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How to predict stick-slip chaos in R^4

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Abstract

Chaotic dynamics in four-dimensional self-excited systems with Coulomb-like friction is predicted analytically. The obtained Melnikov's function yields thresholds for two types stick-slip onsets of chaos.

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After occurence of first reports concerning deterministic chaos (including pioneering works of Lorenz, Toda and Rössler), a naturally motivated attempt to detect onset of chaos in possibly simplest dynamical systems appeared. There exists a series of papers initiated by Gottlieb [5] focused on existence and detection of simple jerk equations exhibiting deterministic chaos [4,7,10]. Examples of dissipative jerk equations with only one quadratic [10] and only one cubic [7] nonlinearities have been reported. A systematic numerical examination of third order ordinary differential equation with nonlinearity form $x\dot{x}^2$ carried out by Malasoma [8] yields deterministic chaos. In all mentioned cases the exhaustive numerical investigations to detect deterministic chaos have been carried out. However, one may give remark that it is rather impossible to scan numerically all possible parameter ranges of an investigated equation. Note that difficulties increase while analysing high-dimensional systems. Therefore, we have decided to choose an analytical approach to predict an intersection of stable and unstable manifolds associated with a saddle in our four-dimensional autonomous mechanical system governed by the following non-dimensional

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equations

$$\begin{cases} \dot{x} = u, \\ \dot{u} = x - x^3 + f_\xi(x, y) - \varepsilon_1 T_1'(u - w), \\ \dot{y} = v, \\ \dot{v} = y - y^3 - f_\xi(x, y) - \varepsilon_2 T_2'(v - w), \end{cases} \tag{1}$$

where

$$T_i'(X - w) = T_{i0}'(X - w') - B_{i1}'(X - w') + B_{i2}'(X - w')^3, \tag{2}$$

$$f_\xi(x, y) = \xi(x - y) - \xi(x - y)^3. \tag{3}$$

The following relations hold between dimensional and non-dimensional equations

$$T_{i0}' = T_{i0}\sqrt{\frac{\tilde{k}}{k^3}}, \qquad B_{i1}' = \frac{B_{i1}}{\sqrt{mk}}, \qquad B_{i2}' = \frac{k^2 B_{i2}}{\sqrt{m^3\tilde{k}^3}}, \qquad w' = w\sqrt{\frac{\tilde{k}m}{k^2}}, \quad i = 1, 2, \tag{4}$$

where w is the tape velocity,[1] and B_{i1}, B_{i2}, T_{i0} are the friction coefficients. For $\varepsilon_1 = \varepsilon_2 = 0$ we obtain the unperturbed system, which possesses a homoclinic orbit of a hyperbolic point $a = 0$. The investigated system belongs to classical ones, since it approximates many real engineering high-dimensional self-excited objects (see monograph [2] and references therein).

This Letter is mainly motivated by two references, i.e., [1,6]. In the first one [6] an extension to R^n of the original Melnikov's approach [9] is proposed, which is applied in our analysis and this method is further referred as the Melnikov–Gruendler approach. In the second reference [1] for the first time an analytical prediction for both smooth and stick-slip onset of deterministic chaos in R^2 is reported. Note that in the latter case it is possible to get a critical state of autonomous one degree-of-freedom mechanical system with Coulomb-like friction cast by the Melnikov's function. In other words, an infinitely small external harmonic perturbation convert immediately the system into chaotic state. In what follows we match ideas given in two cited references to predict chaotic threshold in two self-excited coupled oscillators with friction analytically using the Melnikov–Gruendler technique.

Our research includes three fundamental steps. First, a linearization along a homoclinic orbit is applied and the fundamental solutions are formulated. Let us denote by $\gamma(t)$ the homoclinic orbit of the point a. It has the following form

$$\gamma(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \\ q(t) \\ \dot{q}(t) \end{pmatrix}, \quad \text{where } q(t) = \sqrt{2}\,\mathrm{sech}(t). \tag{5}$$

The linearized system of the unperturbed equations (1) in vicinity of the homoclinic orbit $\gamma(t)$ reads

$$\begin{cases} \dot{\psi}_1 = \psi_2, \\ \dot{\psi}_2 = (1 + \xi - 3q^2(t))\psi_1 - \xi\psi_3, \\ \dot{\psi}_3 = \psi_4, \\ \dot{\psi}_4 = (1 + \xi - 3q^2(t))\psi_3 - \xi\psi_1. \end{cases} \tag{6}$$

[1] See Fig. 1 in [1].

We seek the fundamental solution of the above equations. It can be shown that $\psi^{(4)} = \dot{\gamma}(t)$ is a solution to Eqs. (6). Next, applying the following substitution: $\dot{q}(t) \to r(t)\dot{q}(t)$ and substituting into (6), one gets the second solution

$$\psi^{(2)} = \begin{pmatrix} r(t)\dot{q}(t) \\ \dot{r}(t)\dot{q}(t) + r(t)\ddot{q}(t) \\ r(t)\dot{q}(t) \\ \dot{r}(t)\dot{q}(t) + r(t)\ddot{q}(t) \end{pmatrix}, \tag{7}$$

where

$$r(t) = \frac{3}{4}C_1 t - \frac{1}{2}C_1 \operatorname{ctgh}(t) + \frac{1}{8}C_1 \sinh(2t) + C_2. \tag{8}$$

Two remaining solutions are the following (see [3]):

$$\psi^{(1)} = \begin{pmatrix} \psi^{11} \\ \psi^{12} \\ \psi^{13} \\ \psi^{14} \end{pmatrix}, \qquad \psi^{(3)} = \begin{pmatrix} \psi^{31} \\ \psi^{32} \\ \psi^{33} \\ \psi^{34} \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
\psi^{11} &= \frac{e^{\delta t}}{2}\big(-3\delta \operatorname{tgh}(t)\big(1+\operatorname{tgh}(t)\big) + (\delta+2)(\delta+1) - 3(\delta+1)\operatorname{sech}^2(t)\big),\\
\psi^{12} &= \frac{e^{\delta t}}{2}\Big(-3\delta\big(\big(1+2\operatorname{tgh}(t)\big)\operatorname{sech}^2(t) + \delta \operatorname{tgh}(t)\big(1+\operatorname{tgh}(t)\big)\big)\Big)\\
&\quad + \delta(\delta+2)(\delta+1) - 3(\delta+1)\big(\delta - 2\operatorname{tgh}(t)\big)\operatorname{sech}^2(t),\\
\psi^{13} &= -\psi^{11}, \qquad \psi^{14} = -\psi^{12},\\
\psi^{31} &= \frac{e^{-\delta t}}{2}\big(3\delta \operatorname{tgh}(t)\big(1+\operatorname{tgh}(t)\big) + (\delta-2)(\delta-1) + 3(\delta-1)\operatorname{sech}^2(t)\big),\\
\psi^{32} &= \frac{3\delta e^{-\delta t}}{2}\big(\big(1+2\operatorname{tgh}(t)\big)\operatorname{sech}^2(t) - \delta \operatorname{tgh}(t)\big(1+\operatorname{tgh}(t)\big)\big)\\
&\quad - \frac{e^{-\delta t}}{2}\big(\delta(\delta-2)(\delta-1) + 3(1-\delta)\big(\delta - 2\operatorname{tgh}(t)\big)\operatorname{sech}^2(t)\big),\\
\psi^{33} &= -\psi^{31}, \qquad \psi^{34} = -\psi^{32}.
\end{aligned}$$

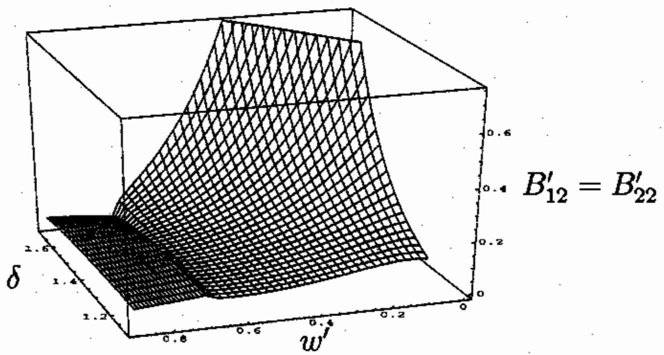

Fig. 1. Critical surface in three-dimensional parameter space.

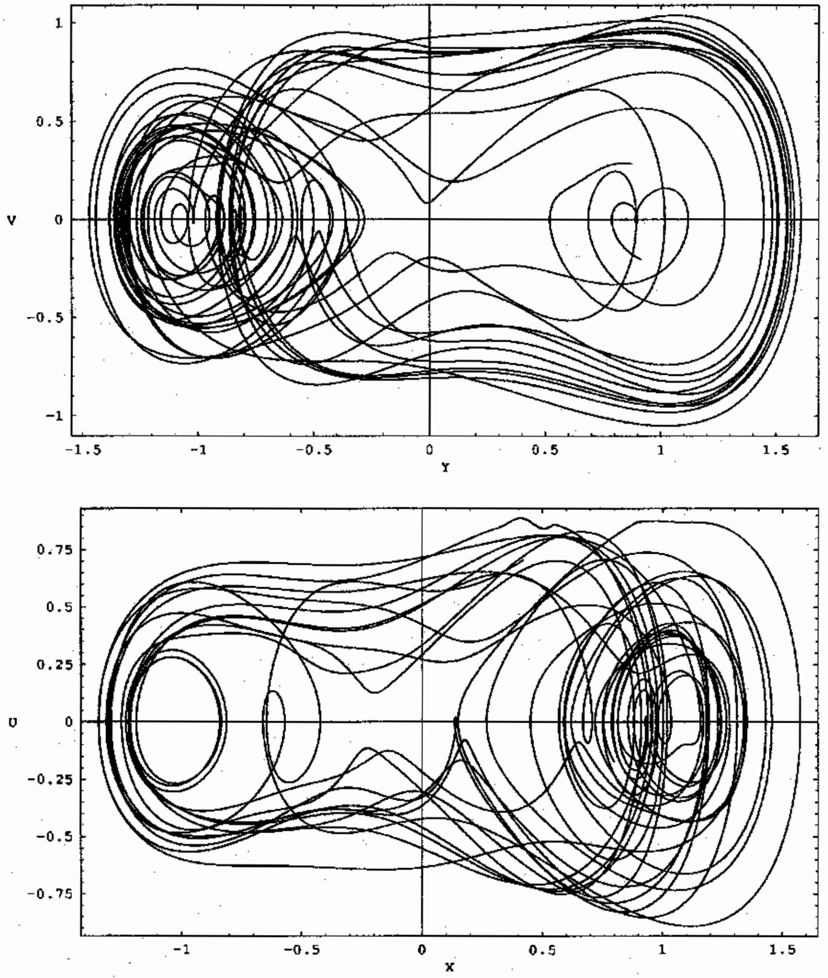

Fig. 2. Phase portraits of (a) type 1 ($w' = 0.87$, $B'_{22} = B'_{12} = 0.09$) and of (b) type 2 ($w' = 0.6$, $B'_{22} = B'_{12} = 1.7$).

The second step ineludes computation of the Melnikov–Gruendler function (see [3,6] for details), which is the following

$$0 = -\frac{\delta(\delta-2)(\delta+1)(\delta+2)}{8}(V_{11}+V_{21}) - \frac{\delta(5\delta^3+5\delta^2-20\delta+52)}{8}(V_{12}+V_{22}) - \frac{\delta(5\delta^3+5\delta^2-20\delta-56)}{4}(V_{13}+V_{23}) + 12\big(1+2\delta^2\big)(V_{14}+V_{24}), \tag{10}$$

where $\delta = \sqrt{1+2\xi}$ and

$$V_{i1} = \frac{128\sqrt{2}}{21}B'_{i1} - \frac{64\sqrt{2}}{3465}\big(990w'^2+103\big)B'_{i2} + \frac{T'_{i0}}{3}\theta\left(\frac{1}{\sqrt{2}} - w'\right)\Big[\big(79-3w'^2\big)\big(\operatorname{sech}(t_1) - \operatorname{sech}(t_2)\big) - 17\sqrt{1-2w'^2}\,\big(\operatorname{sech}(t_1) + \operatorname{sech}(t_2)\big)\Big], \tag{11}$$

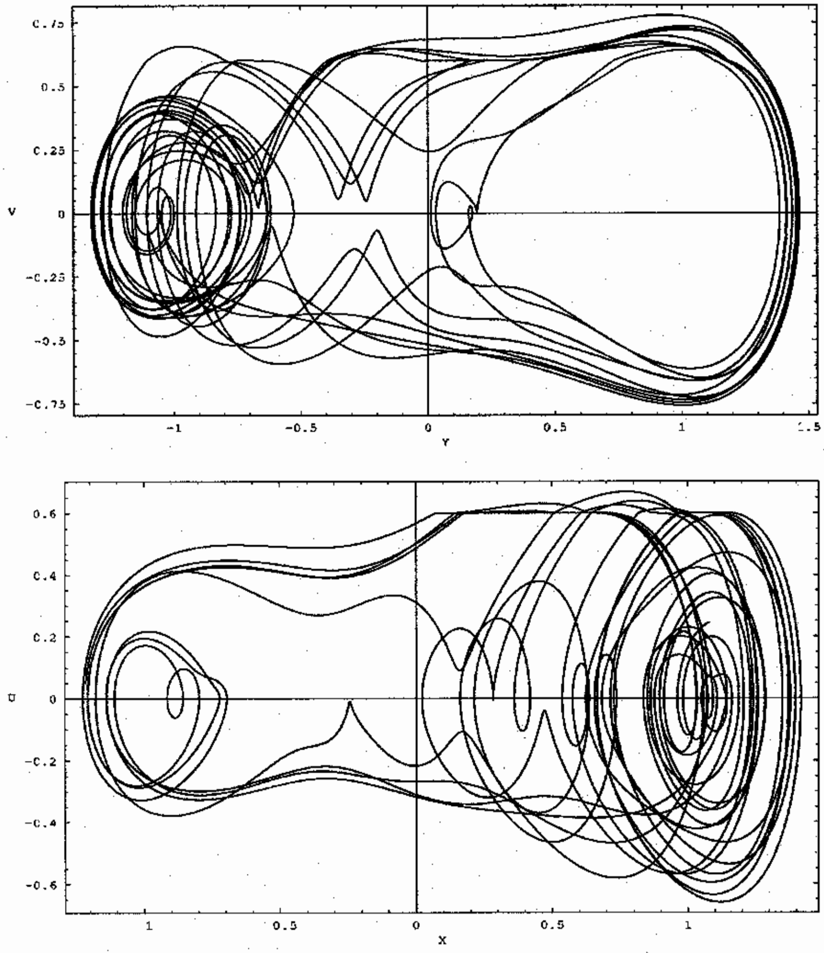

Fig. 2. (*continued*)

$$V_{i2} = \frac{64\sqrt{2}}{105}B'_{i1} - \frac{64\sqrt{2}}{3465}\left(99w'^2+13\right)B'_{i2} + \frac{2T'_{i0}}{3}\theta\left(\frac{1}{\sqrt{2}} - w'\right)\left[\frac{11}{5}\left(\operatorname{sech}^5(t_1) - \operatorname{sech}^5(t_2)\right) + 4\left(\operatorname{sech}^7(t_1) - \operatorname{sech}^7(t_2)\right)\right], \tag{12}$$

$$V_{i3} = \frac{16\sqrt{2}}{105}B'_{i1} - \frac{16\sqrt{2}}{1155}\left(33w'^2+4\right)B'_{i2} + \frac{2T'_{i0}}{5}\theta\left(\frac{1}{\sqrt{2}} - w'\right)\left[\operatorname{sech}^5(t_1) - \operatorname{sech}^5(t_2)\right], \tag{13}$$

$$V_{i4} = -\frac{\pi}{16}T'_{i0} + \frac{\pi w'}{16}B'_{i1} - \frac{\pi w'}{128}\left(8w'^2+9\right)B'_{i2} + \frac{T'_{i0}}{4}\theta\left(\frac{1}{\sqrt{2}} - w'\right)\left[\operatorname{arctg}\left(\operatorname{tgh}\left(\frac{t_2}{2}\right)\right) - \operatorname{arctg}\left(\operatorname{tgh}\left(\frac{t_1}{2}\right)\right)\right]. \tag{14}$$

In the above $\theta(x)$ is the Heaviside's function and

$$t_1 = \ln\left(\frac{1}{w'}\sqrt{1+\sqrt{1-2w'^2}}\left(1 - \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1-2w'^2}}\right)\right),$$

$$t_2 = \ln\left(\frac{1}{w'}\sqrt{1-\sqrt{1-2w'^2}}\left(1-\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-2w'^2}}\right)\right).$$

The third step consists of a graphical representation of the Melnikov–Gruendler function in three dimensional parameter space (see Fig. 1), where we set $B'_{11} = T'_{10} = 0.2$, $B'_{21} = 0.3$, $T'_{20} = 0.1$.

The surface consists of two subsurfaces which correspond to stick-slip chaos of type 1 and stick-slip chaos of type 2. Two sets of parameters (two points) correspond to type 1 (Fig. 2a) and type 2 (Fig. 2b) stick-slip chaos. Type 1 stick-slip motion corresponds to the case where a sign of w' is not changed. The reported numerical computations have been carried out using the Runge–Kutta method of order 8 with step 0.001.

In conclusion we have applied the Melnikov–Gruendler technique to predict two types of non-smooth chaotic behaviour in R^4 of two self-excited coupled oscillators with friction. The obtained analytically chaotic threshold has been verified numerically showing surprisingly good agreement. Furthermore, our results give good prognosis for investigations of chaos occurence in wide class of continuous and discontinuous coupled oscillators.

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References

[1] J. Awrejcewicz, M.M. Holicke, Int. J. Bifur. Chaos 9 (3) (1999) 505.
[2] J. Awrejcewicz, C.-H. Lamarque, Bifurcation and Chaos in Nonsmooth Dynamical Systems, World Scientific, Singapore, 2003.
[3] J. Awrejcewicz, D. Sendkowski, Construction of homoclinic orbits and stick-slip chaos prediction in two degrees-of-freedom mechanical system with friction (submitted).
[4] R. Eichhorn, S.J. Linz, P. Hänggi, Phys. Rev. E 58 (1998) 7151.
[5] H.P.W. Gottlieb, Am. J. Phys. 64 (1996) R525.
[6] J. Gruendler, SIAM J. Math. Anal. 16 (1985).
[7] S.J. Linz, Am. J. Phys. 65 (1997) 523.
[8] J.-M. Malasoma, Phys. Lett. A 264 (2000).
[9] V.K. Melnikov, Trans. Moscow Math. Soc. 12 (1963) 1.
[10] J.C. Sprott, Am. J. Phys. 65 (1997) 537.