_3, \tag{2.10}$$

with the following associated transformation:

$$x = x^*, \qquad y = y^*, \qquad a = \delta^{\gamma_3} a^*, \qquad b = \delta^{\gamma_3} b^*, \qquad c = \delta^{\gamma_3} c^*. \tag{2.11}$$

In other words, (2.1) is invariant with respect to the three-parameter group of extensions, defined by relations (2.7), (2.9), and (2.11). Occurrence of these admitted transformations defines a criterion for the relative equation simplicity and, as will be shown further, may bring the right reduction for input equations.

3. Extension group invariants

Note that in all transformations in the group of extensions that have been found, there are five quantities taking part. Any combination of these quantities that does not change in transformations is referred to as the group-invariant property. We find these invariants in the considered case. To this aim, from transformations (2.7), (2.9), and (2.11), parameters of transformations $\gamma_1, \gamma_2, \gamma_3$ should be excluded.

The first method. The superposition of transformations (2.7), (2.9), (2.11) yields

$$x = \delta^{\gamma_1} x^*, \qquad y = \delta^{\gamma_2} y^*, \qquad a = \delta^{-2\gamma_1+\gamma_2+\gamma_3} a^*, \qquad b = \delta^{\gamma_1-2\gamma_2+\gamma_3} b^*, \qquad c = \delta^{\gamma_3} c^*, \tag{3.1}$$

and hence

$$\delta^{\gamma_3} = \frac{c}{c^*} \Longrightarrow \frac{x}{x^*} = \delta^{\gamma_1}, \qquad \frac{y}{y^*} = \delta^{\gamma_2}, \qquad \frac{a}{a^*} = \delta^{-2\gamma_1+\gamma_2} \frac{c}{c^*}, \qquad \frac{b}{b^*} = \delta^{\gamma_1-2\gamma_2} \frac{c}{c^*}. \tag{3.2}$$

On the other hand,

$$\delta^{\gamma_2} = \frac{ac^*}{a^*c} \delta^{2\gamma_1} \Longrightarrow \frac{x}{x^*} = \delta^{\gamma_1}, \qquad \frac{y}{y^*} = \frac{ac^*}{a^*c} \delta^{2\gamma_1}, \qquad \frac{b}{b^*} = \frac{(a^*)^2 c^3}{a^2 (c^*)^3} \delta^{-3\gamma_1}, \tag{3.3}$$

and finally

$$\delta^{\gamma_1} = \frac{(a^*)^{2/3} (b^*)^{1/3} c}{a^{2/3} b^{1/3} c^*} \Longrightarrow \frac{x}{x^*} = \frac{(a^*)^{2/3} (b^*)^{1/3} c}{a^{2/3} b^{1/3} c^*}, \qquad \frac{y}{y^*} = \frac{(a^*)^{1/3} (b^*)^{2/3} c}{a^{1/3} b^{2/3} c^*}. \tag{3.4}$$

To conclude, invariants are defined in the following way:

$$X = x \frac{a^{2/3} b^{1/3}}{c} = x^* \frac{(a^*)^{2/3} (b^*)^{1/3}}{c^*}, \qquad Y = y \frac{a^{1/3} b^{2/3}}{c} = y^* \frac{(a^*)^{1/3} (b^*)^{2/3}}{c^*}. \tag{3.5}$$

The second method. Transformations (2.1) are given the form

$$\frac{x}{x^*} = \delta^{\gamma_1}, \qquad \frac{y}{y^*} = \delta^{\gamma_2}, \qquad \frac{a}{a^*} = \delta^{-2\gamma_1+\gamma_2+\gamma_3}, \qquad \frac{b}{b^*} = \delta^{\gamma_1-2\gamma_2+\gamma_3}, \qquad \frac{c}{c^*} = \delta^{\gamma_3}. \tag{3.6}$$

In an analogous way, transformations (2.2) take the form

$$\frac{x}{x^*} = \delta^{\beta_1}, \qquad \frac{y}{y^*} = \delta^{\beta_2}, \qquad \frac{a}{a^*} = \delta^{\beta_3}, \qquad \frac{b}{b^*} = \delta^{\beta_4}, \qquad \frac{c}{c^*} = \delta^{\beta_5}, \tag{3.7}$$

with arbitrary values $\beta_1,\ldots,\beta_5$. It is assumed, owing to (3.7), that these values give relations of nontransformed and transformed quantities occurring in (2.1). Comparing (3.6) and (3.7), one arrives at

$$\delta^{\beta_1} = \delta^{\gamma_1}, \qquad \delta^{\beta_2} = \delta^{\gamma_2}, \qquad \delta^{\beta_3} = \delta^{-2\gamma_1+\gamma_2+\gamma_3}, \qquad \delta^{\beta_4} = \delta^{\gamma_1-2\gamma_2+\gamma_3}, \qquad \delta^{\beta_5} = \delta^{\gamma_3}, \tag{3.8}$$

and hence

$$\beta_1 = \gamma_1, \qquad \beta_2 = \gamma_2, \qquad \beta_3 = -2\gamma_1 + \gamma_2 + \gamma_3, \qquad \beta_5 = \gamma_3. \tag{3.9}$$

The obtained equations can be treated as equations with respect to quantities γ_1, γ_2, and γ_3. In this case, one may find these quantities from three arbitrary equalities in (3.8) and then put the obtained result into the two remaining equalities. In particular, this procedure corresponds to the case of removing γ_1, γ_2, and γ_3 in the first method. Namely, to repeat the earlier procedure, the following system of three equations should be solved:

$$-2\gamma_1 + \gamma_2 + \gamma_3 = \beta_3, \qquad \gamma_1 - 2\gamma_2 + \gamma_3 = \beta_4, \qquad \gamma_3 = \beta_5. \tag{3.10}$$

The above operation corresponds to the description of γ_1, γ_2, γ_3 in terms of constants a, b, c. The solution to (3.10) yields

$$\gamma_1 = -\frac{2}{3}\beta_3 - \frac{1}{3}\beta_4 + \beta_5, \qquad \gamma_2 = -\frac{1}{3}\beta_3 - \frac{2}{3}\beta_4 + \beta_5, \qquad \gamma_3 = \beta_5. \tag{3.11}$$

Using the results in the first two equalities of (3.8), one receives

$$\beta_1 + \frac{2}{3}\beta_3 + \frac{1}{3}\beta_4 - \beta_5 = 0, \qquad \beta_2 + \frac{1}{3}\beta_3 + \frac{2}{3}\beta_4 - \beta_5 = 0, \tag{3.12}$$

and hence

$$\delta^{\beta_1+(2/3)\beta_3+(1/3)\beta_4-\beta_5} = 1, \qquad \delta^{\beta_2+(1/3)\beta_3+(2/3)\beta_4-\beta_5} = 1. \tag{3.13}$$

Returning to (3.7), powers of δ are substituted by the corresponding fractions, that is,

$$\frac{x}{x^*}\left(\frac{a}{a^*}\right)^{2/3}\left(\frac{b}{b^*}\right)^{1/3}\frac{c^*}{c}=1, \qquad \frac{y}{y^*}\left(\frac{a}{a^*}\right)^{1/3}\left(\frac{b}{b^*}\right)^{2/3}\frac{c^*}{c}=1. \tag{3.14}$$

It is worth noticing that again quantities (3.4) remain invariant.

Although the two methods of determination of invariants are equivalent, the second one has an advantage of being easier in programming. The problem is actually reduced to comparison of exponents in transformations (2.2) and (3.1), which directly yields (3.9). Then three arbitrarily chosen equations (3.10) are solved and relations (3.12) are obtained.

4. Formulation of initial equation by invariants

Applying all three transformations (2.9), (2.8), and (2.11) to (2.1), one arrives at

$$\delta^{\gamma_1+\gamma_2+\gamma_3}\left[a^*(x^*)^3+b^*(y^*)^3-c^*x^*y^*\right]=0. \tag{4.1}$$

Taking into account (3.10) and (3.6), the following relation is obtained:

$$\delta^{\gamma_1+\gamma_2+\gamma_3}=\delta^{-\beta_3-\beta_4-3\beta_5}=\frac{c^3}{ab}\frac{a^*b^*}{(c^*)^3}. \tag{4.2}$$

Consequently, the effect common to the three transformations is equivalent to the occurrence of general multiplier (4.2). From (4.1), one gets

$$\frac{(a^*)^2b^*}{(c^*)^3}(x^*)^3+\frac{a^*(b^*)^2}{(c^*)^3}(y^*)^3-\frac{a^*b^*}{(c^*)^2}x^*y^*=0. \tag{4.3}$$

It is worth noticing that the first (second) term of (4.3) represents the third power of invariant $X(Y)$, whereas the third term represents multiplication of the mentioned two invariants. Therefore, instead of (4.3), one obtains

$$X^3+Y^3-XY=0. \tag{4.4}$$

Note that such a transition to invariants can also be expressed explicitly, that is, by computing

$$x=X\frac{c}{a^{2/3}b^{1/3}}, \qquad y=Y\frac{c}{a^{1/3}b^{2/3}} \tag{4.5}$$

from (3.4), and substituting (4.5) into (2.1). The reduction by means of the general multiplier $c^3/(ab)$ yields (4.4).

Concluding, the investigation of the transformation group from initial equation (2.1) has revealed the possibility of unification of five quantities as well as two combinations (3.4)—invariants of the admitted group. Consequently, while exhibiting the dependence between invariants (4.4), the equation becomes essentially simpler in its form.

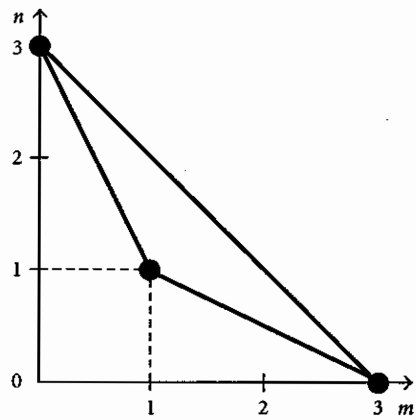

Figure 5.1. Newton's triangle for the initial equation.

5. Asymptotic-group analysis

In reply to the question addressed earlier on how coefficients of equation terms in (2.1) influence weights of those terms, it has been shown that these coefficients should be considered as components of invariants (3.4) rather than separate quantities if their role is to be of any importance.

It has also been demonstrated that the group properties of (2.1) enable the reduction in the number of occurring quantities from five to two. In this way, (4.4) is derived. The capabilities of the group properties become exhausted; (4.4) cannot be subject to any extension (scaling) transformations, except for the trivial identity. However, further simplification of the obtained equation can be achieved by the application of the group approximated properties.

Equation (4.4) consists of three terms whose weights vary depending on the conditions. Thus, for instance, for large (resp., small) values of terms X and Y, large (resp., small) exponents prevail. Investigations of relative weights of terms of similar equations were carried out already by Newton who made use of the Newton polygon. We construct the polygon for the case under consideration. Denote exponents of X by m and let n stand for the exponent of Y. For the first term in (4.4), $m = 0$, $n = 0$; for the second term, $m = 0$, $n = 3$; for the third one, $m = 1$, $n = 1$. By linking the points with the corresponding coordinates marked by m, n on the plane, the triangle shown in Figure 5.1 is obtained.

Each side of this triangle corresponds to two terms from (4.4). Consideration of two of the terms with the omission of the third one leads to a simplified equation. Consequently, the three triangle sides correspond to three simplification cases.

However, a more suitable and updated way to determine simplified equations is offered by the application of the asymptotical analysis. Usually, within this method, relative weights of the terms are investigated via introduction of an arbitrary real small parameter, that is, a small coefficient standing by one or a few of the terms. In (4.4), however, coefficients at all terms are equal (in modulus) to one, that is, a real small parameter does not exist.

Therefore a formal small parameter $\delta < 1$ is introduced. We define the following transformations:

$$X = \delta^{\alpha_1} X^*, \qquad Y = \delta^{\alpha_2} Y^*. \tag{5.1}$$

It is assumed that the transformations yield quantities of order one:

$$X^* \sim 1, \qquad Y^* \sim 1, \tag{5.2}$$

which are equivalent to the following relations:

$$X \sim \delta^{\alpha_1}, \qquad Y \sim \delta^{\alpha_2}. \tag{5.3}$$

Thus the weights of quantities X and Y are now defined by parameters α_1, α_2. Positive (resp., negative) values of the parameters are associated with small (resp., large) quantities.

On applying transformations (5.1), (4.4) takes the following form:

$$\delta^{3\alpha_1}(X^*)^3 + \delta^{3\alpha_2}(Y^*)^3 - \delta^{\alpha_1+\alpha_2} X^* Y^* = 0. \tag{5.4}$$

Owing to relations (5.2), the weights of terms in (5.4) (and also in (4.4)) are fully defined by the multipliers in the form of certain powers of δ occurring as the result of the transformations. On the other hand, the weights of these multipliers are defined via exponents. Below, the multipliers under discussion are written in the same order as the equation terms:

$$3\alpha_1, 3\alpha_2, \alpha_1 + \alpha_2. \tag{5.5}$$

All three exponents of (5.5) can be simultaneously equal only if $\alpha_1 = \alpha_2 = 0$. In fact, this property agrees with our earlier observation that (4.4) allows only the identity transformation to be applied.

Consider now the case when only two exponents given in (5.5) are equal. It is also required that two equal exponents be smaller than the third one (smaller exponents correspond to large weights of terms, and vice versa). The following three cases should be taken into consideration:

$$\begin{aligned} 3\alpha_1 = 3\alpha_2 < \alpha_1 + \alpha_2 &\Longrightarrow \alpha_1 = \alpha_2 < 0, \\ 3\alpha_1 = \alpha_1 + \alpha_2 < 3\alpha_2 &\Longrightarrow 2\alpha_1 = \alpha_2 > 0, \\ 3\alpha_2 = \alpha_1 + \alpha_2 < 3\alpha_1 &\Longrightarrow \alpha_1 = 2\alpha_2 > 0. \end{aligned} \tag{5.6}$$

These three cases are represented by the three radii on the α_1, α_2 plane (see Figure 5.2), which in turn correspond to the Newton triangle sides (see Figure 5.1). The radii are orthogonal to triangle sides and directed into the triangle inside.

We focus on each of the radii in more detail.

The first of the cases in (5.6) corresponds to the following form of (5.5):

$$3\alpha_1, 3\alpha_1, 2\alpha_1, \quad \alpha_1 < 0. \tag{5.7}$$

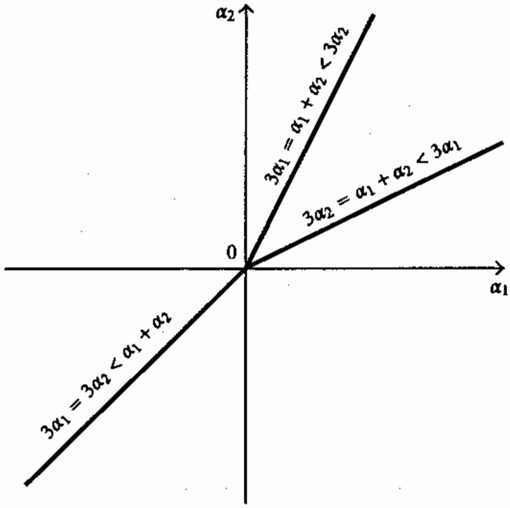

Figure 5.2. Three radii on the α_1, α_2 plane.

We choose the smallest in modulo value α_1 for which all components of (5.7) are rational, that is, the value $\alpha_1 = -1$. This gives

$$\alpha_1 = -1, \qquad \alpha_2 = -1 \tag{5.8}$$

and (5.5) assumes the form

$$-3, -3, -2. \tag{5.9}$$

The values of α_1, α_2 given in (5.8) are associated with the following asymptotic estimations:

$$X \sim \delta^{-1} > 1, \qquad Y \sim \delta^{-1} > 1. \tag{5.10}$$

(5.9) corresponds to the simplified equation

$$(X^*)^3 + (Y^*)^3 = 0. \tag{5.11}$$

In this equation, the terms resulting from transformations (5.1) remain and correspond to the same (smallest) powers of δ. This corresponds to asymptotical estimation (5.10) and also exhibits an important property. Namely, the simplified equation (5.11) is invariant with respect to transformations (5.1) with exponents (5.8), that is, to the transformations

$$X = \delta^{-1}X^*, \qquad Y = \delta^{-1}Y^*. \tag{5.12}$$

This invariance makes it possible to return from the transformed quantities to the initial ones keeping the form of the simplified equation:

$$X^3 + Y^3 = 0. \tag{5.13}$$

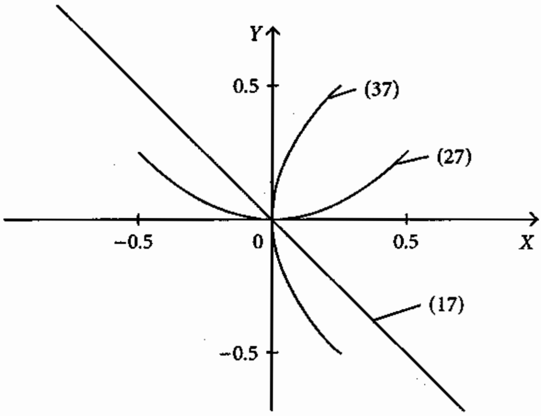

Figure 5.3. First approximation results for the initial equation.

Consequently, the simplified equation (5.13) can be subjected to extension group (5.12) operations. Note the lack of this property with respect to initial equation (2.11). Occurrence of such an additional extension group can be treated as a criterion for a real simplification, that is, (5.13) is more simplified than (2.11). In particular, again (as in the case of (2.1)) the number of terms occurring in the equation can be decreased. For this purpose, again, invariants may be obtained. In the case under discussion there is only one invariant

$$z = \frac{Y}{X}, \tag{5.14}$$

hence

$$Y = zX, \tag{5.15}$$

and, finally, (5.13) yields

$$1 + z^3 = 0. \tag{5.16}$$

Observe that, practically, the simplified equation includes only one quantity z. Solving it, one obtains

$$z = -1 \Longrightarrow Y = -X. \tag{5.17}$$

Equation (5.17) defines the asymptotical direction of infinity (Figure 5.3).

It is worth noticing that the simplification procedure carried out here bases on asymptotical estimations (5.12). The obtained result (5.17) shows that the estimations are automatically satisfied for large values of X. Note that X plays the role of a real large parameter. As the transition to invariant (5.14) matches two quantities X and Y, only one of them may be estimated, and thus the estimation of the second one is performed automatically.

The second case in (5.6) is represented by (5.5) of the form

$$3\alpha_1, 6\alpha_1, 3\alpha_1, \quad \alpha_1 < 0. \tag{5.18}$$

Choosing

$$\alpha_1 = \frac{1}{3}, \qquad \alpha_2 = \frac{2}{3}, \tag{5.19}$$

the following integer components of (5.5) are obtained:

$$1, 2, 1. \tag{5.20}$$

The values of parameters (5.19) correspond to the asymptotical estimations

$$X \sim \delta^{1/3} < 1, \qquad Y \sim \delta^{2/3} < 1, \tag{5.21}$$

that is, small X and Y of a smaller order.

The simplified equation reads

$$X^3 + XY = 0. \tag{5.22}$$

It is invariant with respect to the transformations

$$X = \delta^{1/3} X^*, \qquad Y = \delta^{2/3} Y^*. \tag{5.23}$$

The obtained invariance allows transition from the transformed quantities to the initial ones. There is only one invariant of transformations (5.23), which reads

$$z = \frac{Y}{X^2}. \tag{5.24}$$

Introducing

$$Y = zX^2 \tag{5.25}$$

and substituting it into (5.22), the following equation, with respect to invariants, is obtained:

$$1 - z = 0, \tag{5.26}$$

which means that, again, an equation possessing only one quantity is found. This results in

$$z = 1 \Longrightarrow Y = X^2. \tag{5.27}$$

The corresponding parabolic graph is shown in Figure 5.3.

Estimation (5.21) is carried out automatically for small values of X, which plays here the role of a real small parameter.

The third case in (5.6) yields (5.5) of the form

$$6\alpha_2, 3\alpha_2, 3\alpha_2, \quad \alpha_2 > 0. \tag{5.28}$$

Choosing

$$\alpha_1 = \frac{2}{3}, \qquad \alpha_2 = \frac{1}{3}, \tag{5.29}$$

the following is obtained:

$$2, 1, 1. \tag{5.30}$$

The asymptotical estimations follow:

$$X \sim \delta^{2/3} < 1, \qquad Y \sim \delta^{1/3} < 1, \tag{5.31}$$

and the simplified equation reads

$$Y^3 - YX = 0. \tag{5.32}$$

The allowed group has the form

$$X = \delta^{2/3} X^*, \qquad Y = \delta^{1/3} Y^*, \tag{5.33}$$

and the associated invariant is of the form

$$z = \frac{X}{Y^2}, \tag{5.34}$$

which yields

$$X = zY^2. \tag{5.35}$$

The equation with respect to the invariant

$$1 - z = 0 \tag{5.36}$$

gives the solution

$$z = 1 \Longrightarrow X = Y^2. \tag{5.37}$$

Asymptotical estimations (5.31) are satisfied automatically for small values of Y which plays the role of a real small parameter.

6. Rotation of coordinates

The results obtained in the previous section shed some light on a curve governed by (4.4). In order to acquire a deeper insight, we concentrate on the following observations. The graphs shown in Figure 5.3 exhibit a symmetry axis, which is the bisectrix between

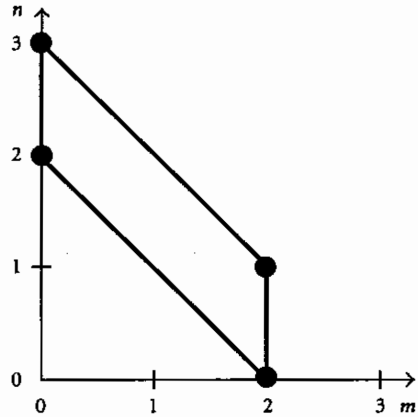

Figure 6.1. Newton's polygon for the equation with rotated axes.

axes X and Y. In what follows, axes rotation by $\pi/4$ is performed, and one of the coordinates becomes the symmetry axis. To achieve the other, the following transformations are introduced:

$$\begin{aligned} X &= u\cos\frac{\pi}{4} + v\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(u+v), \\ Y &= -u\sin\frac{\pi}{4} + v\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}(-u+v). \end{aligned} \tag{6.1}$$

Substitution of X and Y in (4.4) by (6.1) gives

$$\sqrt{2}(3u^2v + v^3) + u^2 - v^2 = 0. \tag{6.2}$$

The investigation to be carried out for (6.2) is similar to that for (4.4). Denoting by $m(n)$ the exponent with respect to $u(v)$, the Newton polygon is constructed (see Figure 6.1). Its analysis shows that the four terms of (6.2) can be matched into pairs only in four ways, although the number of possibilities is six. The Newton polygon, composed of the mesh of convex points, representing exponents of mono-terms occurring in (6.2), clearly explains this conclusion.

Similarly to the previous case, however, we apply a more suitable and challenging approach to estimate the weights of the terms of (6.2) representing an extensional mapping. The following transformations are introduced:

$$u = \delta^{\alpha_1}u^*, \qquad v = \delta^{\alpha_2}v^*, \tag{6.3}$$

and it is required that the following relations be satisfied:

$$u^* \sim 1, \qquad v^* \sim 1. \tag{6.4}$$

As a result of transformations (6.3), the terms of (6.2) are accompanied by multipliers δ. Owing to relation (6.4), these multipliers describe fully the weights of the equation

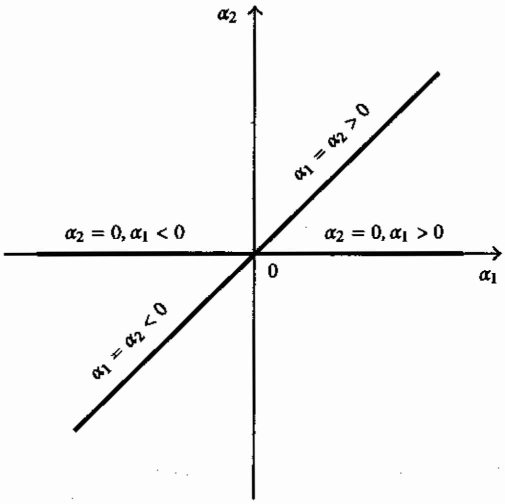

Figure 6.2. Radii on the plane α_1, α_2 for the equation with rotated axes.

terms. Presentation of the transformed equation being neglected, only the following result with the exponents is reported:

$$2\alpha_1 + \alpha_2, 3\alpha_2, 2\alpha_1, 2\alpha_2. \tag{6.5}$$

Consider now pair equalities composed by the components in (6.5):

$$\begin{aligned}
&2\alpha_1 + \alpha_2 = 3\alpha_2, \quad 2\alpha_1 + \alpha_2 < 2\alpha_1, \quad 2\alpha_1 + \alpha_2 < 2\alpha_2 \Longrightarrow \alpha_1 = \alpha_2 < 0,\\
&2\alpha_1 + \alpha_2 = 2\alpha_1, \quad 2\alpha_1 + \alpha_2 < 3\alpha_2, \quad 2\alpha_1 + \alpha_2 < 2\alpha_2 \Longrightarrow \alpha_2 = 0,\ \alpha_1 < 0,\\
&2\alpha_1 + \alpha_2 = 2\alpha_2, \quad 2\alpha_1 + \alpha_2 < 3\alpha_2, \quad 2\alpha_1 + \alpha_2 < 2\alpha_1 \Longrightarrow \alpha_2 = 2\alpha_1,\ \alpha_1 > 0,\ \alpha_1 < 0,\\
&3\alpha_2 = 2\alpha_1, \quad 3\alpha_2 < 2\alpha_1 + \alpha_2, \quad 3\alpha_2 < 2\alpha_2 \Longrightarrow 3\alpha_2 = 2\alpha_1,\ \alpha_1 > 0,\ \alpha_1 < 0,\\
&3\alpha_2 = 2\alpha_2, \quad 3\alpha_2 < 2\alpha_1 + \alpha_2, \quad 3\alpha_2 < 2\alpha_1 \Longrightarrow \alpha_2 = 0,\ \alpha_1 > 0,\\
&2\alpha_1 = 2\alpha_2, \quad 2\alpha_1 < 2\alpha_1 + \alpha_2, \quad 2\alpha_1 < 3\alpha_2 \quad \alpha_1 = \alpha_2 > 0.
\end{aligned} \tag{6.6}$$

The third and fourth cases are omitted since they include contradictory inequalities. The remaining four cases yield radii on the plane α_1, α_2 as shown in Figure 6.2. These radii are perpendicular to the polygon sides and go into its inside.

The following four cases are more deeply analyzed.

For the first case in (6.6), (6.5) takes the form

$$3\alpha_1, 3\alpha_1, 2\alpha_1, 2\alpha_1, \quad \alpha_1 < 0. \tag{6.7}$$

On choosing

$$\alpha_1 = \alpha_2 = -1, \tag{6.8}$$

the resulting form is

$$-3,-3,-2,-2. \tag{6.9}$$

Asymptotical estimations are as follows:

$$u \sim \delta^{-1} > 1, \qquad v \sim \delta^{-1} > 1, \tag{6.10}$$

and the simplified equation reads

$$3u^2v + v^3 = 0. \tag{6.11}$$

Since this equation does not possess real solutions satisfying asymptotical estimations (6.10), this case will not be further considered.

For the second case in (6.6), (6.5) reads

$$2\alpha_1,0,2\alpha_1,0, \quad \alpha_1 < 0. \tag{6.12}$$

By choosing

$$\alpha_1 = -0.5, \qquad \alpha_2 = 0, \tag{6.13}$$

the following final form is obtained:

$$-1,0,-1,0. \tag{6.14}$$

Asymptotical estimations are as follows:

$$u \sim \delta^{-0.5} > 1, \qquad v \sim 1, \tag{6.15}$$

and the simplified equation has the form

$$3\sqrt{2}u^2v + u^2 = 0. \tag{6.16}$$

Transformations via extension assume the form

$$u = \delta^{-0.5}u^*, \qquad v = v^*. \tag{6.17}$$

Here v is an invariant quantity. Through reduction by a nonzero value u, the following equation (with respect to the invariant) is reached:

$$3\sqrt{2}v + 1 = 0, \tag{6.18}$$

with the corresponding solution

$$v = -\frac{1}{3\sqrt{2}}. \tag{6.19}$$

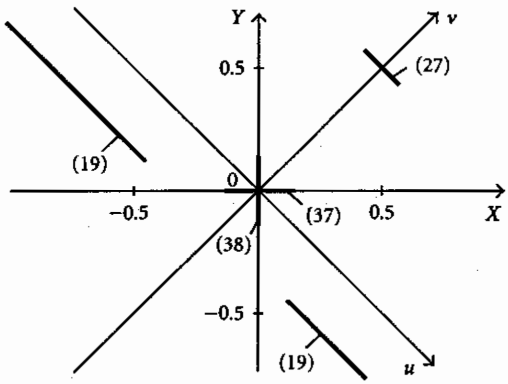

Figure 6.3. First approximation results for the equation with rotated axes.

The last equation represents an asymptote (Figure 6.3). It is worth reminding that due to the application of initial axes X, Y performed before, the direction of the asymptotic became known, which has now been completed by determination of its final position.

Owing to (6.19), asymptotic estimations (6.15) are satisfied automatically and for large values of u, which plays the role of a real large parameter here.

For the fifth case in (6.6), (6.5) is

$$2\alpha_1, 0, 2\alpha_1, 0, \quad \alpha_1 > 0. \tag{6.20}$$

The choice of

$$\alpha_1 = 0.5, \qquad \alpha_2 = 0 \tag{6.21}$$

results in

$$1, 0, 1, 0. \tag{6.22}$$

The asymptotical estimations are of the form

$$u \sim \delta^{0.5} < 1, \qquad v \sim 1, \tag{6.23}$$

and the simplified equation reads

$$\sqrt{2}v^3 - v^2 = 0. \tag{6.24}$$

Extension transformations take the form

$$u = \delta^{0.5} u^*, \qquad v = v^*. \tag{6.25}$$

Quantity v is invariant, and the equation with respect to it, which reads

$$\sqrt{2}v - 1 = 0, \tag{6.26}$$

has the following solution:

$$\nu = \frac{1}{\sqrt{2}}. \tag{6.27}$$

In Figure 6.3, the point where the curve intersects axis ν is marked. Equation (6.22) can be said to describe a tangent to the curve at the corresponding point.

Asymptotical estimations (6.23) are automatically realized (see (6.27)) for small values of u that plays the role of a real small parameter.

For the sixth case, the considered sequence reads

$$3\alpha_1, 3\alpha_1, 2\alpha_1, 2\alpha_1, \quad \alpha_1 > 0. \tag{6.28}$$

Assuming

$$\alpha_1 = \alpha_2 = 1, \tag{6.29}$$

one obtains

$$3, 3, 2, 2. \tag{6.30}$$

The asymptotical estimations are

$$u \sim \delta^1 < 1, \qquad \nu \sim \delta^1 < 1, \tag{6.31}$$

and the simplified equation reads

$$u^2 - \nu^2 = 0. \tag{6.32}$$

The associated transformations are

$$u = \delta u^*, \qquad \nu = \delta \nu^*, \tag{6.33}$$

and the invariant is

$$z = \frac{\nu}{u}. \tag{6.34}$$

Hence

$$\nu = zu \tag{6.35}$$

and, when put into (6.32), it yields

$$1 - z^2 = 0. \tag{6.36}$$

The obtained equation possesses two solutions, the first of which is

$$z = 1 \Longrightarrow \nu = u, \tag{6.37}$$

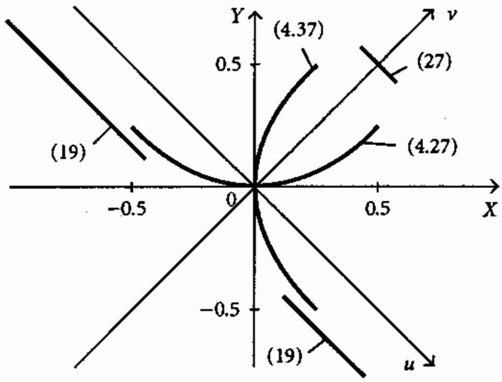

Figure 6.4. The best first approximation results within two investigated variants.

whereas the second one is

$$z = -1 \Longrightarrow v = -u. \tag{6.38}$$

The corresponding lines are shown in Figure 6.3. Asymptotical estimations (6.31) are satisfied automatically (see (6.37) or (6.38)) and for small values of u, which plays the role of a real small parameter.

The comparison of Figures 6.3 and 5.3 admits the conclusion that the new coordinate system makes it possible to define the asymptotic position and to find the intersection with axis v. At present, however, the neighborhood of the origin is described by tangents to the curve only, whereas earlier deeper results were obtained (formulas (5.27) and (5.37)).

Matching two cases, one obtains the picture shown in Figure 6.4.

In conclusion, the investigation of the simplified variants of the equations in two different coordinate systems has provided an efficient amount of information on the curve under analysis, showing it to be a loop with asymptotically linear branches in infinity.

7. Combined equation system

The investigation results with respect to XY and uv have exhibited advantages and disadvantages of each of the considered coordinates. Observe that various parts of the curve are best investigated by various equations, which is evidently inconvenient. In what follows, two systems of coordinates will be matched in order to improve the analysis. Transformation of coordinates (6.1) is applied only to the first two terms of (4.4), leaving the last term unchanged. As a result, the following system, composed of three equations with four unknowns X, Y, u, v, is obtained:

$$\begin{gathered} 3u^2v + v^3 - \sqrt{2}XY = 0, \\ X = \frac{u+v}{\sqrt{2}}, \qquad Y = -\frac{u+v}{\sqrt{2}}. \end{gathered} \tag{7.1}$$

Analogously to the cases already discussed, the following transformations are introduced to estimate the weights of terms in (7.1):

$$X = \delta^{\alpha_1} X^*, \qquad Y = \delta^{\alpha_2} Y^*, \qquad u = \delta^{\alpha_3} u^*, \qquad v = \delta^{\alpha_4} v^*, \tag{7.2}$$

with the assumption that the relations to be satisfied are

$$X^* \sim 1, \qquad Y^* \sim 1, \qquad u^* \sim 1, \qquad v^* \sim 1. \tag{7.3}$$

On substituting (7.2) into (7.1), by all terms, there occur coefficients as power of δ. Owing to (7.3), these coefficients fully define the weights of the terms. The following shows the exponents of δ arranged in the order corresponding to the position of the terms in (7.1):

$$2\alpha_3 + \alpha_4, 3\alpha_4, \alpha_1 + \alpha_2, \qquad \alpha_1, \alpha_3, \alpha_4, \qquad \alpha_2, \alpha_3, \alpha_4. \tag{7.4}$$

Searching for different simplification variants of (7.1), we try finding $\alpha_1, \ldots, \alpha_4$ in a formal way. Simultaneous equality of all exponents in the intervals of each row of (7.4) for exponents $\alpha_1, \ldots, \alpha_4$ different form zero is impossible.

If at least one of quantities $\alpha_1, \ldots, \alpha_4$ is different from zero, then not all exponents of each row in (7.4) will be the same. Smaller values of exponents are associated with large values of the corresponding terms of the equations and vice versa. Following the previous steps, $\alpha_1, \ldots, \alpha_4$ are sought using the minimal simplification criterion. As the group approach, applied earlier for the case of two parameters, is not suitable in the case under investigation, the analytical approach will be used here. In what follows, the exponents in each row of (7.4) are compared by means of different methods. It is also required that the equal exponents should not exceed the third exponent for a given row. As a result, the following system of algebraic equations and inequalities is obtained:

$$\begin{aligned}
2\alpha_3 + \alpha_4 &= 3\alpha_4 \le \alpha_1 + \alpha_2,\\
2\alpha_3 + \alpha_4 &= \alpha_1 + \alpha_2 \le 3\alpha_4,\\
3\alpha_4 &= \alpha_1 + \alpha_2 \le 2\alpha_3 + \alpha_4,\\
\alpha_1 &= \alpha_3 \le \alpha_4,\\
\alpha_1 &= \alpha_4 \le \alpha_3,\\
\alpha_3 &= \alpha_4 \le \alpha_1,\\
\alpha_2 &= \alpha_3 \le \alpha_4,\\
\alpha_2 &= \alpha_4 \le \alpha_3,\\
\alpha_3 &= \alpha_4 \le \alpha_2.
\end{aligned} \tag{7.5}$$

Minimal simplification of (7.1) is associated with the cases when, in the simplified equations, a maximal number of terms remain (and the number of neglected terms is possibly small). That means that in order to find four quantities $\alpha_1, \ldots, \alpha_4$, the largest possible number of equations out of the nine equations in (7.5) should be considered, which corresponds to the largest number of the terms that are taken into account.

Since all (7.5) are homogeneous, the maximal number of linearly independent equations that can be used to reach a nonzero solution is equal to three. On choosing three linearly independent equations, a solution with the sign defined by inequalities and accuracy up to an arbitrary multiplier is obtained. A procedure concerning various variants of the minimal simplification search is reduced to a choice involving all possible sets of three equations with their solutions, formed from nine equations (7.5). Although the total number of such sets equals $C_9^3 = 84$, the actual number of different variants is equal to eight. There are a few obvious reasons for this. First of all, in many cases, various three equation sets give the same solutions. Moreover, some of the sets include linearly dependent equations, while some others do not satisfy the required inequalities. We consider the solutions satisfying all the mentioned requirements. Since these solutions are found with accuracy up to a possible real multiplier, the smallest digital values of quantities $\alpha_1, \ldots, \alpha_4$ yield integer values of the components included in (7.4):

(1)

$$\alpha_1 = 1, \qquad \alpha_2 = 2, \qquad \alpha_3 = 1, \qquad \alpha_4 = 1. \tag{7.6}$$

This case corresponds to the following estimations of weights of the variables:

$$Y < X \sim u \sim v < 1. \tag{7.7}$$

(7.4) for these values of exponents is of the form

$$3,3,3, \qquad 1,1,1, \qquad 2,1,1. \tag{7.8}$$

When the first term in the third equation is neglected, the following simplified equations are obtained:

$$\begin{gathered} 3u^2v + v^3 - \sqrt{2}XY = 0, \\ X = \frac{u+v}{\sqrt{2}}, \qquad 0 = -u + v. \end{gathered} \tag{7.9}$$

This system of equations is invariant with respect to the transformations

$$X = \delta X^*, \qquad Y = \delta^2 Y^*, \qquad u = \delta u^*, \qquad v = \delta v^*. \tag{7.10}$$

The invariants of transformations (7.10) are as follows:

$$z_1 = \frac{Y}{X^2}, \qquad z_2 = \frac{u}{X}, \qquad z_3 = \frac{v}{X}. \tag{7.11}$$

In the transition to invariants, (7.9) acquire the form

$$3(z_2)^2 z_3 + (z_3)^3 - \sqrt{2}z_1 = 0; \qquad 1 = \frac{z_2 + z_3}{\sqrt{2}}, \qquad 0 = -z_2 + z_3. \tag{7.12}$$

Therefore, for simplified equations (7.9), occurrence of an additional one-parameter extension group, not present in exact equations (7.1), is the simplification criterion and allows transition from four quantities X, Y, u, v to three quantities z_1, z_2, z_3.

Equation (7.12) has the following solution:

$$z_1 = 1, \qquad z_2 = z_3 = \frac{1}{\sqrt{2}}, \qquad Y = X^2. \tag{7.13}$$

(2)

$$\alpha_1 = 2, \qquad \alpha_2 = 1, \qquad \alpha_3 = 1, \qquad \alpha_4 = 1. \tag{7.14}$$

This case is associated with the following weights of variables:

$$X < Y \sim u \sim v < 1. \tag{7.15}$$

(7.4) takes the form

$$3,3,3, \qquad 2,1,1, \qquad 1,1,1. \tag{7.16}$$

The simplified equations read

$$\begin{aligned} &3u^2v + v^3 - \sqrt{2}XY = 0, \\ &0 = u + v, \qquad Y = -\frac{u+v}{\sqrt{2}}. \end{aligned} \tag{7.17}$$

The extension group is

$$X = \delta^2 X^*, \qquad Y = \delta Y^*, \qquad u = \delta u^*, \qquad v = \delta v^*, \tag{7.18}$$

and the invariants are of the form

$$z_1 = \frac{X}{Y^2}, \qquad z_2 = \frac{u}{Y}, \qquad z_3 = \frac{v}{Y}. \tag{7.19}$$

Expressed in terms of the invariants, the equations are as follows:

$$3(z_2)^2 z_3 + (z_3)^3 - \sqrt{2}z_1 = 0, \qquad z_2 + z_3 = 0, \qquad 1 = -\frac{z_2 + z_3}{\sqrt{2}}, \tag{7.20}$$

and have the following solutions:

$$z_1 = 1, \qquad z_3 = -z_2 = \frac{1}{\sqrt{2}}, \qquad X = Y^2. \tag{7.21}$$

(3)

$$\alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = 1, \qquad \alpha_4 = 0. \tag{7.22}$$

In this case, the weights of the variables are

$$u < X \sim Y \sim v \sim 1 \tag{7.23}$$

and (7.4) reads

$$2,0,0, \qquad 0,1,0, \qquad 0,1,0. \tag{7.24}$$

The simplified equations are

$$v^3 - \sqrt{2}XY = 0; \qquad X = \frac{v}{\sqrt{2}}; \qquad Y = \frac{v}{\sqrt{2}}, \tag{7.25}$$

and a supplemented subgroup has the form

$$X = X^*, \qquad Y = Y^*, \qquad u = \delta u^*, \qquad v = v^*. \tag{7.26}$$

Its invariants are X, Y, and v since (7.25) are already written in the form of invariants. Their solution yields

$$X = Y = \frac{1}{2}, \qquad v = \frac{1}{\sqrt{2}}. \tag{7.27}$$

(The variant $v = 0$ is not considered since it defined estimation (7.23).)

(4)

$$\alpha_1 = -1, \qquad \alpha_2 = -1, \qquad \alpha_3 = -1, \qquad \alpha_4 = 0. \tag{7.28}$$

In this case, the weights of the variables are

$$X \sim Y \sim u > v \sim 1. \tag{7.29}$$

(7.4) reads

$$-2,0,-2, \qquad -1,-1,0, \qquad -1,-1,0. \tag{7.30}$$

The simplified equations are of the form

$$3u^2 v - \sqrt{2}XY = 0, \quad X = \frac{u}{\sqrt{2}}, \quad Y = -\frac{u}{\sqrt{2}}, \tag{7.31}$$

and a supplemented subgroup is

$$X = \delta^{-1}x^*, \qquad Y = \delta^{-1}y^*, \qquad u = \delta^{-1}u^*, \qquad v = v^*. \tag{7.32}$$

The invariants are

$$z_1 = \frac{Y}{u}, \qquad z_2 = \frac{Y}{u}, \qquad z_3 = v, \tag{7.33}$$

and the equations expressed in the invariants read

$$3v - \sqrt{2}z_1z_2 = 0, \qquad z_1 = \frac{1}{\sqrt{2}}, \qquad z_2 = -\frac{1}{\sqrt{2}}. \tag{7.34}$$

The solutions are

$$z_1 = -z_2 = \frac{1}{\sqrt{2}}, \qquad v = -\frac{1}{3\sqrt{2}}. \tag{7.35}$$

Note that all the solutions obtained now have already been obtained before but with the use of a different system of equations; here they are determined under one unique approach.

Only four out of the eight choices of parameters $\alpha_1,\ldots,\alpha_2$ have been discussed so far. Attained formally, the choices correspond to the minimal simplification of (7.1). We consider now the remaining four possibilities.

(5)

$$\begin{aligned} &\alpha_1 = 1, \qquad \alpha_2 = 1, \qquad \alpha_3 = 1, \qquad \alpha_4 = 1, \qquad X \sim Y \sim u \sim v < 1, \\ &3,3,2, \qquad XY = 0, \\ &1,1,1, \qquad X = \frac{u+v}{\sqrt{2}}, \\ &1,1,1, \qquad Y = -\frac{u+v}{\sqrt{2}}. \end{aligned} \tag{7.36}$$

First of all, simplified equations (7.36) possess two solutions: $X = 0$ and $Y = 0$. Solution $Y = 0$ reduces the problem to case (1) and to (7.9) in the second approximation. Solution $X = 0$ reduces the problem to case (2) and to (7.17). Thus, the fifth case yields the results studied earlier assuming a less suitable form, and requiring, in addition, the application of the successive approximation procedure.

(6)

$$\begin{aligned} &\alpha_1 = -1, \qquad \alpha_2 = -1, \qquad \alpha_3 = -1, \qquad \alpha_4 = -1, \qquad X \sim Y \sim u \sim v > 1, \\ &-3,-3,-2, \qquad 3u^2v + v^3 = 0, \\ &-1,-1,-1, \qquad X = \frac{u+v}{\sqrt{2}}, \\ &-1,-1,-1, \qquad Y = -\frac{u+v}{\sqrt{2}}. \end{aligned} \tag{7.37}$$

The first equation in (7.37) possesses only one real solution $v = 0$, but it does not satisfy the asymptotic estimations. No more real solutions either to this equation or to the whole system exist.

(7)

$$\alpha_1 = -1, \quad \alpha_2 = -2, \quad \alpha_3 = -1, \quad \alpha_4 = -1, \quad Y > X \sim u \sim v > 1,$$
$$\begin{aligned} -3,-3,-3, &\quad 3u^2v + v^3 - \sqrt{2}XY = 0, \\ -1,-1,-1, &\quad X = \frac{u+v}{\sqrt{2}}, \\ -2,-1,-1, &\quad Y = 0. \end{aligned} \tag{7.38}$$

The first equation in (7.38), with the third one taken into account, coincides with the first equation in (7.37). Consequently, all the earlier results hold.

(8)

$$\alpha_1 = -2, \quad \alpha_2 = -1, \quad \alpha_3 = -1, \quad \alpha_4 = -1, \quad X > Y \sim u \sim v > 1,$$
$$\begin{aligned} -3,-3,-3, &\quad 3u^2v + v^3 - \sqrt{2}XY = 0, \\ -2,-1,-1, &\quad X = 0, \\ -1,-1,-1, &\quad Y = -\frac{u+v}{\sqrt{2}}. \end{aligned} \tag{7.39}$$

This is again a particular case of variant (6).

To conclude, the last four variants studied do not provide any additional information in comparison to the first four cases.

8. Procedures of successive approximations

In order to improve the results obtained earlier, a procedure of successive approximations will be applied. Considering (4.4), X and Y are presented in the form

$$X = X_1 + X_2, \qquad Y = Y_1 + Y_2. \tag{8.1}$$

We introduce the transformations

$$X_1 = \delta^{\alpha_1} X_1^*, \qquad X_2 = \delta^{\alpha_1+1} X_2^*, \qquad Y_1 = \delta^{\alpha_2} Y_1^*, \qquad Y_2 = \delta^{\alpha_2+1} Y_2^*, \tag{8.2}$$

and require satisfaction of the following relations:

$$X_1^* \sim 1, \qquad X_2^* \sim 1, \qquad Y_1^* \sim 1, \qquad Y_2^* \sim 1. \tag{8.3}$$

From (7.1), one obtains

$$X = \delta^{\alpha_1} X_1^* + \delta^{\alpha_1+1} X_2^*, \qquad Y = \delta^{\alpha_2} Y_1^* + \delta^{\alpha_2+1} Y_2^*. \tag{8.4}$$

Observe that, in accordance with (7.3), the second terms in the expressions for X and Y are of one order lower than the first terms.

Substituting (8.4) into (4.4), one arrives at

$$(\delta^{\alpha_1} X_1^* + \delta^{\alpha_1+1} X_2^*)^3 + (\delta^{\alpha_2} Y_1^* + \delta^{\alpha_2+1} Y_2^*)^3 - (\delta^{\alpha_1} X_1^* + \delta^{\alpha_1+1} X_2^*)(\delta^{\alpha_2} Y_1^* + \delta^{\alpha_2+1} Y_2^*) = 0. \tag{8.5}$$

Further investigation is carried out for fixed values of α_1 and α_2. The values of α_1, α_2, that earlier led to the smallest simplification of (4.4), are applied, and the study focuses on the cases that provided the best results on axes X, Y.

(i) $\alpha_1 = 1/3$, $\alpha_2 = 2/3$. Equation (8.5) has the form

$$(\delta^{1/3}X_1^* + \delta^{4/3}X_2^*)^3 + (\delta^{2/3}Y_1^* + \delta^{5/3}Y_2^*)^3 - (\delta^{1/3}X_1^* + \delta^{4/3}X_2^*)(\delta^{2/3}Y_1^* + \delta^{5/3}Y_2^*) = 0. \tag{8.6}$$

Performing the operations of involution and multiplication and leaving only the terms with δ and δ^2, one arrives at

$$\delta(X_1^*)^3 + 3\delta^2(X_1^*)^2X_2^* + \delta^2(Y_1^*)^3 - \delta X_1^*Y_1^* - \delta^2X_1^*Y_2^* - \delta^2X_2^*Y_1^* = 0. \tag{8.7}$$

The following splitting with respect to δ is introduced:

$$\begin{aligned} \delta &: (X_1^*)^3 - X_1^*Y_1^* = 0, \\ \delta^2 &: 3(X_1^*)^2X_2^* + (Y_1^*)^3 - X_1^*Y_2^* - X_2^*Y_1^* = 0. \end{aligned} \tag{8.8}$$

In each equation in (8.8), the splitting procedure has kept the terms resulting from transformation (8.2). Thus, (8.8) are invariant with respect to transformations (8.2) that, for the considered values of α_1, α_2, have the form

$$X_1 = \delta^{1/3}X_1^*, \qquad X_2 = \delta^{4/3}X_2^*, \qquad Y_1 = \delta^{2/3}Y_1^*, \qquad Y_2 = \delta^{5/3}Y_2^*. \tag{8.9}$$

The invariance makes it possible to return, through (8.8), from the transformed variables to the initial ones:

$$\begin{aligned} &X_1^3 - X_1Y_1 = 0, \\ &(3X_1^2 - Y_1)X_2 - X_1Y_2 + Y_1^3 = 0. \end{aligned} \tag{8.10}$$

Thus, as a result of application of successive approximations, the equation system (8.10), invariant with respect to transformations (8.9), is obtained. The associated invariants read

$$z_1 = \frac{Y_1}{X_1^2}, \qquad z_2 = \frac{X_2}{X_1^4}, \qquad z_3 = \frac{Y_2}{X_1^5}, \tag{8.11}$$

and hence

$$Y_1 = z_1X_1^2, \qquad X_2 = z_2X_1^4, \qquad Y_2 = z_3X_1^5. \tag{8.12}$$

We recall here the discussion concerning initial equation (4.4) that does not include any small parameters and all estimations of the equation term weights are carried out only with the use of the weights of quantities X and Y. A certain analogy can be drawn between that problem and the procedure of successive approximations that begins from summation (8.1). No small parameters occur in the summation. However, the second

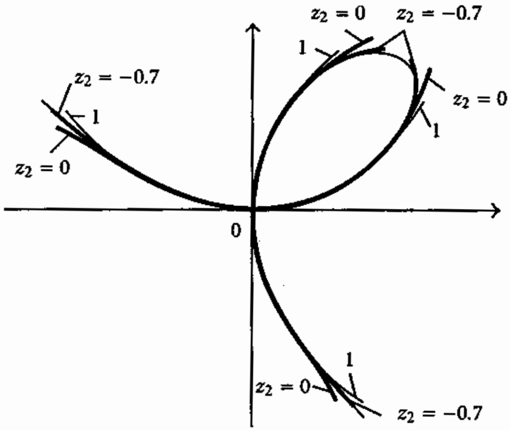

Figure 8.1. Variants of two approximation results for a zone in the origin vicinity.

term in each sum should be of one order lower than the first one, which follows directly from the asymptotical estimations obtained from (8.9) and (8.3) (for $\delta < 1$):

$$X_1 < 1, \qquad X_2 \sim X_1^4, \qquad Y_1 \sim X_1^2, \qquad Y_2 \sim X_1^5. \tag{8.13}$$

These estimations are automatically satisfied with respect to (8.12) for small values of X_1.

Substituting (8.12) into (8.10), the following equations with respect to invariants are obtained:

$$\begin{aligned} 1 - z_1 &= 0, \\ (3 - z_1)z_2 - z_3 + z_1^3 &= 0. \end{aligned} \tag{8.14}$$

Solving the first equation of (8.14), one obtains

$$z_1 = 1 \Longrightarrow Y_1 = X_1^2, \tag{8.15}$$

and hence the second equation reads

$$2z_2 - z_3 + 1 = 0. \tag{8.16}$$

To solve this equation with two unknowns, express z_3 through z_2 to receive

$$z_3 = 2z_2 + 1 \Longrightarrow X_2 = z_2 X_1^4, \qquad Y_2 = (2z_2 + 1)X_1^5. \tag{8.17}$$

An application of the results of two approximations yields an equation that describes the curve parametrically by means of parameter X_1, and the shape of the curve depends on z_2. In Figure 8.1, the curve governed by (4.4) is determined by the first approximation (two parabolas denoted by 1), by two curves constructed with respect to (8.17) for $z_2 = 0$, and by two curves for $z_2 = -0.7$. Note that the curve corresponding to $z_2 = 0$ approximates the considered curve better that the parabola. However, it is unable to approximate the loop "bending" since, for $z_2 = 0$, quantity X_1 overlaps with X, and the improved dependence $Y = Y(X)$ is obtained. The latter gives a unique value of Y for each value of X. For $z_2 = -0.7$, parametric equations (8.17) allow a description of the loop bending yielding much more suitable results:

$$Y = Y_1 + z_2 Y_1^4, \qquad X = Y_1^2 + (2z_2 + 1) Y_1^5. \tag{8.18}$$

The corresponding curves are presented in Figure 8.1. Two variants of successive approximations using (4.4) are considered taking into account those simplification cases that yielded better results in comparison to (6.2). We now consider the cases that are more accurately approximated by (6.2).

For this purpose, represent u and v in the form

$$u = u_1 + u_2, \qquad v = v_1 + v_2. \tag{8.19}$$

Introduce the transformations

$$u_1 = \delta^{\alpha_1} u_1^*, \qquad u_2 = \delta^{\alpha_1 + 1} u_2^*, \qquad v_1 = \delta^{\alpha_2} v_1^*, \qquad v_2 = \delta^{\alpha_2 + 1} v_2^*, \tag{8.20}$$

and assume the following relations to be satisfied:

$$u_1^* \sim 1, \qquad u_2^* \sim 1, \qquad v_1^* \sim 1, \qquad v_2^* \sim 1. \tag{8.21}$$

From (8.19), one obtains

$$u = \delta^{\alpha_1} u_1^* + \delta^{\alpha_1 + 1} u_2^*, \qquad v = \delta^{\alpha_2} v_1^* + \delta^{\alpha_2 + 1} v_2^*. \tag{8.22}$$

Placing (8.22) in (6.2), one arrives at

$$\begin{aligned} &\sqrt{2}\left[3(\delta^{\alpha_1} u_1^* + \delta^{\alpha_1+1} u_2^*)^2 (\delta^{\alpha_2} v_1^* + \delta^{\alpha_2+1} v_2^*) + (\delta^{\alpha_2} v_1^* + \delta^{\alpha_2+1} v_2^*)^3\right] \\ &\quad + (\delta^{\alpha_1} u_1^* + \delta^{\alpha_1+1} u_2^*)^2 - (\delta^{\alpha_2} v_1^* + \delta^{\alpha_2+1} v_2^*)^2 = 0. \end{aligned} \tag{8.23}$$

Consider now fixed values of α_1 and α_2.

(ii) $\alpha_1 = -0.5$, $\alpha_2 = 0$. Equation (8.23) takes the form

$$\begin{aligned} &\sqrt{2}\left[3(\delta^{-0.5} u_1^* + \delta^{0.5} u_2^*)^2 (v_1^* + \delta v_2^*) + (v_1^* + \delta v_2^*)^3\right] \\ &\quad + (\delta^{-0.5} u_1^* + \delta^{0.5} u_2^*)^2 - (v_1^* + \delta v_2^*)^2 = 0. \end{aligned} \tag{8.24}$$

Taking into account the involution procedure, multiplication, and keeping only the terms including δ^{-1} and δ^0, one has

$$\sqrt{2}\left\{3\left[\delta^{-1}(u_1^*)^2 v_1^* + (u_1^*)^2 v_2^* + 2u_1^* u_2^* v_1^*\right] + (v_1^*)^3\right\} - \delta^{-1}(u_1^*)^2 + 2u_1^* u_2^* - (v_1^*)^2 = 0. \tag{8.25}$$

Splitting with respect to δ gives

$$\begin{gathered}\delta^{-1} : 3\sqrt{2}(u_1^*)^2 v_1^* + (u_1^*)^2 = 0,\\ \delta^0 : \sqrt{2}\left\{3\left[(u_1^*)^2 v_1^* + 2u_1^* u_2^* v_1^*\right] + (v_1^*)^3\right\} + 2u_1^* u_2^* - (v_1^*)^2 = 0.\end{gathered} \tag{8.26}$$

Equation (8.26) is invariant with respect to the transformations

$$u_1 = \delta^{-0.5} u_1^*, \qquad u_2 = \delta^{0.5} u_2^*, \qquad v_1 = v_1^*, \qquad v_2 = \delta v_2^*, \tag{8.27}$$

which makes it possible to return from the transformed values to the initial ones

$$\begin{gathered}3\sqrt{2}u_1^2 v_1 + u_1^2 = 0,\\ \sqrt{2}[3(u_1^2 v_1 + 2u_1 u_2 v_1) + v_1^3] + 2u_1 u_2 - v_1^2 = 0.\end{gathered} \tag{8.28}$$

The invariants of transformations (8.27) are

$$z_1 = v_1, \qquad z_2 = u_1 u_2, \qquad z_3 = u_1^2 v_2, \tag{8.29}$$

and hence

$$v_1 = z_1, \qquad u_2 = \frac{z_2}{u_1}, \qquad v_2 = \frac{z_3}{u_1^2}. \tag{8.30}$$

For this case, asymptotical estimations (8.20), (8.21) have the form

$$u_1 \sim \delta^{-0.5} > 1, \qquad u_2 \sim \delta^{0.5} < 1, \qquad v_1 \sim 1, \qquad v_2 \sim \delta < 1. \tag{8.31}$$

Substituting (8.30) into (8.28), the following equations with respect to invariants are obtained:

$$\begin{gathered}3\sqrt{2}z_1 + 1 = 0,\\ \sqrt{2}[3(z_3 + 2z_1 z_2) + z_1^3] + 2z_2 - z_1^2 = 0.\end{gathered} \tag{8.32}$$

Solving the first equation

$$z_1 = -\frac{1}{3\sqrt{2}} \Longrightarrow v_1 = -\frac{1}{3\sqrt{2}}, \tag{8.33}$$

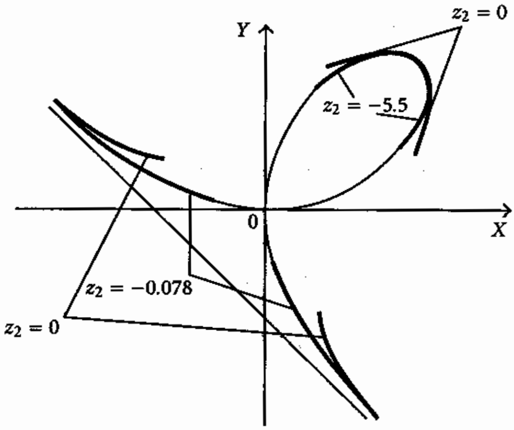

Figure 8.2. Variants of two approximation results in infinity and on the loop.

and substituting the obtained results for the appropriate quantities in the second equation, one has

$$3\sqrt{2}z_3 = \frac{2}{27} \Longrightarrow z_3 = \frac{2}{81\sqrt{2}}. \tag{8.34}$$

Note that z_2 is reduced and remains undefined.

Consequently, in view of (8.30) and (8.19),

$$u = u_1 + \frac{z_2}{u_1}, \qquad v = -\frac{1}{3\sqrt{2}} + \frac{\sqrt{2}}{81u_1^2}. \tag{8.35}$$

For $z_2 = 0$, u_1 coincides with u, and for sufficiently large values of u_1, the formula for v gives a deviation of the curve from the asymptote. For nonzero values of z_2, (8.35) yields a parametric equation of the curve with parameter u_1. Furthermore, a suitable choice of z_2 may give a better approximation of the curve than that for $z_2 = 0$. The corresponding results are shown in Figure 8.2:

$$\begin{gathered}\sqrt{2}\left[3\left(\delta^{0.5}u_1^* + \delta^{1.5}u_2^*\right)^2\left(v_1^* + \delta v_2^*\right) + \left(v_1^* + \delta v_2^*\right)^3\right] \\ + \left(\delta^{0.5}u_1^* + \delta^{1.5}u_2^*\right)^2 - \left(v_1^* + \delta v_2^*\right)^2 = 0.\end{gathered} \tag{8.36}$$

Application of the involution operations, multiplication, and keeping only the terms including δ^0 and δ results in

$$\sqrt{2}\left[3\delta\left(u_1^*\right)^2v_1^* + \left(v_1^*\right)^3 + 3\delta\left(v_1^*\right)^2v_2^*\right] + \delta\left(u_1^*\right)^2 - \left(v_1^*\right)^2 - 2\delta v_1^* v_2^* = 0. \tag{8.37}$$

Splitting with respect to δ yields

$$\begin{gathered}\delta^0 : \sqrt{2}\left(v_1^*\right)^3 - \left(v_1^*\right)^2 = 0, \\ \delta : 3\sqrt{2}\left[\left(u_1^*\right)^2 v_1^* + \left(v_1^*\right)^2 v_2^*\right]^2 + \left(u_1^*\right)^2 - 2v_1^* v_2^* = 0.\end{gathered} \tag{8.38}$$

Equations (8.38) are invariant with respect to the transformations

$$u_1 = \delta^{0.5} u_1^*, \qquad u_2 = \delta^{1.5} u_2^*, \qquad v_1 = v_1^*, \qquad v_2 = \delta v_2^*, \tag{8.39}$$

and therefore one may return from the transformed variables to the initial ones:

$$\begin{gathered} \sqrt{2} v_1^3 - v_1^2 = 0, \\ 3\sqrt{2}(u_1^2 v_1 + v_1^2 v_2) + u_1^2 - 2 v_1 v_2 = 0. \end{gathered} \tag{8.40}$$

The invariants of transformations (8.39) read

$$z_1 = v_1, \qquad z_2 = \frac{u_2}{u_1^3}, \qquad z_3 = \frac{v_2}{u_1^2}, \tag{8.41}$$

and hence

$$v_1 = z_1, \qquad u_2 = u_1^3 z_2, \qquad v_2 = u_1^2 z_3. \tag{8.42}$$

Asymptotical estimations (8.20), (8.21), taking the form

$$u_1 \sim \delta^{0.5} < 1, \qquad u_2 \sim \delta^{1.5} < 1, \qquad v_1 \sim 1, \qquad v_2 \sim \delta < 1, \tag{8.43}$$

are automatically satisfied (see (8.42)) for small values of u_1 playing the role z_2 of a real small parameter.

Substituting (8.42) into (8.40), one achieves the following equations with respect to the invariants:

$$\begin{gathered} \sqrt{2} z_1 - 1 = 0, \\ 3\sqrt{2}(z_1 + z_1^2 z_3) + 1 - 2 z_1 z_3 = 0. \end{gathered} \tag{8.44}$$

The solution to the first equation is

$$z_1 = \frac{1}{\sqrt{2}} \Longrightarrow v_1 = \frac{1}{\sqrt{2}}, \tag{8.45}$$

and, on appropriate substitution with the obtained results, the second equations yields

$$\frac{1}{\sqrt{2}} z_3 = -4 \Longrightarrow z_3 = -4\sqrt{2}. \tag{8.46}$$

Quantity z_2 does not appear in (8.44) and remains undefined. Taking into account (8.42) and (8.19), one finally obtains

$$u = u_1 + u_1^3 z_2, \qquad v = \frac{1}{\sqrt{2}} - 4\sqrt{2} u_1^2. \tag{8.47}$$

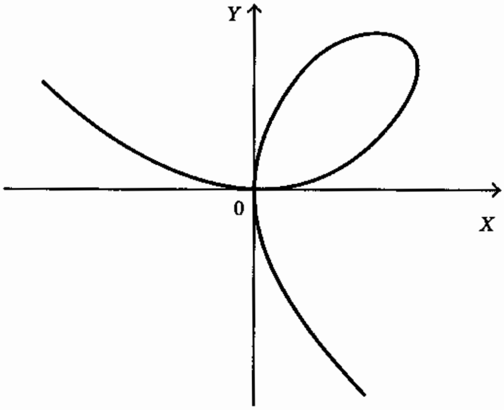

Figure 8.3. The loop shape with respect to the sum of two approximations, with all the best results taken into account.

Under the assumption $z_2 = 0$, u_1 coincides with u and for sufficiently small values of u_1 the formula for v describes the bending of the loop top. For nonzero values of z_2, (8.47) yields the curve equation in a parametric form with parameter u_1. Again, by a suitable choice of z_2, one may achieve a better approximation of the curve than that for $z_2 = 0$. The corresponding results are reported in Figure 8.2.

With the matching of all suitable results provided by the procedure of successive approximations, the exact curve is practically reached. Disadvantageously, it consists of as many as five parts, two of which are asymptotical, corresponding to large (in modulo) values of u, another two of which are in the vicinity of the origin of the coordinate system, and one part of which is in a neighborhood of the loop top. The corresponding summarized results are shown in Figure 8.3.

9. Improved variant of successive approximations

The variant of successive approximations described earlier is burdened with an essential drawback. Namely, some of the parameters (invariants) are sought via visual choice. In what follows, an improved variant of successive approximations is proposed.

The operations carried out in (8.6) keep the supplemented terms including δ^3, and result in

$$\begin{aligned}\delta(X_1^*)^3 + 3\delta^2(X_1^*)^2X_2^* + 3\delta^3X_1^*(X_2^*)^2 + \delta^2(Y_1^*)^3 + 3\delta^3(Y_1^*)^2Y_2^* \\ - \delta X_1^*Y_1^* - \delta^2X_1^*Y_2^* - \delta^2X_2^*Y_1^* - \delta^3X_2^*Y_2^* = 0.\end{aligned} \tag{9.1}$$

Splitting with respect to the same powers of δ in (8.8) yields additionally

$$\delta^3 : 3X_1^*(X_2^*)^2 + 3(Y_1^*)^2Y_2^* - X_2^*Y_2^* = 0. \tag{9.2}$$

The system composed of three equations (8.8) and (9.2) is invariant with respect to transformations (8.9). Hence, a transition to untransformed quantities can be realized, and two equations (8.10) as well as the equation

$$3X_1X_2^2+3Y_1^2Y_2-X_2Y_2=0 \tag{9.3}$$

are obtained.

The application of new (8.11) and (8.12) results in two equations (8.14) accompanied by

$$3z_2^2+3z_1^2z_3-z_2z_3=0. \tag{9.4}$$

Substituting the solution of the first equation of (8.14) into the second one and into (9.4), one obtains the following system of equations with respect to z_2 and z_3:

$$2z_2-z_3+1=0, \qquad 3z_2^2+3z_3-z_2z_3=0. \tag{9.5}$$

The reduction of z_3 gives the equation

$$z_2^2+5z_2+3=0. \tag{9.6}$$

Observe that the smaller (in modulo) root of this equation reads

$$z_2=-0.69722, \tag{9.7}$$

which is very close to the one found earlier through the visual choice of $z_2=-0.7$.

We apply an analogous approach to two procedures of successive approximations constructed on the basis of (6.2). In the first one, with the operations carried out in (8.24), the terms including not only δ^{-1} and δ^0 but also δ are kept:

$$\sqrt{2}\left\{3\left[\delta^{-1}(u_1^*)^2v_1^*+(u_1^*)^2v_2^*+2u_1^*u_2^*v_1^*+2\delta u_1^*u_2^*v_2^*+\delta(u_2^*)^2v_1^*\right]+(v_1^*)^3+3\delta(v_1^*)^2v_2^*\right\}$$
$$+\delta^{-1}(u_1^*)^2+2u_1^*u_2^*+\delta(u_2^*)^2-(v_1^*)^2-2\delta v_1^*v_2^*=0. \tag{9.8}$$

Splitting with respect to δ yields, apart from the two equations (8.26), the equation

$$\delta:3\sqrt{2}\left[2u_1^*u_2^*v_2^*+(u_2^*)^2v_1^*+(v_1^*)^2v_2^*\right]+(u_2^*)^2-2v_1^*v_2^*=0. \tag{9.9}$$

This equation is invariant with respect to transformations (8.28). Hence, one may return to the initial variables

$$3\sqrt{2}(2u_1u_2v_2+u_2^2v_1+v_1^2v_2)+u_2^2-2v_1v_2=0. \tag{9.10}$$

On applying invariants (8.29) besides (8.32) the third equation is obtained:

$$3\sqrt{2}(2z_2z_3+z_2^2z_1+z_1^2z_3)+z_2^2-2z_1z_3=0. \tag{9.11}$$

Substituting here the solution (8.33) of the first equation of (8.32), one obtains

$$12z_2z_3 + z_3 = 0 \Longrightarrow z_2 = -\frac{1}{12} = -0.083333. \tag{9.12}$$

Note that also in this case the obtained value z_2 is close to the one obtained earlier ($z_2 = -0.078$) with the help of a proper choice.

In (8.36) the terms δ^0, δ, and δ^2 are kept:

$$\begin{aligned}\sqrt{2}\Big\{3\Big[\delta(u_1^*)^2v_1^* + \delta^2(u_1^*)^2v_2^* + 2\delta^2u_1^*u_2^*v_1^*\Big] + (v_1^*)^3 + 3\delta(v_1^*)^2v_2^* + 3\delta^2v_1^*(v_2^*)^2\Big\}\\ + \delta(u_1^*)^2 + 2\delta^2u_1^*u_2^* - (v_1^*)^2 - 2\delta v_1^*v_2^* - \delta^2(v_2^*)^2 = 0.\end{aligned} \tag{9.13}$$

Splitting with respect to F yields two equations (8.38) and the equation

$$\delta^2 : 3\sqrt{2}\Big[(u_1^*)^2v_2^* + 2u_1^*u_2^*v_1^* + v_1^*(v_2^*)^2\Big] + 2u_1^*u_2^* - (v_2^*)^2 = 0. \tag{9.14}$$

It is transformed, using (8.39), to the form

$$3\sqrt{2}(u_1^2v_2 + 2u_1u_2v_1 + v_1v_2^2) + 2u_1u_2 - v_2^2 = 0, \tag{9.15}$$

and then (8.41) is received in the form

$$3\sqrt{2}(z_3 + 2z_1z_2 + z_1z_3^2) + 2z_2 - z_3^2 = 0. \tag{9.16}$$

Substituting here the solution of the first of equations (8.44), that is, (8.45), one obtains

$$8z_2 + 3\sqrt{2}z_3 + 2z_3^2 = 0. \tag{9.17}$$

Taking z_2 in (8.46) into account, one obtains

$$8z_2 + 40 = 0 \Longrightarrow z_2 = -5. \tag{9.18}$$

To conclude, again a value close to that obtained via the visual choice is found.

10. Full variant of successive approximations

In order to enable deeper apprehension of all peculiarities of the successive approximations procedure, its full (general) variant will be rigorously considered.

The quantities occurring in (4.4) are expressed in the form of the series

$$x = \sum_{i=1}^{\infty} x_i, \qquad y = \sum_{i=1}^{\infty} y_i. \tag{10.1}$$

The following transformations are introduced:

$$x_i = \delta^{\alpha_1+i-1}x_i^*, \quad y_i = \delta^{\alpha_2+i-1}y_i^* \quad (i = 1,2,\ldots), \tag{10.2}$$

and it is required that they yield satisfaction to the following relations:

$$x_i^* \sim 1, \quad y_i^* \sim 1 \quad (i = 1,2,\ldots). \tag{10.3}$$

Substituting (10.2) into (10.1), one arrives at

$$x = \delta^{\alpha_1} \sum_{i=1}^{\infty} x_i^* \delta^{i-1}, \qquad y = \delta^{\alpha_2} \sum_{i=1}^{\infty} y_i^* \delta^{i-1}. \tag{10.4}$$

Consequently, the series composed of the terms with the increasing exponents of small parameter δ are obtained. If conditions (10.3) are satisfied, then the necessary conditions of the series convergence are satisfied as well.

Substitution of (10.4) into (4.4) yields

$$\begin{aligned}\delta^{3\alpha_1} \sum_{i=1}^{\infty} x_i^* \delta^{i-1} \sum_{j=1}^{\infty} x_j^* \delta^{j-1} \sum_{k=1}^{\infty} x_k^* \delta^{k-1} + \delta^{3\alpha_2} \sum_{i=1}^{\infty} y_i^* \delta^{i-1} \sum_{j=1}^{\infty} y_j^* \delta^{j-1} \sum_{k=1}^{\infty} y_k^* \delta^{k-1} \\ - \delta^{\alpha_1+\alpha_2} \sum_{i=1}^{\infty} x_i^* \delta^{i-1} \sum_{j=1}^{\infty} y_j^* \delta^{j-1} = 0.\end{aligned} \tag{10.5}$$

For the purpose of further analysis, (10.5) should be simplified with respect to the same powers of δ. However, this splitting is carried out for fixed values of α_1, α_2. We take the values of α_1, α_2 considered in the previous sections.

(iii) $\alpha_1 = 1/3$, $\alpha_2 = 2/3$. The values of α_1, α_2 correspond to the following form of (10.5):

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_i^* x_j^* x_k^* \delta^{i+j+k-3} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} y_i^* y_j^* y_k^* \delta^{i+j+k-2} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i^* y_j^* \delta^{i+j-2} = 0. \tag{10.6}$$

The splitting yields

$$\sum_{i+j+k=p+2} x_i^* x_j^* x_k^* + \sum_{i+j+k=p+1} y_i^* y_j^* y_k^* - \sum_{i+j=p+1} x_i^* y_j^* = 0 \quad (p = 1,2,\ldots). \tag{10.7}$$

In the above, in each term the summation is carried out with respect to all values of indices i, j, k up to $p+1$ or $p+2$.

Each of equations (10.7) is obtained via extraction, from (10.6), of the terms obtained as a result of transformations (10.2). Consequently, (10.7) are invariant with respect to transformations (10.2) for chosen values of α_1, α_2, that is, the following transformations are applied:

$$x_i = \delta^{i-2/3} x_i^*, \quad y_i = \delta^{i-1/3} y_i^* \quad (i = 1,2,\ldots). \tag{10.8}$$

This variance makes it possible to return in (10.7) from the transformed quantities to the initial ones:

$$\sum_{i+j+k=p+2} x_i x_j x_k + \sum_{i+j+k=p+1} y_i y_j y_k - \sum_{i+j=p+1} x_i y_j = 0 \quad (p = 1,2,\ldots). \tag{10.9}$$

Therefore, the solution to (4.4) is cast in the form of series (10.1) with the terms satisfying (10.9). Observe that neither the series nor the equations include a formally introduced small parameter δ.

Owing to (10.3), the following asymptotical estimations hold:

$$x_i \sim \delta^{i-2/3}, \quad y_i \sim \delta^{i-1/3} \quad (i = 1,2,\ldots) \tag{10.10}$$

and, in particular,

$$x_1 \sim \delta^{1/3}. \tag{10.11}$$

Therefore, the role of a real (not formal) small parameter introduced here is played by the following quantity:

$$\delta = x_1^3. \tag{10.12}$$

The invariants of transformations (10.8) are as follows:

$$I_{xi} = x_1^{2-3i} x_i, \quad I_{yi} = x_1^{1-3i} y_i \quad (I_{x1} = 1;\ i = 1,2,\ldots). \tag{10.13}$$

Hence

$$x_i = x_1^{3i-2} I_{xi}, \quad y_i = x_1^{3i-1} I_{yi} \quad (i = 1,2,\ldots). \tag{10.14}$$

For constant values of invariants I_{xi}, I_{yi} and the chosen small parameter (10.12), relations (10.10) hold automatically. Thus, the infinite choice of estimations (10.10) follows from the estimation

$$x_1^3 < 1. \tag{10.15}$$

That means that giving sufficiently small values x_1, one can verify all asymptotical estimations. The series of the form (10.1) are the series along the increasing exponents of quantity x_1^3.

Substituting (10.14) into (10.9), the following equations with respect to invariants are obtained:

$$\sum_{i+j+k=p+2} I_{xi} I_{xj} I_{xk} + \sum_{i+j+k=p+1} I_{yi} I_{yj} I_{yk} - \sum_{i+j=p+1} I_{xi} I_{yj} = 0 \quad (p = 1,2,\ldots). \tag{10.16}$$

We give some examples of the first approximations:

$$p = 1 : I_{x1}^3 - I_{x1} I_{y1} = 0. \tag{10.17}$$

It has been assumed that for $p = 1$ there are no terms of the second sum in (10.16) corresponding to the equality $i + j + k = p + 1$ for $i \geq 1$, $j \geq 1$, $k \geq 1$. Furthermore,

$$\begin{aligned}
&p = 2 : 3I_{x1}^2 I_{x2} + I_{y1}^3 - I_{x1} I_{y2} - I_{x2} I_{y1} = 0,\\
&p = 3 : 3I_{x1}^2 I_{x3} + 3I_{x1} I_{x2}^2 + 3I_{y1}^2 I_{y2} - I_{x1} I_{y3} - I_{x2} I_{y2} - I_{x3} I_{y1} = 0,\\
&p = 4 : 3I_{x1}^2 I_{x4} + 6I_{x1} I_{x2} I_{x3} + I_{x2}^3 + 3I_{y1}^2 I_{y3} + 3I_{y1} I_{y2}^2 - I_{x1} I_{y4} - I_{x2} I_{y3} - I_{x3} I_{y2} - I_{x4} I_{y1} = 0.
\end{aligned} \tag{10.18}$$

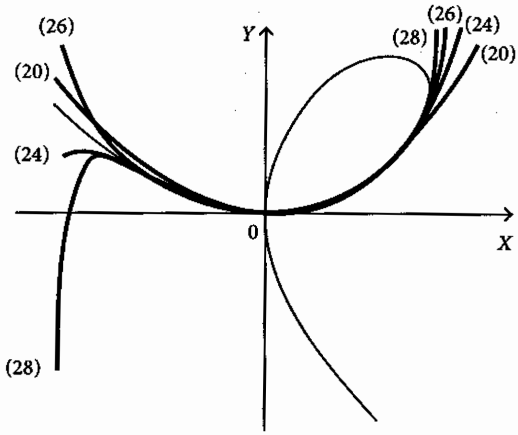

Figure 10.1. Results of application of many approximations using the standard approach.

A solution to the approximating equation of first order (10.17) is easy to find. In accordance with (10.13) and (10.14),

$$I_{x1} = 1, \qquad I_{y1} = 1 \Longrightarrow y_1 = x_1^2. \tag{10.19}$$

If only the first approximation is taken, then

$$x = x_1, \quad y = y_1, \quad y = x^2. \tag{10.20}$$

This parabolic dependence has already been considered earlier; the corresponding graph is shown in Figure 10.1.

Substituting $I_{x1} = 1$, $I_{y1} = 1$ into the second-order approximation equation ((10.18), $p = 2$), one obtains

$$2I_{x2} - I_{y2} + 1 = 0, \tag{10.21}$$

which is an equation with two unknowns. To solve this equation as well as the other equations governing successive approximations obtained eventually, two different approaches will be considered.

The first approach. Since, in the first approximation, the dependence $y = y(x)$ (10.20) is found, the successive approximations improve its exactness. Quantity x, playing the role of the argument, can be improved, that is,

$$x_i = 0, \quad I_{xi} = 0 \quad (i \geq 2). \tag{10.22}$$

Then (10.21) yields

$$I_{y2} = 1 \Longrightarrow y_2 = x_1^5. \tag{10.23}$$

Owing to the results of the two approximations, one obtains

$$x = x_1, \quad y = y_1 + y_2 = x^2 + x^5. \tag{10.24}$$

The corresponding graph, together with graph (10.20), is shown in Figure 10.1.

Substituting $I_{x1} = 1$, $I_{y1} = 1$, $I_{x2} = 0$, $I_{y2} = 1$, $I_{x3} = 0$ into the equation of the third approximation ((10.18), $p = 3$), one obtains

$$3 - I_{y3} = 0 \Longrightarrow I_{y3} = 3 \Longrightarrow y_3 = 3x_1^8. \tag{10.25}$$

The results of the three approximations give

$$x = x_1, \quad y = y_1 + y_2 + y_3 = x^2 + x^5 + 3x^8. \tag{10.26}$$

The corresponding graph is shown in Figure 10.1.

With the results of the first three approximations taken into account, the equation of the fourth approximation ((10.18), $p = 4$) takes the form

$$12 - I_{y4} = 0 \Longrightarrow I_{y4} = 12, \quad y_4 = 12x_1^{11}. \tag{10.27}$$

The common results of the four approximations are as follows:

$$x = x_1, \quad y = y_1 + y_2 + y_3 + y_4 = x^2 + x^5 + 3x^8 + 12x^{11} \tag{10.28}$$

and they are shown graphically in Figure 10.1.

Although, with respect to the proposed approach, the procedures of the successive approximations can be extended, the results obtained so far prove sufficient for analysis. Comparing the graphs (cf. Figure 10.1) obtained using one, two, three, and four approximations, one can observe that in fact they coincide in a certain neighborhood of the coordinate system origin and become significantly different with the increase of the distance from the origin. The reason for this lies in the fact that the series for quantity (10.1) has a bounded convergence space, inside which the additional new series terms improve slightly the accuracy, and at the same time outside the boundaries, the computational results obtained via various numbers of approximations (i.e., the number of series terms) differ strongly from one another.

It should be emphasized that here the theorem of the series convergence is not formulated and the series radius of convergence is not computed. An image of the bounded space of the series convergence is obtained somehow in an experimental manner on the basis of analysis of Figure 10.1. This figure exhibits the reasons for boundness of the convergence space and even shows its dimension, as the initial curve is given in an implicit form by (4.4) and has the shape of a loop in the neighborhood of the coordinate system origin.

Moving away from the origin to the right along axis x, a point of loop rotation is approached. The function $y = y(x)$ cannot describe this rotation since the loop gives two values of y, while, by assumption, the function $y = y(x)$ is unique. Therefore, the loop rotation point is a real boundary of the series convergence space for the function

$y = y(x)$. The distance from the origin to the rotation point along axis x is equal to the radius of the series convergence. Since the convergence space is symmetric with respect to the coordinate system origin, therefore to the left from the origin, at a distance larger than the radius, the series is divergent.

To conclude, the first approach gives good results in the neighborhood of the origin bounded by the loop dimension, but it is unsuitable for the curve description outside this space. This corresponds fully with the asymptotical estimations on the basis of which the applied procedure of successive approximations was constructed (smallness of x).

The second approach. It is possible to obtain more accurate results than those achieved in the first approach. We turn once more to the approach discussed in the previous section. Developing both y and x into series, in the second approximation one obtains two additional (and sought after) quantities x_2, y_2 as well as two corresponding new invariants I_{x2}, I_{y2}, but only one new equation ((10.18), $p = 2$).

In the above, the considerations are reduced to the following observation: part of the third approximation, in which terms I_{x3}, I_{y3} have been neglected, is added to the second approximated equation. Taking into account that $I_{x1} = 1$, $I_{y1} = 1$, the following system is obtained:

$$2I_{x2} - I_{y2} + 1 = 0, \qquad 3I_{x2}^2 + 3I_{y2} - I_{x2}I_{y2} = 0. \tag{10.29}$$

The first equation in (10.29) yields

$$I_{y2} = 2I_{x2} + 1, \tag{10.30}$$

which, substituted into the second equation, gives the following squared equation:

$$I_{x2}^2 + 5I_{x2} + 3 = 0. \tag{10.31}$$

From the two roots of this equation, the smaller one (in modulo) is taken, that is, the one nearest to the zero value of I_{x2}, and corresponding to the first approach:

$$I_{x2} = \frac{\sqrt{13} - 5}{2} = -0.69722. \tag{10.32}$$

The corresponding value I_{y2} is found from (10.30). Owing to (10.14) and (10.1), one obtains

$$x = x_1 + x_2 = x_1 + I_{x2}x_1^4, \qquad y = y_1 + y_2 = x_1^2 + I_{y2}x_1^5. \tag{10.33}$$

The results are graphically presented in Figure 10.2 together with the parabola corresponding to the first approximation.

We construct the third approximation. With three terms in series (10.1) being kept, again two additional sought after quantities x_3, y_3 together with corresponding invariants I_{x3}, I_{y3} are obtained. The corresponding equation of the third approximation ((10.18), $p = 3$) was already applied in the second approximation in the form of (10.29).

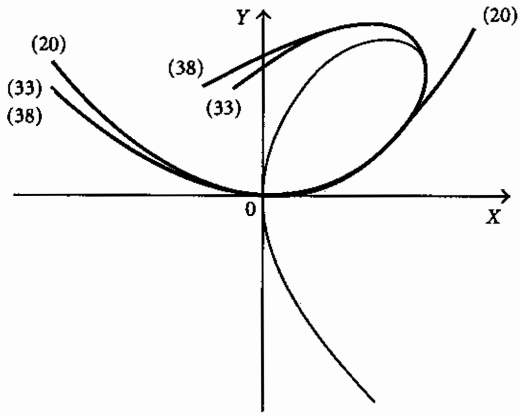

Figure 10.2. Results of one, two, and three approximations within the improved approach.

The part of the third approximation equation that remains reads

$$3I_{x1}^2 I_{x3} - I_{x1} I_{y3} - I_{x3} I_{y1} = 0. \tag{10.34}$$

In consequence, the equation of the third approximation has been split into two parts, each of which is separately equal to zero, that is, two equations are obtained instead of one.

In the equation of the fourth approximation ((10.18), $p = 4$), the terms with I_{x4}, I_{y4} are neglected to yield

$$6I_{x1} I_{x2} I_{x3} + I_{x2}^3 + 3I_{y1}^2 I_{y3} + 3I_{y1} I_{y2}^2 - I_{x2} I_{y3} - I_{x3} I_{y2} = 0. \tag{10.35}$$

Finally, taking the results of the first two approximations into account, the following system of two equations is obtained:

$$\begin{gathered} 2I_{x3} - I_{y3} = 0, \\ (6I_{x2} - I_{y2}) I_{x3} + (3 - I_{x2}) I_{y3} + I_{x2}^3 + 3I_{y2}^2 = 0. \end{gathered} \tag{10.36}$$

Solving the given system of linear equations, one gets

$$I_{x3} = \frac{I_{x2}^3 + 3I_{y2}^2}{I_{y2} - 4I_{x2} - 6}, \qquad I_{y3} = 2I_{x3}. \tag{10.37}$$

The result of the three approximations is as follows:

$$\begin{gathered} x = x_1 + x_2 + x_3 = x_1 + I_{x2} x_1^4 + I_{x3} x_1^7, \\ y = y_1 + y_2 + y_3 = x_1^2 + I_{y2} x_1^5 + I_{y3} x_1^8. \end{gathered} \tag{10.38}$$

The corresponding graphs are shown in Figure 10.2, which will be further analyzed. It is clearly seen that an increase in the number of approximations does not practically yield any improvement in the accuracy. In other words, from the point of view of accuracy, the

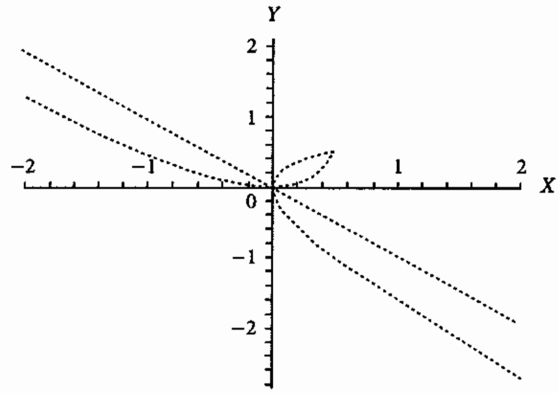

Figure 11.1. The solution obtained applying two-point Padé approximants.

second approach to the successive approximation procedure is the same as the first one. The most significant result—a possibility of bending of the approximating curve outside the loop boundary—is already sufficiently well reported in the second approximation; further approximations do not offer anything of actual importance.

However, there exist problems when the third and further approximations are required. In what follows, the importance of the results obtained through the second approach will be addressed briefly. Beginning from the third approximation equation, each of the occurring equations is split into two. The first of them often includes all terms related to previous approximations whereas the second one contains only the quantities related to a given approximation; moreover, each part separately is equal to zero. The number of equations becomes twice larger, and for each approximation there appears a possibility of finding two of the occurring quantities in the form I_{xi}, I_{yi}.

Within the problem under discussion, there is no need to consider other variants of the successive approximation procedure, since, with two approximations taken into account, the problem becomes reduced to the one considered in Section 9. The applied method, being a match of the asymptotical and group analyses, divides the curve into characteristic parts. Each of the parts is associated with a sufficiently exact description. As a result, one complex curve is divided into (a series of) three simpler ones, each of which, while possessing its own characteristic properties, describes a certain initial curve.

In conclusion, it should be pointed out that making use of a relatively simple problem, as an example, an effective method for the complex system investigation, including decomposition and interactions of separate parts, has been suggested and discussed.

11. Using two-point Padé approximants

It is worth noticing that an application of two-point Padé approximation yields very good results in this case. Matching the solutions $y = x = 0.5$, $y = -x$, $y = x^2$, one obtains the branch $y = x^2/(1-x)$, which is shown in Figure 11.1. By changing $x \Leftrightarrow y$, the second solution branch is attained. However, vicinity of point (0.5,0.5) requires additional investigation.

References

[1] I. V. Andrianov and J. Awrejcewicz, *New trends in asymptotic approaches: summation and interpolation methods*, Applied Mechanics Reviews **54** (2001), no. 1, 69–92.

[2] J. Awrejcewicz and V. A. Krys'ko, *Nonclassical Thermoelastic Problems in Nonlinear Dynamics of Shells*, Scientific Computation, Springer-Verlag, Berlin, 2003.

[3] J. Awrejcewicz, G. Kudra, and C.-H. Lamarque, *Investigation of triple pendulum with impacts*, to appear in IJBC.

[4] ———, *Nonlinear dynamics of triple pendulum with impacts*, J. Tech. Phys. **43** (2002), no. 2, 97–112.

[5] ———, *Dynamics investigation of three coupled rods with a horizontal barrier*, Meccanica **38** (2003), no. 6, 687–698.

[6] A. D. Bruno, *Local Methods in Nonlinear Differential Equations*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1989.

[7] ———, *Power Geometry in Algebraic and Differential Equations*, North-Holland Mathematical Library, vol. 57, North-Holland Publishing, Amsterdam, 2000.

[8] ———, *Self-similar solutions and power geometry*, Russian Math. Surveys **55** (2000), no. 1, 1–42.

[9] G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, *Linear functionals of idempotent spaces: an algebraic approach*, Dokl. Math. **58** (1998), no. 3, 389–391.

[10] V. P. Maslov and A. D. Shamrovskii (eds.), *Idempotent Analysis*, Advances in Soviet Mathematics, vol. 13, American Mathematical Society, Rhode Island, 1992.

[11] A. D. Shamrovskii, *Asymptotic integration of static equation of the theory of elasticity in Cartesian coordinates with automated search of integration parameters*, PMM J. Appl. Math. Mech. **43** (1979), no. 5, 925–934.

[12] ———, *Asymptotic Group Analysis of the Theory of Elasticity Differential Equations*, Zaporoshye State Engineering Academy, Zaporoshye, 1997.

A. D. Shamrovskii: Zaporoshye State Engineering Academy, Department of Software for Automatic Systems, pr. Lenina 226, Zaporoshye, UA-69006, Ukraine

E-mail address: shamr@comint.net

I. V. Andrianov: Institut für Allgemeine Mechanik der RWTH Aachen, Templergraben 64, 52062 Aachen, Germany

E-mail address: igor-andrianov@hotmail.com

J. Awrejcewicz: Department of Automatics and Biomechanics, Technical University of Łódź, 90-924 Łódź, Poland

E-mail address: awrejcew@p.lodz.pl