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Complex Parametric Vibrations of Flexible Rectangular Plates

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Abstract. In this paper we consider parametric oscillations of flexible plates within the model of von Kármán equations. First we propose the general iterational method to find solutions to even more general problem governed by the von Kármán–Vlasov–Mushtari equations. In the language of physics the found solutions define stress–strain state of flexible shallow shell with a bounded convex space $\Omega \in R^2$ and with sufficiently smooth boundary Γ. The new variational formulation of the problem has been proposed and his validity and application has been discussed using precise mathematical treatment. Then, using the earlier introduced theoretical results, an effective algorithm has been applied to convert problem of finding solutions to hybrid type partial differential equations of von Kármán form to that of the ordinary differential (ODEs) and algebraic (AEs) equations. Mechanisms of transition to chaos of deterministic systems with infinite number of degrees of freedom are presented. Comparison of mechanisms of transition to chaos with known ones is performed. The following cases of longitudinal loads of different sign are investigated: parametric load acting along X direction only, and parametric load acting in both directions X and Y with the same amplitude and frequency.

Key words: Flexible plates, Parametric vibrations, von Kármán equations, Periodic and quasi-periodic motion, Chaos.

1. Introduction

The dynamics of mechanical objects governed by von Kármán, Vlasov or Mushtari non-linear partial differential equations (PDEs) belongs to less investigated in mechanics and physics [1–3]. Observe that the aim of plates and shells theory is to describe the behaviour of a thin three-dimensional solid layer (plate or shell). However, this general treatment is too difficult and various two-dimensional approximations are used [4–9]. In addition, still exist rather old models given by Kirchhoff [10] and Love [11]. More detailed treatment of this problem can be found in [12], where also 305 references related to geometrical non-linear theories of thin elastic shells are given. When a plate is thin, its material is elastic and both Hook and Kirchhoff hypotheses are valid, then the mathematical model is governed by von Kármán equations, which are considered in this paper. It is clear that the plate (shell) equations are expected to possess much more interesting dynamics in comparison to lumped physical systems governed by ordinary differential equations (ODEs), since their dynamics depends on both spatial and time variables.

We would like to mention different kinds of internal and external resonances, exchange of energy between the modes of internal and external resonances, coupled-mode response when only one is excited, standing and travelling waves, periodic, quasi-periodic and chaotic

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behaviour, various bifurcations, steady state, transitional, spatial and temporal chaos, and other [13–17].

Observe that depending on the approximate method used to solve the PDEs we again make a step away from a real behaviour of the starting mechanical model of the considered plate. In general, one deals either with asymptotical (semi-asymptotical) or purely numerical approaches as well as combined asymptotical–numerical ones.

Although there are many advantages of the asymptotical techniques, there exist also some disadvantages [14], and our idea of choosing the numerical method is as follows. Since we decided to consider the von Kármán equations, then the best approximation is that one of difference scheme. It is known that usually the governing PDEs describing behaviour of continuous systems are transformed to a non-linear set of ODEs using the Bubnov–Galerkin procedure in most studies. However, this approximation depends strongly on the eigenfunctions used, and is inconvenient during analysis of quasi-periodic or chaotic response. The main drawback concerns always open question, that is, how the obtained averaged equations (multi-degree-of-freedom models) approximate the initial infinite dimensional model.

Our approach does not include the mentioned potential drawbacks since we present a new iterational method and its convergence conditions are rigorously discussed and estimated. Also the used spatial numerical approximations of $O(h^2)$ or $O(h^4)$ give the best (in some sense) approximation to the considered problem.

Note that the model discussed by us represented by von Kármán equations has been recently also reconsidered from a point of view of mathematical physics (see [18–24] and literature therein). Observe that the uniqueness of weak solution to von Kármán equations with an aerodynamic pressure and the thermal stresses had been an open problem for a long time and only recently it has been solved [20]. In addition, although there are many important results concerning asymptotic behaviour of solutions to the parabolic-like dynamics which has inherently smoothing effect, but it cannot always be applied to hyperbolic-like dynamics (i.e. to plate (shell) equations, among others). In the case of von Kármán equations the dynamics does not introduce any regularising effects for the initial data and the compactness requirement of the non-linear term does not hold (see more details in [25]).

The thermo-elastic plate system of equations are analysed in [26, 27], when the sufficient conditions of existence, unity and continuity dependence on initial data of the Cauchy problem for differential–operational equation of mixed hyperbolic–parabolic type are given. If the operational coefficients are suitably chosen, the investigated system can be used to obtain the modified Germain–Lagrange hyperbolic type plate equation. Moreover, one can use three-dimensional parabolic equation of thermal conductivity in order to define the temperature field.

The paper is organised in the following manner. First (Section 2) a general iterational method of solution to von Kármán–Vlasov–Mushtari equations is introduced and its mathematical background is stated. Then (Section 3) the fundamental hypotheses, as well as von Kármán equations with attached initial and boundary conditions and further used algorithms are given. Section 4 includes results, whereas discussion is addressed in Section 5.

2. Iterational Method of Solution to the Kármán–Vlasov–Mushtari Equations

This part is devoted to the analysis of convergence conditions of the solution method to the system of equations governing a stress–strain state of flexible plates. Observe that on each iterational step only linear equations are solved.

2.1. Statement of the Problem

The problem of finding a stress–strain state of the flexible shallow shells with the bounded convex space $\Omega \subset R^2$ and with enough regular boundary Γ can be reduced to that of finding the functions w and F satisfying the following system of non-linear differential equations

$$a_1\Delta^2 w - L(w, F) - \tilde{\nabla}_k^2 F = q, \tag{1}$$

$$a_2\Delta^2 F + \tfrac{1}{2}L(w, w) + \tilde{\nabla}_k^2 w = 0, \tag{2}$$

where a_i are positive constants, and the following boundary conditions attached

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{in } \Gamma, \qquad F = \frac{\partial F}{\partial n} = 0 \quad \text{in } \Gamma, \tag{3}$$

where $\partial/\partial n$ denotes the derivative along a normal to Γ.

In (1) and (2) the following differential operators are used

$$\tilde{\nabla}_k^2(\cdot) = k_2\frac{\partial^2(\cdot)}{\partial x_1^2} + k_1\frac{\partial^2(\cdot)}{\partial x_2^2}, \qquad \nabla_k^2(\cdot) = \frac{\partial^2(k_2\cdot)}{\partial x_1^2} + \frac{\partial^2(k_1\cdot)}{\partial x_2^2},$$

$$L(u, v) = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2},$$

where k_1 and k_2 are the main shell curvatures.

Let $H^2(\Omega)$ denote the Sobolev space of functions v with the properties

$$v \in L^2(\Omega), \qquad \frac{\partial v}{\partial x_i} \in L^2(\Omega), \qquad \frac{\partial^2 v}{\partial x_i \partial x_j} \in L^2(\Omega), \quad i, j = 1, 2,$$

where $L^2(\Omega)$ denotes the space of functions being summed with the square power in Ω.

Let $H_0^2(\Omega)$ denote the closure of the functions from $D(\Omega)$ with the norm

$$H^2(\Omega) : H^2 + O(\Omega) = D(\Omega)^{H^2(\Omega)} = v | v \in H^2(\Omega), v|_\Gamma = \frac{\partial v}{\partial n}|_\Gamma = 0.$$

Since the space Ω is bounded and its boundary is enough regular then the mapping $v \to \|\Delta v\|_{0,\Omega}$ defines a norm in $H_0^2(\Omega)$, which is equivalent to a norm generated by the space $H^2(\Omega)$.

We assume that $q \in H^{-2}(\Omega)$, where $H^{-2}(\Omega)$ is topologically conjugated to $H_0^2(\Omega)$, and $k_1, k_2 \in L^\infty(\Omega)$. It is known, that in this case the problem (1) and (2) possesses at least one solution.

2.2. New Variational Formulation of the Problem

Let $(\cdot, \cdot)$ denotes a scalar $(u, v) = \int_\Omega uv\mathrm{d}\Omega$ product in $L^2(\Omega)$, and let $\beta(w, F, \mu)$ be a triple-linear form

$$\beta(w, F, \mu) = a_2(\Delta F, \Delta\mu) + \tfrac{1}{2}(L(w, w), \mu) + (\nabla^2 - kw, \mu), \tag{4}$$

defined in $(H_0^2(\Omega))^3$.

Let us define the following manifold

$$M = \{w, F \in H_0^2(\Omega) | \forall \mu \in H_0^2(\Omega), \beta(w, F, \mu) = 0\} \tag{5}$$

and the following bilinear form $J(w, F) : M \to R$

$$J(w, F) = \tfrac{1}{2}a_1\|\Delta w\|_{0,\Omega}^2 + \tfrac{1}{2}a_2\|\Delta F\|_{0,\Omega}^2 - (q, w). \tag{6}$$

THEOREM 2.1. *The problem of minimalisation of* (4) *on the manifold* (5) *possesses at least one solution.*

Proof. Let $\{w_n, F_n\} \in M$ be a minimising series, that is,

$$J(w_n, F_n) \to \inf_{\{w,F\}\in M} J(w, F). \tag{7}$$

For any $w, F \in H^2O(\Omega)$ the following inequality is satisfied

$$J(w, F) \geqslant C_1\|w\|_{2,\Omega}^2 + C_2\|F\|_{2,\Omega}^2 - C_3\|w\|_{2,\Omega},$$

where $\|\cdot\|_{2,\Omega}$ denotes a norm in $H^2(\Omega)$, and C_i are the positive constants. The inequality (7) yields

$$C_1\|w_n\|_{2,\Omega}^2 + C_2\|F_n\|_{2,\Omega}^2 - C_3\|w_n\|_{2,\Omega} \leqslant J(w_n, F_n) \leqslant J(w_0, F_0) = A.$$

and we get

$$C_1\left(\|w_n\|_{2,\Omega}^2 - \frac{C_3}{2C_1}\right)^2 + C_2\|F_n\|_{2,\Omega}^2 \leqslant A + \frac{C_3^2}{4C_1}, \quad \|w_n\|_{2,\Omega} \leqslant C_4, \quad \|F_n\|_{2,\Omega} \leqslant C_5,$$

which means that the series is bounded in $(H_0^2(\Omega))^2$. Consequently, one can choose a series $\{w_k, F_k\}$ in a way that $w_k \to \tilde{w}$ and $F_k \to \tilde{F}$ are strong in $L^2(\Omega)$.

Now we show that the limit $\{\tilde{w}, \tilde{F}\}$ of the minimising series belongs to M, that is, that the following relation is satisfied:

$$\beta(\tilde{w}, \tilde{F}, \mu) = 0, \quad \forall \mu \in H_0^2(\Omega).$$

Since $k_1, k_2 \in L^2(\Omega)$, and owing to a weak convergence of $w_k \to \tilde{w}$ and $F_k \to \tilde{F}$, we obtain

$$(\nabla_k^2 w_k, \mu) \to (\nabla_k^2 \tilde{w}, \mu), \qquad (\Delta F_k, \Delta\mu) \to (\Delta\tilde{F}, \Delta\mu), \quad \forall \mu \in H_0^2(\Omega).$$

Besides, because $(L(w_k, w_k), \mu) = (L(w_k, \mu)w_k)$, $\forall \mu \in H_0^2(\Omega)$ and $L(w_k, \mu) \to L(\tilde{w}, \mu)$ is weak in $H_0^2(\Omega)$, then taking into account that $w_k \to \tilde{w}$ is strong in $L^2(\Omega)$ we get $(L(w_k, w_k), \mu) = (L(\tilde{w}, \tilde{w}), \mu)$, which yields $\beta(\tilde{w}, \tilde{F}, \mu) = 0$, $\forall \mu \in H_0^2(\Omega)$. This means that

$$\{\tilde{w}, \tilde{F}\} \in M. \tag{8}$$

However, $J(w, F)$ is halfway continuous in a weak topology on $(H_0^2(\Omega))^2$, and therefore the following estimation holds

$$\lim_{k\to\infty} J(w_k, F_k) \geqslant J(\tilde{w}, \tilde{F}).$$

Taking into account (7) and (8), we obtain that $J(\tilde{w}, \tilde{F}) \leqslant \inf_{\{w,F\}\in M} J(w, F)$. Therefore, the following relation holds $J(\tilde{w}, \tilde{F}) = \inf_{\{w,F\}\in M} J(w, F)$, because $\{\tilde{w}, \tilde{F}\} \in M$ is exactly a solution to the problem of minimalisation. □

Let us now explain how the points being the minimum of the functional (6) are related to the solutions of the problem (1)–(3). For this aim we need to introduce the definition of a weak solution. A pair of functions $\{w, F\} \in M$ satisfying the equation

$$a_1(\Delta w, \Delta\mu - Lw, F)(\mu) - (\tilde{\nabla}_k^2 F, \mu) = (q, \mu), \quad \forall\mu \in H_0^2(\Omega), \tag{9}$$

is called the weak solution to the problem (1)–(3). □

THEOREM 2.2. *The points of the functional* (6) *minimum are the weak solutions to the problem* (1)–(3).

Proof. Let $\{w, F\} \in M$ be one of the minimum points of the functional (6). Let $u = w+t\delta w$, $\delta w \in H_0^2(\Omega)$, and let us choose $v = F+\delta w$, $\delta F \in H_0^2(\Omega)$ in a way that $\{u, v\} \in M$, that is, that $\beta(w, F, \mu) = 0$, $\forall\mu \in H_0^2(\Omega)$. Then $J(w, F) = J(u, v)$. Therefore, we get

$$\begin{aligned} &ta_1(\Delta w, \Delta\delta w) + a_2(\Delta F, \Delta\delta F) - t(q, \Delta w) + \tfrac{1}{2}t_2 a_1\|\Delta\delta w\|_{0,\Omega}^2 + \\ &\quad + \tfrac{1}{2}a_2\|\Delta\delta F\|_{0,\Omega}^2 \geqslant 0, \quad \forall t \in R, \quad \delta w \in H_{0,\Omega}^2. \end{aligned} \tag{10}$$

From the relation $\beta(u, v, \mu) = 0$ we take $\mu = F$ and we obtain

$$a_2(\Delta F, \Delta\delta F) = -t(L(w, \delta w), F) - t(\nabla_k^2\delta w, F) - \tfrac{1}{2}t^2(L(\delta w, \delta w), F). \tag{11}$$

Substituting (11) into (10), dividing both sides by t and approaching a limit for $t \to 0$ we get

$$a_1(\Delta w, \Delta\delta w) - (L(w, f), \delta w) - (\tilde{\nabla}_k^2 F, \mu) \geqslant (q, \mu). \tag{12}$$

Changing δw by $-\delta w$ in (12) we get equality, that is (8).

Let us denote

$$(\Phi(w, F), \mu) = a_1(\Delta w, \Delta\mu) - (L(w, F), \mu) - (\tilde{\nabla}_k^2 F, \mu) - (q, \mu).$$

It is clear that $\Phi(w, F) \in H_{-2}(\Omega)$ is a projection of the functional $J(w, F)$ gradient into hyperplane generated by the equation $\beta(w, F, \mu) = 0$, $\forall\mu \in H_0^2(\Omega)$. Then the equality (9) can be written in the form

$$\Phi(w, F, \mu) = 0. \tag{13}$$

Therefore, each point of the minimum of the functional (6) on M satisfies (13), that is, is a weak solution to the problem (1)–(3). □

2.3. Iterational Method

In Section 2.2 we have shown that a solution to the problem (1)–(3) is equivalent to a solution of the minimalisation problem (6) with the occurrence of constraints $\{w, F\} \in M$. In order to solve this problem different methods of minimum search can be applied. Depending on the choice of a solution to the extremal problem the various algorithms to solve the problem (1)–(3) can be used.

Let us construct the iterational minimalisation process of $J(w, F)$ on M within the following scheme:

(a) we choose arbitrary $w_0 \in H_0^2(\Omega)$;
(b) after computations of w_n we find $F_n \in H_0^2(\Omega)$ and $w_{n+1} \in H_0^2(\Omega)$ as the solutions to the following problems

$$\beta(w_n, F_n, \mu) = 0, \quad F_n \in H_0^2(\Omega), \ \forall \mu \in H_0^2(\Omega), \tag{14}$$

$$a_1(\Delta w_{n+1}, \Delta\mu) = a_1(\Delta w_n, \Delta\mu) - \rho_n(\Phi(w_n, F_n), \mu), \quad \forall \mu \in H_0^2(\Omega); \tag{15}$$

(c) the coefficient ρ_n on the step (b) is chosen from the condition

$$J(w_{n+1}, F_{n+1}) - J(w_n, F_n) \leqslant \varepsilon(\Phi(w_n, F_n), \quad w_{n+1} - w_n), \quad 0 < \varepsilon < 1. \tag{16}$$

THEOREM 2.3. *For the iterational process* (14)–(16) $(\Phi(w_n, F_n), \mu) \to 0$ *for* $n \to \infty$ *for any initial point* $\{w_0, F_0\} \in M$. *In addition, the obtained series* $\{w_n, F_n\}$ *includes the subseries converging to a weak solution to the problem* (1)–(3).

Proof. A possibility of construction of the series $\{w_n, F_n\}$ results from the observation that for any arbitrary ρ_n, $w_{n+1} \in H_0^2(\Omega)$, and what follows, $L(w_{n+1}, w_{n+1}) \in H^{-2}(\Omega)$, $\nabla_k^2 w_{n+1} \in H^{-2}(\Omega)$ [28, 29]. It implies that the equation $\beta(w_{n+1}, F_{n+1}, \mu) = 0$ can be solved, and $F_{n+1} \in H_0^2(\Omega)$.

Let us consider the difference

$$\Delta J_n = J(w_{n+1}, F_{n+1}) - J(w_n, F_n) = \tfrac{1}{2}(\Delta(w_{n+1} - w_n), \tag{17}$$

$$\Delta(w_{n+1} + w_n)) + \tfrac{1}{2}(\Delta(F_{n+1} - F_n), \Delta(F_{n+1} + F_n)) = (q, w_{n+1} - w_n).$$

Taking into account that $\{w_n, F_n\} \in M$, $\{w_{n+1}, F_{n+1}\} \in M$ from (17) one obtains

$$\Delta J_n = (\Phi(w_n, F_n), \delta w) + \tfrac{1}{2} a_1 \|\Delta \delta w\|_{0,\Omega}^2 + \tfrac{1}{2} a_2 \|\Delta \delta F\|_{0,\Omega}^2,$$

where $\delta w = w_{n+1} - w_n$ and $\delta F = F_{n+1} - F_n$. In addition, it follows from (15) that δw is a solution to the equation

$$a_1 \Delta^2 \delta w = -\rho_n \Phi(w_n, F_n), \quad \delta w \in H_0^2(\Omega),$$

and we obtain

$$\delta w = -\rho_n G[\Phi(w_n, F_n)], \tag{18}$$

where $G[\bullet] : H^{-2}(\Omega) \to H_0^2(\Omega)$ is the linear and bounded operator inversed to the operator $a_1 \Delta^2(\bullet)$. Therefore, we get

$$\Delta J_n = -\rho_n(\Phi(w_n, F_n), G[\Phi(w_n, F_n)]) + \tfrac{1}{2} a_1 \|\Delta \delta w\|_{0,\Omega}^2 + \tfrac{1}{2} a_2 \|\Delta \delta F\|_{0,\Omega}^2. \tag{19}$$

Let us consider now the second order terms. Let us choose in (15) $\mu = \delta w$ and taking into account (18) we get

$$a_1 \|\Delta \delta w\|_{0,\Omega}^2 = -\rho_n(\Phi(w_n, F_n), \delta w) = \rho_n^2(\Phi(w_n, F_n), G[\Phi(w_n, F_n)]). \tag{20}$$

We are going to estimate last term appearing in (19). Because $\{w_n, F_n\} \in M$ and $\{w_{n+1}, F_{n+1}\} \in M$, then the following equation can be used to define δF:

$$a_2(\Delta \delta F, \Delta \mu) + L(w_n, \delta w) + (\nabla_k^2 \delta w, \mu) + \tfrac{1}{2}(L(\delta w, \delta w), \mu) = 0,$$
$$\delta F \in H_0^2(\Omega), \ \forall \mu \in H_0^2.$$

In particular, we obtain [28]

$$\|\Delta\delta w\|_{0,\Omega}\leqslant C_7(\|L(w_n,\delta w)\|_{L^1(\Omega)})+\|L(\delta w,\delta w)\|_{L^1(\Omega)})+\|\nabla_k^2\delta w\|_{L^1(\Omega)}.$$

Observe that w_n belongs to the bounded manifold in $H_0^2(\Omega)$ for arbitrary n. Consequently, $\|\Delta\delta F\|_{0,\Omega}\leqslant C_8\|\Delta\delta w\|_{0,\Omega}^2$ and

$$\|\Delta\delta F\|_{0,\Omega}^2\leqslant C_9\rho_n^4(\Phi(w_n,F_n),\quad G[\Phi(w_n,F_n)])^2. \tag{21}$$

Substituting (20) and (21) into (19) and taking into account the positive definition as well as the boundness of the operator $G[\bullet]$ we get

$$\Delta J_n=-\rho_n C_{10}\|\Phi(w_n,F_n)\|^2\big(-1+\tfrac{1}{2}a_1\rho_n+C_1\tfrac{1}{2}\rho_n^3\|\Phi(w_n,F_n)\|^2\big).$$

The obtained estimation indicates that there exist values $\rho_n\neq 0$ satisfying the inequalities (16). It is sufficient to take ρ_n satisfying the following inequality

$$\tfrac{1}{2}a_1\rho_n+\tfrac{1}{2}C_1\rho_n^3\|\Phi(w_n,F_n)\|^2\leqslant 1-\varepsilon.$$

Observe that it can be realised always since $\|\Phi(w_n,F_n)\|$ is the finite quantity and $0<\varepsilon<1$. Taking ρ_n due to the described algorithm we obtain on each step

$$\Delta J_n\leqslant-\rho_n\varepsilon\|\Phi(w_n,F_n)\|^2, \tag{22}$$

that is, for arbitrary n we have $J_{n+1}-J_n\leqslant 0$. Since the functional J is bounded from below than taking into account the last inequality we obtain $n\to\infty$, $\Delta J_n\to 0$. Besides, from (22) one obtains

$$\|\Phi(w_n,F_n)\|^2\leqslant\frac{-\Delta J_n}{\varepsilon\rho_n}. \tag{23}$$

It is important to note that the described algorithm how to choose ρ_n also guaranties that for arbitrary n the following inequality holds: $\rho_n\geqslant\rho_0>0$. Indeed, because $\Delta J_n\leqslant 0$, then

$$J(w_n,F_n)\leqslant J(w_0,F_0)=A. \tag{24}$$

On the other hand, it follows from (24) that the norms $\|w_n\|_{2,\Omega}$, $\|F_n\|_{2,\Omega}$ are bounded. It means that also the norm $\|\Phi(w_n,F_n)\|$ is bounded.

Taking into account the later observation and (23) we obtain $\|\Phi(w_n,F_n)\|\to 0$ for $n\to\infty$, and consequently, $(\Phi(w_n,F_n),\mu)\to 0$ for $n\to\infty$, $\forall\mu\in H_0^2(\Omega)$.

Now, an occurrence of a convergent subseries results from the boundness of the norms $\|w_n\|_{2,\Omega}$, $\|F_n\|_{2,\Omega}$ (see also proof of Theorem 2.1). □

2.4. Decreasing Dimension of Partial Differential Equations

2.4.1. *The Kantorovich–Vlasov (K–V) method*

Two Russian scientists Kantorovich [30] and Vlasov [31] have proposed independently the method of reduction of PDEs to ODEs, which is now-a-days called the K–V method (in fact they considered linear elliptic equations). A proof of convergence of the proposed method is formulated by Kantorovich [32] and Vlasov [33] for the mentioned elliptic equations. A proof of convergence of K–V method is given in [34].

Owing to the K–V method scheme applied to equations (1) and (2), the latter are transformed to the following compact form

$$l_1(w, F) = q, \qquad l_2(w, F) = 0, \tag{25}$$

where $l_1(w, F)$ and $l_2(w, F)$ denote linear and non-linear parts of equations (1) and (2), correspondingly. Following the K–V method steps, a solution of the formulated problem is sought in the following form

$$w_N = \sum_{i=1}^{N} a_i(x_1)b_i(x_2), \qquad F_N = \sum_{j=1}^{N} c_j(x_1)d_j(x_2). \tag{26}$$

Let the analysed space be the rectangular $\Omega = (0, a)x(0, b)$ with the boundary Γ for our considered equations (25).

Observe that the basis functions components are taken with respect to the variable x_2, that is, $b_i(x_2)$ and $d_j(x_2)$. Substituting (26) into (25), and applying the Bubnov–Galerkin procedure with respect to the variable x_2, the following equations are obtained

$$\begin{gathered}(l_1(w_N, F_N) - q, b_m)_{L_2(x_2)} = 0, \qquad (l_2(w_N, F_N), d_k)_{L_2(x_2)} = 0,\\ m = 1, 2, \ldots, N, \quad k = 1, 2, \ldots, N,\end{gathered} \tag{27}$$

with the attached boundary conditions

$$\begin{gathered}a_m(0) = a_m(a) = \frac{\mathrm{d}a_m(0)}{\mathrm{d}x_1} = \frac{\mathrm{d}a_m(a)}{\mathrm{d}x_1} = 0, \quad m = 1, 2, \ldots, N,\\ c_k(0) = c_k(a) = \frac{\mathrm{d}c_k(0)}{\mathrm{d}x_1} = \frac{\mathrm{d}c_k(a)}{\mathrm{d}x_1} = 0, \quad k = 1, 2, \ldots, N.\end{gathered} \tag{28}$$

It is clear that the system (27), (28) is solvable and a set of approximate solutions $\{w_N, F_N\}$ is weakly compact in the space $H^2((0, a) \times (0, b))$. This observation indicates a solvability of the stated problem using the Bubnov–Galerkin method [35]. Since the space Ω is chosen to be the rectangular with the sides parallel to the co-ordinate axes, the solution of the stated problem can be found using the Bubnov–Galerkin method. In words, the basis system of a space, where a solution exists, should satisfy the conditions of the theorem reported in [36], that is, the functions system should be fully defined in a Sobolev space $H^2(0, b)$.

The system of non-linear ODEs (27) is reduced to a system of non-linear algebraic equations (AEs), for instance, applying the finite difference method of any order and then it is solved via Newton's method.

2.4.2. *The variational iterations (VI) method*

This method is modification of the K–V method. For simplicity, the VI scheme is presented using the linear elliptic PDEs.

Assume that solutions to the following equations are sought

$$Aw(x_1, x_2) = q(x_1, x_2), \quad (x_1, x_2) \in \Omega = X_1 \times X_2, \tag{29}$$

where A is a positively defined operator given on a compact set $D(A)$ in the Hilbert space $L_2(\Omega)$, $q(x_1, x_2)$ is the given function of two variables x_1, x_2; $q \in L_2(\Omega)$, $w(x_1, x_2)$ is the being sought function of two variables x_1, x_2; Ω is the space of the variables x_1, x_2 variations; $X_1(X_2)$ is a coupled bounded set of the variables $x_1(x_2)$, $x_1(x_2) \in E_1(E_2)$. By $H_A(X_1 \times X_2)$

we denote the energy space of the operator A. The latter is defined as a closure of the manifold $D(A)$ with respect to the following norm

$$\|w\|^2_{H_A} = (Aw, w)_{L_2(\Omega)}.$$

Observe that $H_A \subset L_2(\Omega)$. Now the space H_A is identified with the space $H_2^m(\Omega)$ being a Sobolev space of the functions $w(x_1, x_2)$ defined in the space Ω. Owing to the VI method, an approximated solution of the equation (29) is sought in the form

$$w_N(x_1, x_2) = \sum_{i=1}^{N} a_i(x_1) b_i(x_2), \tag{30}$$

where the functions $a_i(x_1)$ and $b_i(x_2)$ are defined via the equation

$$(Aw_N - q, a_i)_{L_2(X_1)} = 0, \quad i = 1, 2, \ldots, N, \tag{31}$$

$$(Aw_N - q, b_i)_{L_2(X_2)} = 0, \quad i = 1, 2, \ldots, N, \tag{32}$$

in the following way. First, a certain system composed of N function with respect to one variable is constructed, for example, $a_1^0(x_1), \ldots, a_N^0(x_1)$. Then, the system of N functions $b_1^1(x_2), \ldots, b_N^1(x_2)$ is obtained from (31).

The obtained functions are substituted to the system (32), and they define a new choice of the functions with respect to the variable x_1, that is, $a_1^2(x_1), \ldots, a_N^2(x_1)$, and so on.

DEFINITION. The computational process using one given system functions to get the second one is called a step of the VI method. A number of steps required for determination of any functions set corresponds to a top index (number) of κ of the functions of the mentioned set. The functions $a_i(x_1)$ and $b_i(x_2)$ yielded in the κth step, corresponding, for example, to the functions set $b_1^\kappa(x_2), \ldots, b_N^\kappa(x_2)$, define the following function

$$w_n^\kappa(x, y) = \sum_{i=1}^{N} a_i^{\kappa-1}(x_1) b_i^\kappa(x_2).$$

The obtained function represents an approximated solution of equation (29) of the VI method.

The fundamental background of the VI method devoted to plates and shells theory can be found in [37].

2.5. Numerical Experiment

A simultaneous application of the iterational procedure described in Section 2.3 of this work and the VI method (see Section 2.4) possesses at least three main advantages:

- (i) The order of a being sought equations system is reduced two times (from eighth to fourth order);
- (ii) The linearisation of a being sought non-linear system is carried out;
- (iii) The transition from PDEs with two variables x_1, x_2 into ODEs with constant coefficients holds.

These are remarkable achievements in the field of non-linear elliptic PDEs.

A numerical experiment of the presented approach is outlined applying an example of computations of flexible isotropic squared plates with constant thickness using the three following different boundary conditions:

$$w = \frac{\partial^2 w}{\partial n^2} = F = \frac{\partial^2 F}{\partial n^2} = 0, \qquad x_1 = x_2 = 0, \qquad x_1 = x_2 = 1, \tag{33}$$

$$w = \frac{\partial^2 w}{\partial n^2} = F = \frac{\partial F}{\partial n} = 0, \qquad x_1 = x_2 = 0, \qquad x_1 = x_2 = 1, \tag{34}$$

$$w = \frac{\partial w}{\partial n} = F = \frac{\partial F}{\partial n} = 0, \qquad x_1 = x_2 = 0, \qquad x_1 = x_2 = 1. \tag{35}$$

For computation simplicity, the VI method is applied to (26) for $N = 1$. The obtained ODEs system is reduced to AEs set using the finite difference method with $O(h^2)$ approximation, which is solved via the Gauss technique. The equations (1) and (2) are non-dimensionalised in a typical way:

$$x_1 = a\bar{x}_1, \qquad x_2 = a\bar{x}_2, \qquad w = \bar{w}h, \qquad F = Eh^3\bar{F}, \qquad \lambda = \frac{a}{b}, \qquad q = \frac{Eh^4}{a^2b^2}\bar{q}.$$

The bars standing over the non-dimensional quantities are further omitted. The integration interval [0, 1] is divided into 100 parts. The obtained numerically dependence $q(w)$ is reported in Figure 1. The curves 1 and 2 correspond to the boundary condition (33) and (34), respectively. The curve 3 corresponds to the boundary condition (35). The curves 2, 3(1) are obtained for the Poisson's coefficient $\nu = 0.33$ ($\nu = 0.1$). The marked black circles correspond to experimental results [38], whereas the marked asterisks correspond to the results obtained via the finite difference method [39], when the latter one has been applied directly to equations (1) and (2), and the associated non-linear AE has been solved via the Newton's method. The corresponding mesh is composed of 20×20 parts. The computations are carried out using the step $\Delta q = 10$. Finally, in order to accelerate a convergence of the iterational procedure, the initial values of w and F are taken from previous computational step increasing a load continuously.

A convergence of the iterational procedure described in the Section 2.3 for w and for $q = 60$ is illustrated owing to a solution of the equations (1) and (2) for $q = 20$ and $\nu = 0.28$.

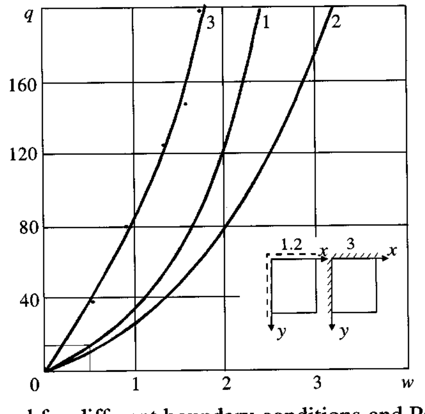


Figure 1. Dependence $q(w)$ obtained for different boundary conditions and Poisson's coefficient (w denotes the maximal deflection in the case of dynamics, see text for more details).

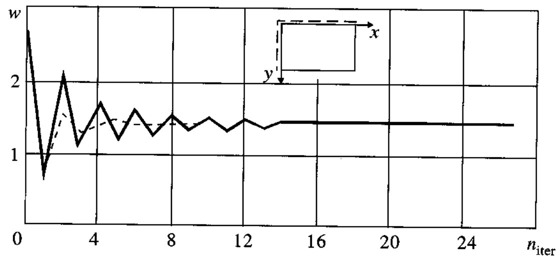


Figure 2. Dependence $w(n_{\text{iter}})$; solid (dashed) curve corresponds to exact (iterational) solution (dynamical case).

In the first approximation of the VI method the integration interval [0, 1] is partitioned into 20 parts owing to the finite difference method, and the boundary conditions (33) are applied. The dependence $w(n)$, where n is the iterations number, is shown in Figure 2. The assumed computational accuracy achieved $\varepsilon = 1 \times 10^{-5}$ for a deflection. In order to get the required accuracy, 26 iterations are needed. Observe that numerically obtained trajectory approaches the exact solution from both sides. Owing to the latter observation, a convergence of the iterational process is improved, since a deflection after an odd (w_{o}) and even (w_{e}) iterations are defined via the following averaged formula

$$w = \frac{w_{\text{o}} + w_{\text{e}}}{2}.$$

An application of the above formula yielded a decrease of the iterations number up to eight (see the dashed curve in Figure 2). One may apply also other iterational formulas.

In the next step, the peculiarities of solutions of the shells theory problems using the VI method will be briefly discussed. As an illustrative example, the linear Germain–Lagrange equation $A = D\,\Delta^2 w$ in (29) is studied (D denotes the cylindrical stiffness). In what follows the boundary conditions (33), (35); $\nu = 0.28$; $q = 1$, $N = 2.0$ (partition of [0, 1] interval) are applied. The obtained ODEs are solved via the finite difference method.

A convergence of the VI method with respect to an initial approximation and the applied boundary conditions for a centre deflection is analysed. The obtained results are reported in Table 1. The included data analysis exhibits a high convergence speed independently on the applied boundary conditions. However, the convergence depends on a choice of initial approximation. Observe that a convergence is achieved also in the cases, when initial approximation does not satisfy the boundary conditions. It is worth noticing that for the Poisson's coefficient $\nu = 0.3$ the solutions obtained via application of the VI method (for free (33) and clamped (35) boundary conditions along a contour) exactly overlap with the known exact solutions already in the first approximation.

Remark 1. The illustrated methods can be effectively applied: (i) for the problems devoted to optimal design; (ii) investigation of elastic–plastic deformations; (iii) plates and shells with variable thickness analysis; (iv) wear processes, and other.

Remark 2. The discussed iterational method in the Section 2.3 can be further developed, and the 8th order of the investigated systems (1), (2) can be decreased four times. A corresponding iterational procedure can be applied solving the Poisson's equations $\Delta_i u(x_1, x_2) = f_i(x_1, x_2)$ ($i = 1, 2, 3, 4$) with the attached arbitrary boundary conditions.

Table 1. Convergence of the VI method with respect to initial and boundary conditions

Boundary conditions	Initial approximation	Approximation order			
		1	2	3	4
(33)	$a_1^0 = \sin \pi x_1$	0.04552	0.04493	*0.044925*	*0.044925*
	$a_1^0 = \sin^2 \pi x_1$	0.02185	0.043965	*0.044925*	*0.044925*
	$a_1^0 = 1.0$	-1×10^{-6}	6×10^{-5}	0.011468	0.042793
(35)	$a_1^0 = \sin \pi x_1$	0.003377	0.011984	0.014077	*0.014078*
	$a_1^0 = \sin^2 \pi x_1$	0.014177	0.014077	*0.014078*	*0.014078*
	$a_1^0 = 1.0$	0.028944	0.01446	0.014018	*0.014078*
$x_1 = 0; 1$ (33)	$a_1^0 = \sin \pi x_1$	0.021827	0.021274	*0.021268*	*0.021268*
$x_2 = 0; 1$ (35)	$a_1^0 = \sin^2 \pi x_1$	0.014561	0.020821	0.021254	*0.021267*
	$a_1^0 = 1.0$	6×10^6	-1.9×10^{-5}	4.4×10^{-4}	0.018546

3. Formulation of the Problem

3.1. Algorithm

The problem of parametric oscillations of flexible plates formulated from position of differential equations' theory is presented in limited number of publications [40–43]. Known von Kármán equations are as follows:

$$\frac{\partial^2 w}{\partial t^2} + \varepsilon \frac{\partial w}{\partial t} = -\frac{1}{12(1-\nu^2)} \Delta^2 w + L(w, F) - \Delta_p w + q, \quad \Delta^2 F = -\tfrac{1}{2} L(w, w). \tag{36}$$

Introducing the following notation we get the equations in non-dimensional form

$$x = a\bar{x}, \qquad y = b\bar{y}, \qquad w = 2H\bar{w}, \qquad \lambda = \frac{a}{b}, \qquad t = t_0\bar{t}, \qquad \varepsilon = (2H)\bar{\varepsilon}$$

$$P_x = \frac{E(2H)^3}{b^2}\bar{P}_x, \qquad P_y = \frac{E(2H)^3}{a^2}\bar{P}_y, \qquad q = \frac{E(2H)^4}{a^2 b^2}\bar{q}.$$

In the above $q(x, y, t)$ is the transversal load function; $P_x(y, t)$, $P_y(x, t)$ are longitudinal load functions in x and y directions, respectively; $2H$ denotes plate thickness; a and b are plate dimensions; ε is damping coefficient; E is Young modulus; ν is Poisson's ratio; $w(x, y, t)$ and $F(x, y, t)$ are deflection and load functions, respectively. Origin of a co-ordinate system is located in the lower left corner of a plate, axes x and y are directed along sides of a plate, z axis is oriented down to the Earth centre, and $(x, y) \in \overline{G} = \{0 \leqslant x \leqslant 1;\ 0 \leqslant y \leqslant 1\}$, $0 \leqslant t \leqslant t_{\text{end}}$. The applied operators have the following form:

$$L(w, F) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2},$$

$$\Delta(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2}, \qquad \Delta_p(\cdot) = P_x\frac{\partial^2(\cdot)}{\partial x^2} + P_y\frac{\partial^2(\cdot)}{\partial y^2}.$$

Let us consider square plates ($\lambda = 1$) with the following initial conditions:

$$w\mid_{t=0} = \varphi_1(x, y), \qquad \dot{w}\mid_{t=0} = \varphi_2(x, y). \tag{37}$$

We assume simple support of an edge on non-compressing and non-stretching ribs in tangent plane. Then boundary conditions have the following forms:

$$w = w''_{xx} = F = F''_{xx} = 0, \quad (x = 0; 1), \;\; x \leftrightarrow y. \tag{38}$$

We apply difference method $O(h^2)$ to convert system of PDEs (36)–(38) to a system of ODEs in time. Note that an application of $O(h^4)$ approximation does not change the results significantly. Partial derivatives on x and y in the equation (36) are approximated by difference equations with accuracy $O(h^2)$. Applying Taylor series in a vicinity of a point (x_i, y_i) with step h, where h is a mesh size, and taking into account

$$\overline{G}_n = \left\{0 \leqslant x_i; y_i \leqslant 1, x_i = ih, y_i = ih, i, j = \overline{O, N}, h = \frac{1}{N}\right\},$$

we obtain known difference equations for partial derivatives of the second and fourth order, which are necessary for converting of PDEs into a system of ODEs in time and AEs for F_{ij}:

$$\frac{\partial^2 w_{ij}}{\partial t^2} + \varepsilon \frac{\partial w_{ij}}{\partial t} = \{A(w_{ij}) + B(w_{ij}, F_{ij})\} + q_{ij}, \qquad D(f_{ij}) = E(w_{ij}). \tag{39}$$

Difference operators $A_2(w_{ij})$, $B_2(w_{ij}, F_{ij})$, $D_2(F_{ij})$ and $E_2(w_{ij})$ in (39) have the following form:

$$A(w_{ij}) = \frac{1}{12(1-\nu^2)}(\lambda^{-2}\Lambda_{x^4}w_{ij} + 2\Lambda_{x^2}\Lambda_{y^2}w_{ij} + \lambda^2\Lambda_{y^4}w_{ij}),$$
$$A(w_{ij}, F_{ij}) = \Lambda_{x^2}w_{ij}(\Lambda_{y^2}F_{ij} - P_x) + \Lambda_{y^2}w_{ij}(\Lambda_{x^2}F_{ij} - P_y) - 2\Lambda_{x^2y^2}w_{ij}\Lambda_{x^2y^2}F_{ij},$$
$$D(F_{ij}) = 12(1-\nu^2)A(F_{ij}), \qquad E(w_{ij}) = \Lambda_{x^2}w_{ij}\Lambda_{y^2}F_{ij} + [\Lambda_{xy}w_{ij}]^2,$$

where $\Lambda_{x^k y^k}$ ($k = 0, 2, 4$) are known difference operators of respective derivatives. Let us convert system of non-linear difference–differential equations for w_{ij} into a vector form:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = Q(W, F, t), \qquad AF = P, \tag{40}$$

where W is a searched vector with components w_{ij}, A is matrix, F is searched vector with components F_{ij}, P is right hand side vector of the second equation from (39), depending on w_{ij}. Algorithm of a solution is presented in [44], where also reliability of results is discussed.

We investigate further stability of an isotropic ($\nu = 0, 3$) square plate ($\lambda = 1$) subjected to longitudinal load of different sign: $P_x = P_x^0 \sin \omega t$, $\omega = 8$ with the initial conditions:

$$w\mid_{t=0} = A \sin \pi x \sin \pi y \;(A = 1 \times 10^{-3}), \qquad \dot{w}\mid_{t=0} = 0.$$

Observe that during a construction of the computational algorithm the theoretical considerations from the Section 2 are applied.

Table 2. A comparison of the numerical results of critical load value P_{x_1} computations using three different approaches (see text)

References	Critical load P_{x_1}	
	BC (34)	BC (35)
[13]	3.60	9.12
[14]	3.48	8.40
[15]	3.61	9.12

3.2. Application of the Set-up Method to Solve Statical Problems of Flexible Plates and Shells

The system of equations (36), besides of investigation of dynamical behaviour of plates and shells, yields also the statical solutions. This approach will be further called the set-up method [44]. In fact, owing to this method a solution of the non-linear statical problem (1) and (2) with eighth PDEs order is reduced to a solution of a linear dynamical problem. In words, the set-up method linearises the initial analysed system (1) and (2) and the studied continuous system is reduced to a discrete one.

From the mathematical point of view, the set-up method can be treated as an iterational method of solution of a system of linear AEs, where each step in time is a new approximation to a being sought exact solution. Since the set-up method belongs to the iterational methods, it is characterised by a high accuracy order. In addition, it does not include one of the main drawbacks of almost all iterational procedures. Namely, it is not sensitive to an initial condition choice. The latter observation is yielded by a physical interpretation of the equations (36) governing vibrations of plates and shells in a viscous medium. Furthermore, the set-up method can be easily realised numerically. However, the main drawback of this method is that only stable solutions can be found, and a problem of an optimal choice of the medium damping ϵ is not solved.

For a purpose of efficiency and realiability estimations of the obtained dynamical responses in the case of dissipative parametrical vibrations of plates, a series of classical examples is solved. A stability of plates with two types of boundary conditions (34) and (35) and subjected to an action of constant loads P_{x_1} are studied.

The solutions reported in [45, 46] are obtained via the spectral ($P_{x_2} = 0$), finite difference with approximation $O(h^2)$ of equation (36), and via the set-up methods, correspondingly. A discretisation procedure of the problem is realised with respect to spatial co-ordinates (x_1, x_2) applying the finite difference method with approximation $O(h^2)$. The space $\Omega = (0,1) \times (0,1)$ is divided into 16×16 parts. The comparison of the reported in Table 2 data indicates a high accuracy of the set-up method.

4. Results

Let us consider scenario of transition to chaos under variation of the parameter $\{P_x^0\}$ and we take into account the following boundary conditions:

$$w \mid_\partial = w_n'' \mid_\partial = F \mid_\partial = F_n'' \mid_\partial = 0,$$

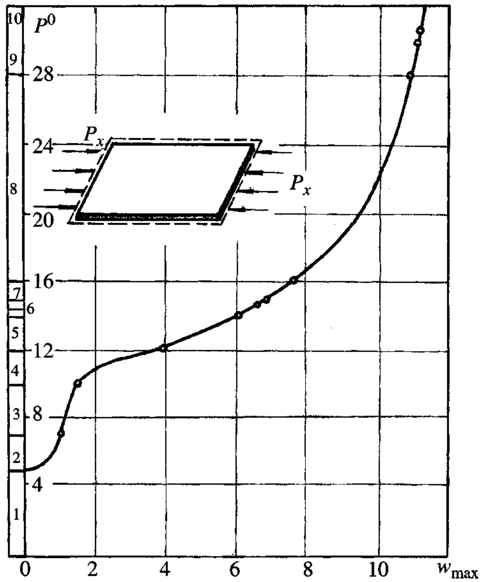


Figure 3. Dependence $P^0(w_{\max})$ and zones of different types of vibrations (see Table 3; dynamical case $P_0 = P_x$; $P_y = 0$).

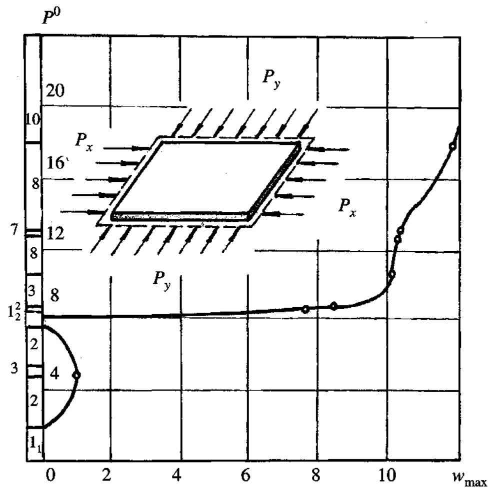


Figure 4. Dependence $P_0(w_{\max})$ and zones of different types of vibrations (see Table 3; dynamical case $P_0 = P_x = P_y$).

where F is the boundary of $G = \{x, y \mid 0 \leqslant x \leqslant 1; 0 \leqslant y \leqslant 1\}$, n is normal to the space G.

The Figures 3 and 4 present graph of $P^0(w_{\max}(0,5; 0,5))$. Vertical axes contain two scales: values of P^0 and zones of types of vibrations. These types are characterised in Table 3. For each point (ij) of the space $\overline{G}_n$ the following characteristics were obtained: $w_{ij}(t)$, $\dot{w}_{ij}(t)$, phase portraits $\dot{w}_{ij}(w_{ij})$, Poincaré sections $w_{ij}^T(w_{ij}^{(t+T)})$ (where T is period of external excitating load) and power spectra $\log(w)$. Numerical calculations showed that above listed characteristics are similar to all points (ij) of the space $\overline{G}_n$, therefore the Figures 5 and 6 present only results for a plate centre.

Table 3. Different vibration types and the associated notations

Region	Type of vibrations
l, l_1, l_2	Steady point
2	Periodic vibrations
3	Andronov–Hopf bifurcations
4	Quasi-periodic vibrations
5	Crisis
6	Post-crisis
7	Periodic vibrations with new frequency
8	Quasi-periodic vibrations with new frequency
9	Intermittency
10	Chaos

Let us analyse results of numerical calculations for a load along x axis only. On the right hand side of the Figure 5 there are numbers of corresponding zones in the Figure 3. Scenario of plate vibration transition is the following. For $0 \leqslant P_x^0 \leqslant 4{,}75$ plate is in a stable equilibrium state (we have one point in the Poincaré map). For $P^0 = 4$ on a phase portrait we have a point attractor. For $P^0 \geqslant 4{,}75$ elastic stability loss occurs and a plate vibrates periodically in an equilibrium state with the fundamental frequency ω_1 (other frequencies are: $3\omega_1, 5\omega_1$). In the phase portrait we have periodic attractor with two small loops, which are also visible in the Poincaré pseudosection. These loops show that soft Andronov–Hopf bifurcation is going to appear, what happens in zone 3 (see Figure 5).

Further increase of P^0 forces the plate to shift to zone 4 of quasi-periodic vibrations, what is easily visible for $P_x^0 = 12$, where quasi-periodic attractor appears in the phase portrait and Poincaré section. Further increase of P_x^0 yields crisis and post-crisis state, described in [43]. The corresponding zone exhibits a certain regularity in results of change of frequency characteristics, as well as of shape of $W_{ij}(*)$ (see $P^0 = 14{,}7$).

The new frequency ω_2 occurs. On high frequencies part of the power spectrum graph, the Andronov–Hopf bifurcation appears. Analysis of power spectrum leads to conclusion that all vibrations are quasi-periodic. Period doubling vibration happens for higher frequencies, and with increase of P_x^0 it moves to lower frequencies, which are then broad band (zone 8, $P^0 = 18$).

The increase of P^0 causes an occurrence of intermittency. For $P^0 = 28$ (see Figure 5), in spite of clearly visible chaotic trajectories of the phase portraits, the broad band power spectrum and the strange distribution of the points of Poincaré pseudosections, there is a zone of regular vibrations. The observed and reported intermittency slightly differs from the already known intermittency types. In the case of the intermittency I vibrations of the system are periodic which are sometimes interrupted by sudden turbulent jumps (it is exhibited by the Lorenz system, Belousov–Zhabotinsky reaction or Rayleigh–Benard convection).

Recall that the intermittency III can be obtained from intermittency I when additional rotation of the orbit of the angle of 180° occurs. The corresponding Floquet multiplicator changes its sign, and intermittency I is substituted by intermittency III. It has been shown during the numerical observation of the Rayleigh–Bernard convection that for $Ra/Ra_c \approx 416.7$ (Ra is

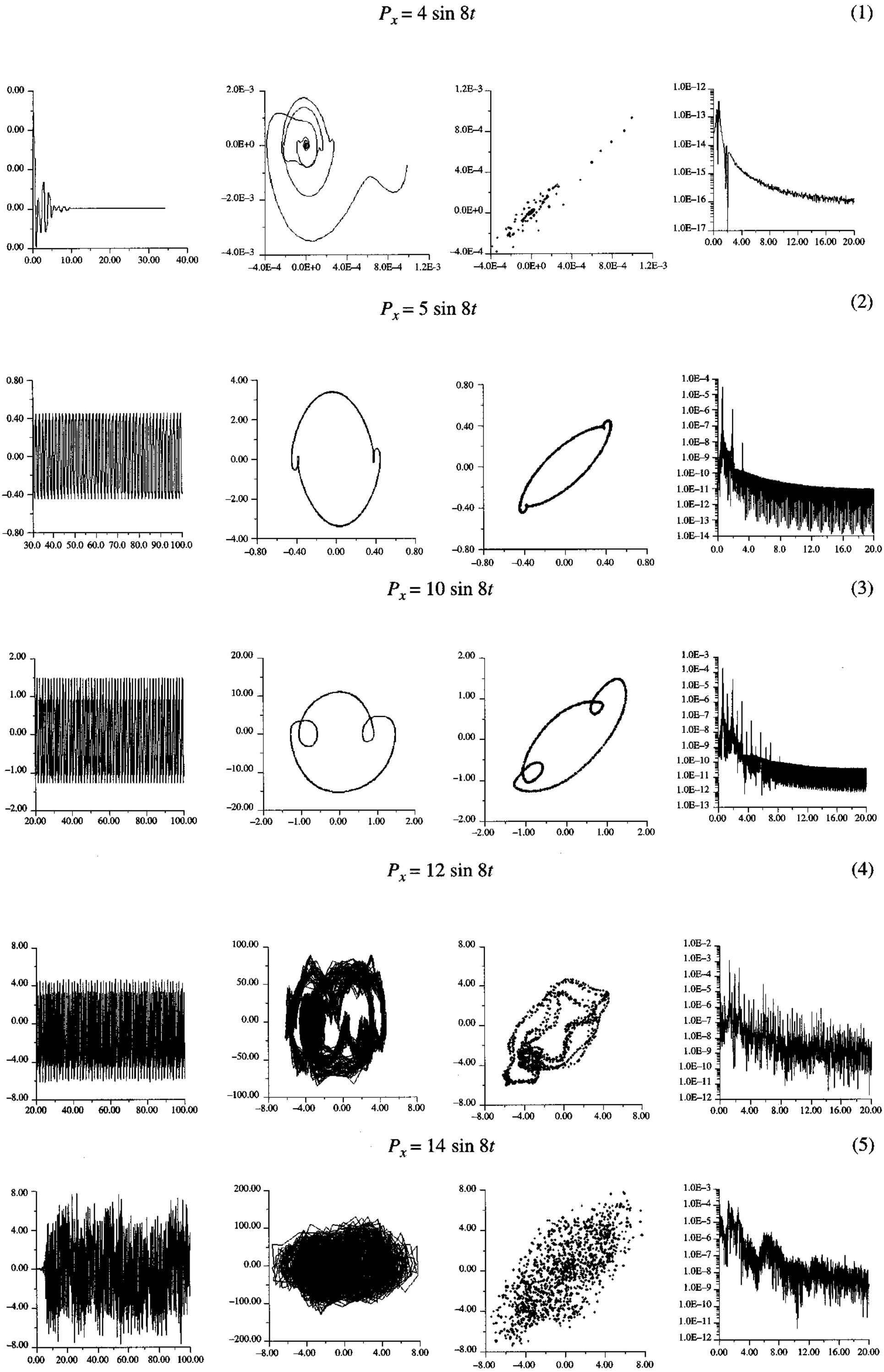
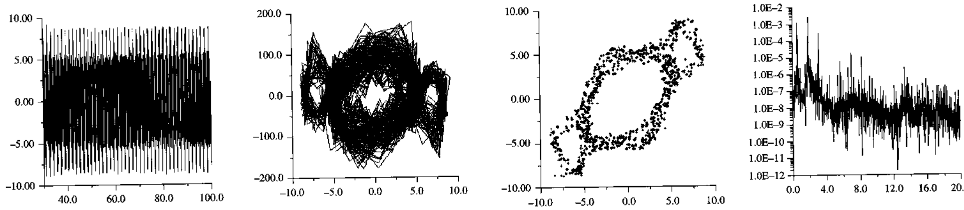
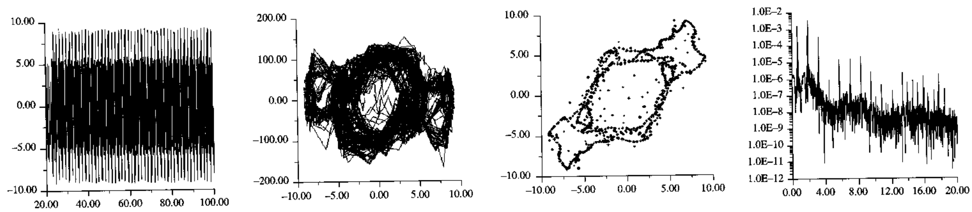


Figure 5. Time histories, phase portraits, Poincaré sections and power spectra for different P_x.

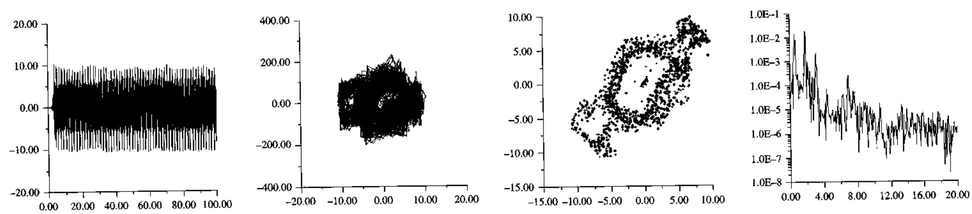
$P_x = 14.7 \sin 8t$ (6)



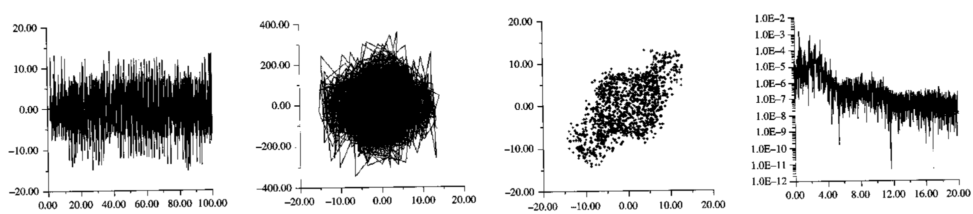
$P_x = 15.1 \sin 8t$ (7)



$P_x = 18 \sin 8t$ (8)



$P_x = 28 \sin 8t$ (9)



$P_x = 30 \sin 8t$ (10)

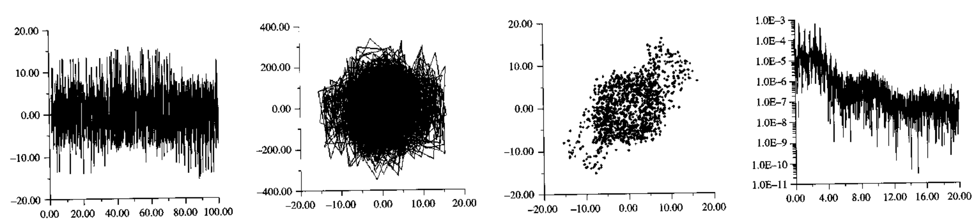


Figure 5. (continued).

Rayleigh number and subscript c denotes its critical value) the fundamental frequency has been significantly increased, as well as the amplitude of subharmonic vibration. A turbulent jump appears when amplitude of subharmonic vibration achieves an order of the amplitude of the fundamental frequency vibrations. After a sudden jump again regular vibration appear.

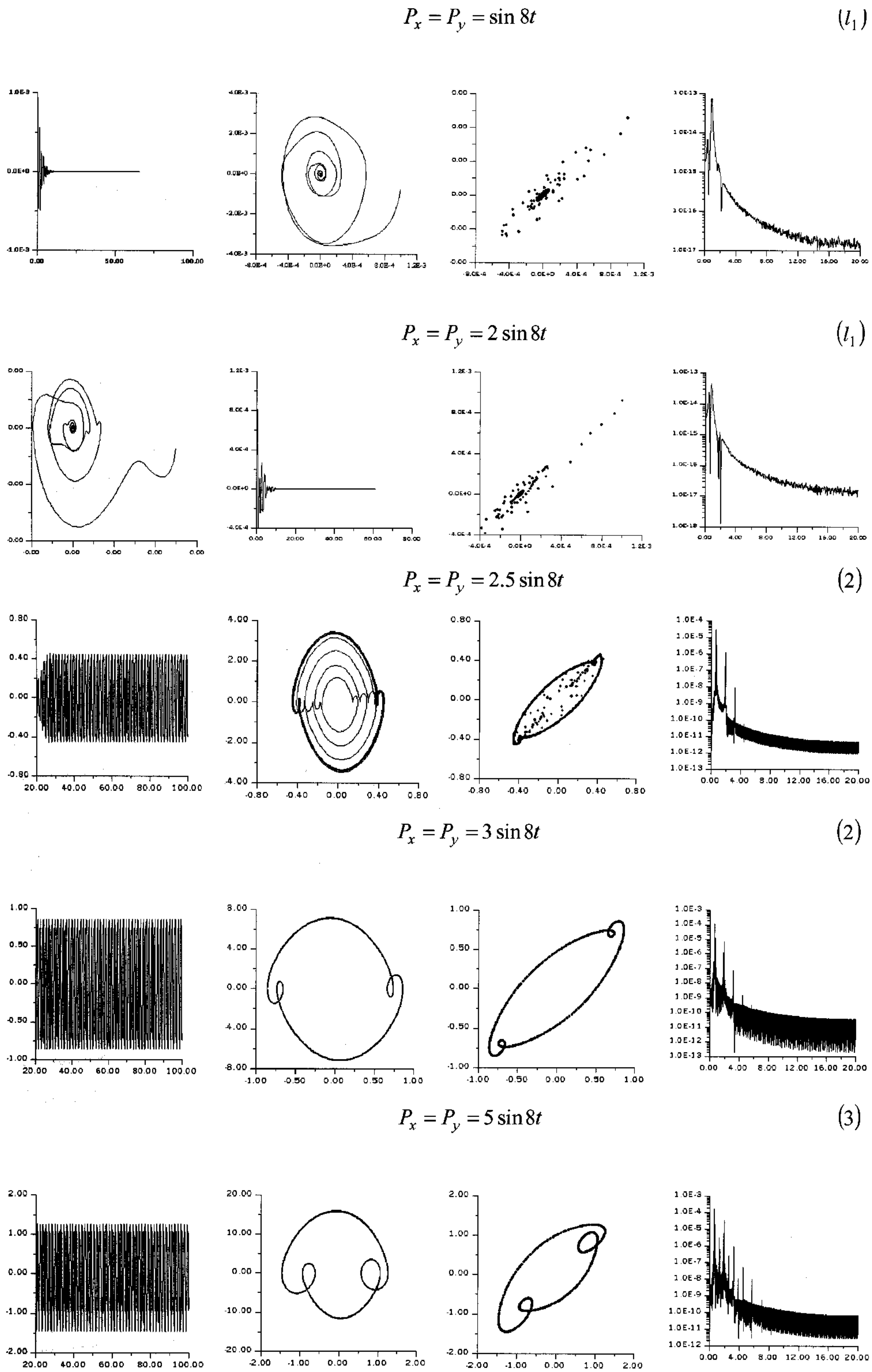
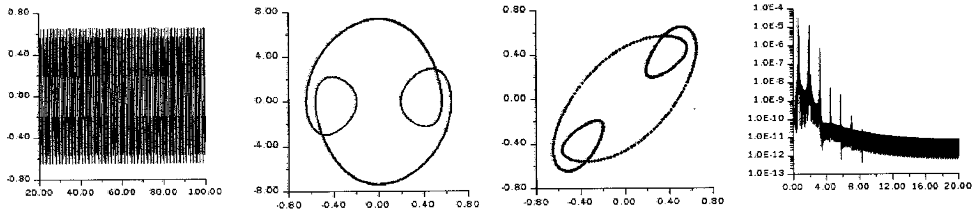
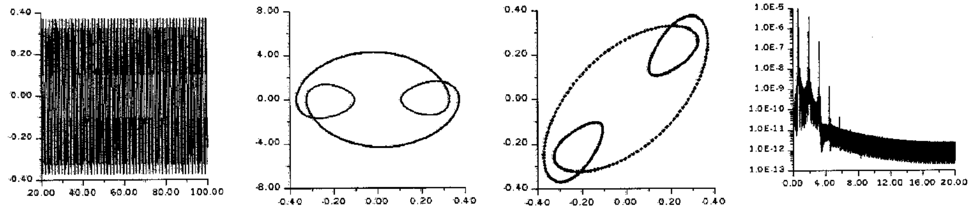


Figure 6. Time histories, phase portraits, Poincaré sections and power spectra for different $P_x = P_y$.

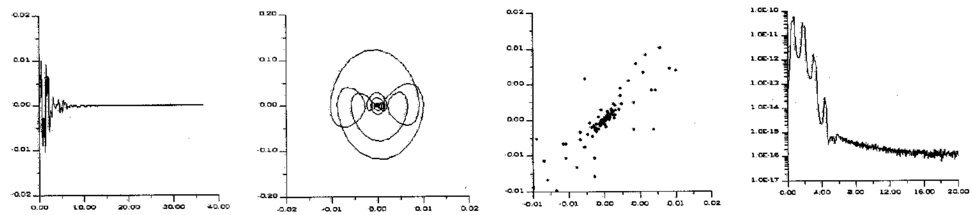
$P_x = P_y = 7\sin 8t$ (2)



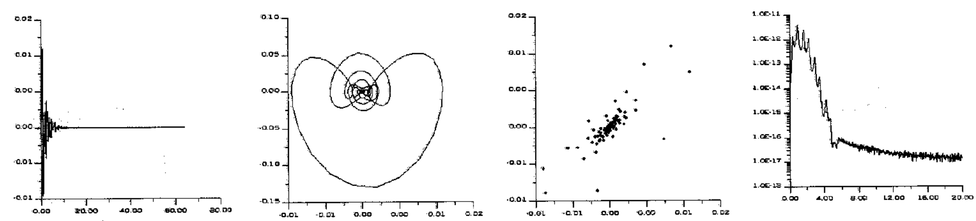
$P_x = P_y = 7.25\sin 8t$ (2)



$P_x = P_y = 7.5\sin 8t$ 1



$P_x = P_y = 7.75\sin 8t$ (l_2)



$P_x = P_y = 8\sin 8t$ (l_2)

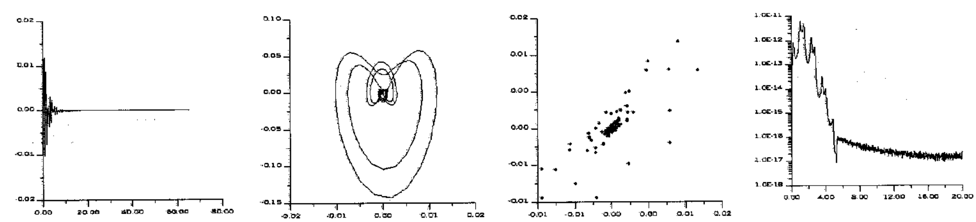
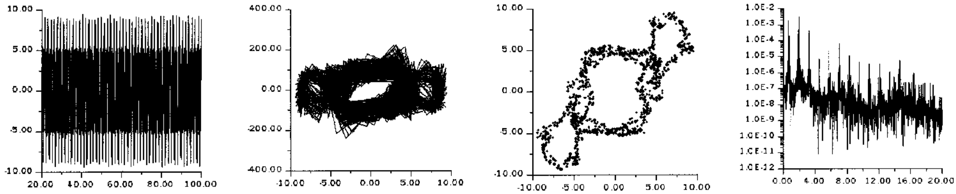
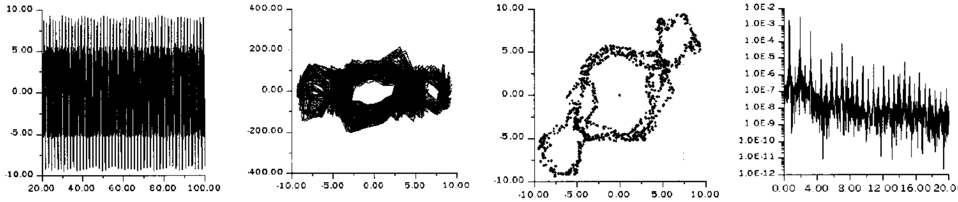


Figure 6. (*continued*).

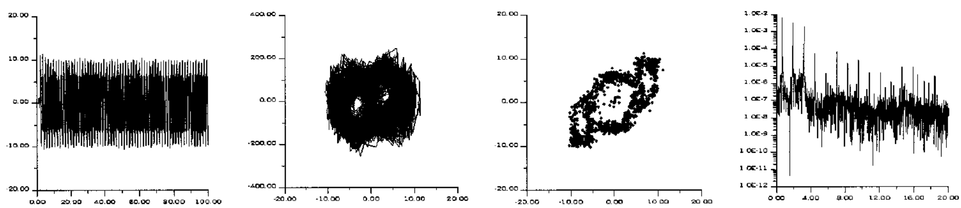
$P_x = P_y = 8.25 \sin 8t$ (2)



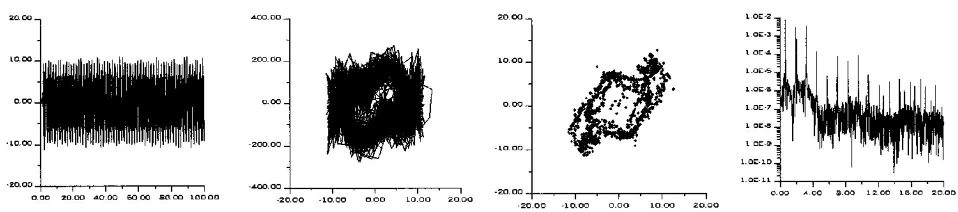
$P_x = P_y = 8.5 \sin 8t$ (2)



$P_x = P_y = 12 \sin 8t$ (4)



$P_x = P_y = 13 \sin 8t$ (7)



$P_x = P_y = 14 \sin 8t$ (8)

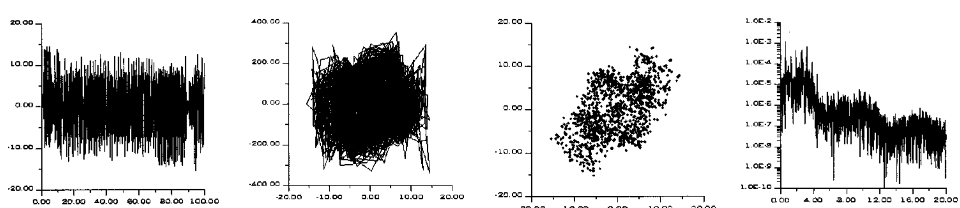


Figure 6. (continued).

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