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Chaos in the three-well potential system

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Abstract

A new approach based on analysis of the wandering trajectories is applied to investigate an appearance of chaotic vibrations in many-well potential systems. The chaotic behavior regions were found in the both amplitude–frequency of excitation and amplitude–damping coefficient plane. The phase plane of initial conditions has been investigated taking into account different values of an external periodic excitation. It demonstrated remarkable agreement with investigations based on homoclinic and heteroclinic bifurcation criteria for chaos, computations of Lyapunov exponents and fractal basin boundaries. The presented technique is very effective, convenient to use, and can be applied to the investigation of a wide class of problems.

Keywords: Chaos; Lyapunov exponents; Wondering trajectories

1. Introduction

There are many prognostic and diagnostic criteria of chaos appearing in non-linear dynamical systems. Some of them were established experimentally and others by numerical simulation. There are also theoretical criteria which in one's turn have reduced to creation of other particular criteria of the chaotic vibrations. At the same time for some comparatively simple equations, the theoretical criteria of appearance of chaos have not been found yet.

Even though the Melnikov's method for the systems in R^n has been presented (Gruendler, 1985), the only periodically forced dynamical systems possessing a small parameter are considered. In addition, for such systems it is necessary to know homoclinic orbit explicitly. But for the most of non-linear dynamical systems, finding an analytical expression of homoclinic orbits is impossible. Even if the analytical expression of homoclinic orbit is known, obtaining of chaotic threshold is still very bulky.

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Thresholds, which correspond to the homoclinic trajectory criterion, are known to be undervalued. Therefore it is necessary to use various theoretical and numerical approaches to confirm a chaotic character of motions in investigated domains.

Li and Moon (1990) have used Melnikov's technique to obtain the bifurcation curves for the following three-well potential oscillator:

$$\ddot{x} + \gamma\dot{x} + x(x^2 - x_0^2)(x^2 - 1) = f\cos\omega t. \tag{1}$$

Beam buckling, deforming of non-linear shells, magnetic dipole engine with many poles, non-linear orbits problems in synchrotron, *etc.* can be related to such problems.

Unlike the two-well potential problem, where the unperturbed homoclinic orbit can be obtained analytically, for comparatively simple Duffing's type equation (1) finding an analytical solution $x_0(t)$ for the homoclinic and heteroclinic orbits (when $\gamma = f = 0$) is impossible. Thus Li and Moon's algorithms of realization of Melnikov's approach for computing the bifurcation conditions for this system are based on numerical simulation.

In this paper, the numerical–analytical approach of investigation of appearance of chaotic vibrations in non-linear dynamical systems is presented. The three-well potential system (1) has been investigated. The obtained domains of chaotic vibrations agree pretty well with the investigations presented by Li and Moon (1990), which based on the homoclinic and heteroclinic bifurcation criteria for chaos, computation of Lyapunov exponents, fractal basin boundaries and experiments.

As it will be shown, our approach is effective, convenient to use, requires tremendously less computational time with compare to other approaches (for example, the standard procedure of computation of Lyapunov exponents) and can be applied to an investigation of a wide class of problems.

In addition, the presented approach can be treated as challenging one to the classical computations of the Lyapunov exponents (see for instance Oseledec, 1968; Wolf et al., 1985; Oden and Martins, 1985; Brogliato, 1996). The applied technique can be used for both smooth (Awrejcewicz et al., 2004) and non-smooth (Awrejcewicz and Dzyubak, in press) dynamical systems.

2. Analysis of the wandering trajectories

A chaotic behavior of non-linear deterministic systems supposes a wandering of trajectories of motion around the various equilibrium states. They are characterized by unpredictability and sensitive dependence on the initial conditions. By analyzing trajectories of motion of these systems, it is possible to find the chaotic vibration regions in control parameters space.

Let us express a dynamical system as the following set of ordinary differential equations:

$$\dot{\mathbf{X}} = f(t, \mathbf{x}), \tag{2}$$

where $\mathbf{x} \in R^n$ is the state vector, $f(t, \mathbf{x})$ is defined in $R \times R^n$ and describing the time derivative of the state vector. $f(t, \mathbf{x})$ is supposed to be smooth enough to guarantee an existence and uniqueness of a solution of the set (2). The right-hand side can be discontinuous while the solution of the set of differential equation (2) remains continuous. The property of continuous dependence on the initial conditions, $\mathbf{x}^{(0)} = \mathbf{x}(t_0)$ of a solution of the set (2) will be used: for every initial conditions $\mathbf{x}^{(0)}$, $\tilde{\mathbf{x}}^{(0)} \in R^n$, for every number $T > 0$ (no matter how large), and for every preassigned arbitrary small $\varepsilon > 0$ it is possible to indicate a positive number $\delta > 0$ such that if the distance ρ between $\mathbf{x}^{(0)}$ and $\tilde{\mathbf{x}}^{(0)}$ $\rho(\mathbf{x}^{(0)}, \tilde{\mathbf{x}}^{(0)}) < \delta$ and $|t| \leqslant T$, the following inequality is satisfied:

$$\rho(\mathbf{x}(t), \tilde{\mathbf{x}}(t)) < \varepsilon.$$

That is if the initial points are chosen close enough, than during the preassigned arbitrary large time interval $-T \leqslant t \leqslant T$, the distance between simultaneous positions of moving points will be less than given positive number ε.

A metric ρ on R^n can be determined in various ways, for example, as $\rho_1(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\sum_{i=1}^{n}(x_i-\tilde{x}_i)^2}$, $\rho_2(\mathbf{x},\tilde{\mathbf{x}}) = \sum_{i=1}^{n}|x_i-\tilde{x}_i|$ or $\rho_3(\mathbf{x},\tilde{\mathbf{x}}) = \max_{1\leqslant i\leqslant n}|x_i-\tilde{x}_i|$, where $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in R^n$, $\tilde{\mathbf{x}} = (\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n) \in R^n$.

It is assumed that with the increase of time the trajectories $\mathbf{x}(t)$ remain in a closed bounded domain of the space R^n, i.e.

$$\exists C_i \in R : \max_t |x_i(t)| \leqslant C_i.$$

To analyze trajectories of the set (2), we introduce the characteristic vibration amplitudes A_i of components of the motion $x_i(t)$ $(i = 1,2\ldots,n)$:

$$A_i = \frac{1}{2}\left|\max_{t_1\leqslant t\leqslant T} x_i(t) - \min_{t_1\leqslant 1\leqslant T} x_i(t)\right| \quad (i = 1,2\ldots,n). \tag{3}$$

Here $[t_1,T] \subset [t_0,T]$ and $[t_0,T]$ is the time interval which the trajectory is considered in. The interval $[t_0,t_1]$ is the time interval which all transient processes are damped in. The characteristic vibration amplitudes A_i can be calculated simultaneously to the integration of the trajectory.

For the sake of our investigations it seems the most convenient to use the embedding theorem and to consider an n-dimensional parallelepiped instead of a hyper-sphere with the center at point $\mathbf{x}$. From the embedding theorem it follows that if $S_\varepsilon(\mathbf{x}) = \{\tilde{x} \in R^n : \rho(\mathbf{x},\tilde{\mathbf{x}}) < \varepsilon\}$ is the hyper-sphere with centre in the point $\mathbf{x}$ and with radius ε and $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x}) = \{\tilde{x} \in R^n : |x_i-\tilde{x}_i| < \varepsilon_i\}$ is the n-dimensional parallelepiped, then for any $\varepsilon > 0$ there is parallelepiped $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x})$ such that $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x}) \subset S_\varepsilon(\mathbf{x})$. And conversely, for any parallelepiped $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x})$ it is possible to indicate such $\varepsilon > 0$ that $S_\varepsilon(\mathbf{x}) \subset P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x})$.

Let us choose such two neighboring initial points $\mathbf{x}^{(0)}$ and $\tilde{\mathbf{x}}^{(0)}$ the parallelepiped $P_{\delta_1,\delta_2,\ldots,\delta_n}(x^{(0)})$ that $|x_i^{(0)}-\tilde{x}_i^{(0)}| < \delta_i$ $(i = 1,2\ldots,n)$. Here δ_i is small in comparison with A_i for $i = 1,2\ldots,n$. In case of regular motion, the ε_i in the inequality $|x_i(t)-\tilde{x}_i(t)| < \varepsilon_i$ is expected to be also small in comparison with A_i for $i = 1,2,\ldots,n$. The wandering orbits attempt to fill some bounded domain of the phase space. Afterwards the neighboring trajectories at the instant t_0 diverge exponentially on the average. Hence, the absolute values of differences $|x_i(t)-\tilde{x}_i(t)|$ for some instant t_1 can take any values in the interval $[0,2A_i]$. If the differences $|x_i(t)-\tilde{x}_i(t)|$ are equal to zero for some instants $\{t_k^*\}$, $(t_k^* \in [t_1,T])$, then the trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ either are intersecting or have points of contact at these instants. Obviously, $2A_i$ are the maximal values for these differences, and for some time instants this value is permissible. Let us introduce an auxiliary parameter α, $0 < \alpha < 1$ and let αA_i be referred to as divergence measures of observable trajectories in the directions of generalized coordinates x_i $(i = 1,2,\ldots,n)$. By analyzing Eq. (2) and its equilibrium states it is easy to choose parameter α, $0 < \alpha < 1$, such that if the following statement:

$$\exists t^* \in [t_1,T] : |x_i(t^*)-\tilde{x}_i(t^*)| > \alpha A_i \quad (i = 1,2,\ldots,n) \tag{4}$$

is satisfied, then there is a time interval (or set of time intervals), for which the representative points of the closed at the initial instant trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ move around various equilibrium states afterwards or these trajectories are sensitive to changing of the initial conditions. Thus these trajectories are wandering.

Indeed as mentioned above, all trajectories are in the closed bounded domain of the space R^n. With the aid of parameter α the divergence measures of the trajectories αA_i have been chosen, which is *inadmissible* for the case of 'regularity' of the motion. Note that this choice is non-unique and the parameter α can take various values in interval $(0,1)$. It is clear, however, that if α is close to 0 and when $|x_i(t)-\tilde{x}_i(t)| < \alpha A_i$ when $t \in [t_0,T]$, then the trajectories do not diverge and the trajectories are regular. There are values of the parameter α, which a priori correspond to inadmissible divergence measures αA_i $(i = 1,2,\ldots,n)$ of the

trajectories in the sense of 'regularity'. For example, $\alpha \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$ or other choices are possible. If the representative points of the observable trajectories move chaotically, then for another choice α from the set of a priori 'appropriate' α, the divergence of the trajectories will be recorded at another time instant t^*. As numerical experiments show, the obtained domains of chaotic behavior with various a priori 'appropriate' values of α are practically congruent. Therefore, in this work figures for different values of α are not presented.

A similar non-unique choice of parameters occurs when applying other criteria for the chaotic oscillations. For instance, according to procedure of calculations of the Lyapunov's exponents. $d(t) = d_0 2^{\lambda t}$. Here λ is Lyapunov exponent, d_0 is the initial distance measure between the starting points and $d(t)$ is the distance between trajectories in instant t. The base 2 is chosen for simplicity. In all other respects the parameter $\alpha > 1$ in the relation $d(t) = d_0 \alpha^{\lambda t}$ is arbitrary. That means α parameter can take values, for example, $\alpha \in \{2, 3, 4, 5\}$ (or other choices are possible). In general, the specificity of numerical approaches is that all parameters have to be concrete and most of them can be non-unique.

Comment that this paper's approach and well-known Wolf's algorithm of determining of the Lyapunov exponents are both realized by computer simulation. According to Wolf's algorithm, the calculation of the Lyapunov exponent λ as the measure of the trajectory divergence begins with the choosing of a basic trajectory $\mathbf{x}^*(t, \mathbf{x}^{(0)})$. At each time step t_k the dynamical system (2) is integrated again with any neighboring points $\mathbf{x}^*(t_k) + \eta$ acting as the initial conditions. Thus, to find λ the governing equation (2) and the corresponding variational equations $\dot{\eta} = \mathbf{A} \cdot \eta$, in which $\mathbf{A}$ is matrix of partial derivatives $\nabla f(\mathbf{x}^*(t_k))$, are solved N times (where N is the number of the time steps). Averaging over a long time results in a reliable value of λ variations of distances between the trajectories. The analogous calculations it is necessary to execute for all nodal points of a sampled space. This procedure is very computationally intensive especially for discontinues systems. The method can often suffice for low-dimensional systems, but in practice, it fails with a chaotic system of high-dimensions. To realize this paper's approach it is enough to solve the equations governing the dynamical system only two times for each selected trajectory. The characteristic vibration amplitudes A_i is calculated simultaneously to the integration of the trajectory. So, for each selected trajectory it do not have to find matrix of partial derivatives $\mathbf{A}$, the equations (2) are solved 2 times instead $2N$ and it do not have to average over a long time the variations of distances between the trajectories.

The parameter α might have another physical interpretation. Lets assume that for the non-linear dynamical system under investigation, it is possible to identify the singular points (equilibria). For instance, in case of two-well potential systems, we have two nodes and one saddle. An external periodic excitation applied to such one-degree-of-freedom system may cause a chaotic response. Chaos is characterized by the unpredictable switches between two potential wells. A phase point may wander between all three singular points. Lets consider two neighboring nodes. As a result of switch, neighboring at the initial instant representative points of the phase trajectories are in motion about various equilibrium states afterward. Hence a choice of α, on the relation $\alpha A_i \cong \frac{1}{2} d$, is related to the distance d between the two nodes separated by a saddle. However, many of non-linear dynamical systems do not have analytical solutions and sometimes it is laborious to find the singular points. This situation occurs very often in non-smooth dynamical systems. In this case it is recommended to take the α parameter from a priori 'appropriate' values.

Our approach has been successfully applied for both smooth and non-smooth systems. By varying parameters and using condition (4), it is possible to find domains of chaotic motion (including transient and alternating chaos) and domains of regular motion.

Remark. All inequalities (4) do not have to be checked for the case, when the equations of motion under investigation can be transformed to a normal form. It means that the inequalities related to velocities $x_j = \dot{x}_i$ may be canceled. In another words, solutions related to regular motion with respect to x_i are also regular in relation to $x_j = \dot{x}_i$. Here $i, j \in \{\overline{1, n}\}$.

3. Investigation of chaos appearance in the three-well potential system

The unperturbed system (1) has five singular points (equilibrium). There exist three sinks, one at the origin $x = 0$ and two at $x = \pm 1$. There are also two saddles at $x = \pm x_0$. In the case of chaotic motion the representative points of phase orbits wander between three attractors in the neighborhoods of $x = 0, \pm 1$.

According to the described technique in Section 2, to find the domains of chaotic behavior for the system (1) the control parameter space have to be sampled. For each nodal point the characteristic vibration amplitudes (3) have to be calculated and the conditions (4) have to be checked. For this system the characteristic vibration amplitudes of the components of the motion x and $\dot{x}$ is introduced:

$$A_x = \frac{1}{2}\left| \max_{t_1 \leqslant t \leqslant T} x(t) - \min_{t_1 \leqslant t \leqslant T} x(t) \right|, \quad A_{\dot{x}} = \frac{1}{2}\left| \max_{t_1 \leqslant t \leqslant T} \dot{x}(t) - \min_{t_1 \leqslant t \leqslant T} \dot{x}(t) \right|,$$

Since Eq. (1) can be transformed to the normal form, according to the remark in Section 2, it is not necessary to check the inequality related to the velocity $\dot{x}$. Therefore for this system only one inequality (corresponding (4)) have to be checked:

$$\exists t^* \in [t_1, T] : |x(t^*) - \tilde{x}(t^*)| > \alpha A_x.$$

If this inequality is satisfied in some nodal point, such motion is relative to chaotic one (including transient and alternating chaos). The manifold of all such nodal points of the investigated control parameter space set up domains where chaotic behavior of the considered system is possible.

Under this condition, the different planes of parameters of Eq. (1) have been investigated. The dynamics of this equation is determined by three parameters f, ω, γ and initial conditions $x(0)$ and $\dot{x}(0)$. In Fig. 1, dots represent the domains of chaotic behavior in the amplitude–frequency of excitation (ω, f) plane for fixed value of parameter $\gamma = 0.1$ and initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0.1$. The time period for the simulation is $\frac{100\pi}{\omega}$ non-dimensional time units. It is accepted that the half of time period corresponds to the time interval $[t_0, t_1]$ in the space where all transient processes are damping. The integration step size is $\frac{\pi}{90\omega}$. The space of parameters is uniformly sampled in rectangle ($0 < \omega \leqslant 2$; $0 < f \leqslant 0.3$) by 60×70 nodal points. Initial conditions of the closed trajectories are distinguished by 0.5% with ratio to characteristic vibration amplitudes A_x and $A_{\dot{x}}$ correspondingly, *e.g.* the starting points of these trajectories are in the 2-dimensional parallelepiped ($|x(t_0) - \tilde{x}(t_0)| < 0.005A_x$, $|\dot{x}(t_0) - \tilde{\dot{x}}(t_0)| < 0.005A_{\dot{x}}$). α parameter is equal 1/3.

The part of (ω, f) plane investigated in the paper (Li and Moon, 1990) is in rectangle bounded by lines ($\omega = 0.6$, $\omega = 1.2$, $f = 0$, $f = 0.16$). Solid line in this rectangle corresponds to the homoclinic bifurcation

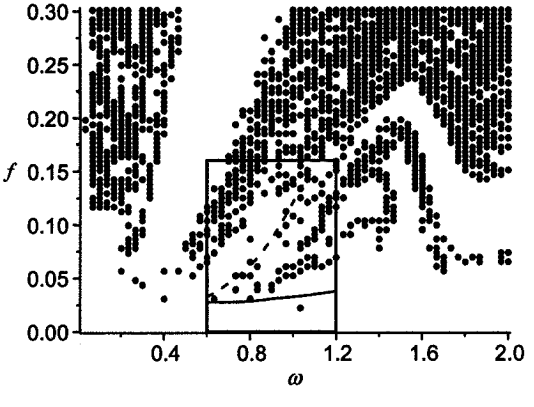

Fig. 1. Domains of chaotic behavior for the three-well potential system in the (ω, f) plane ($\gamma = 0.1$, $x(0) = 0.5$, $\dot{x}(0) = 0.1$).

curve and dash line corresponds to the heteroclinic bifurcation curve. Li and Moon calculated also 100×100 Lyapunov exponents in this part of (ω, f) plane. To find the chaos, the only largest exponent was simulated and every single exponent was averaged over the time period of 650 non-dimensional time units for each motion. The domains of chaotic vibrations, which have been obtained using the approach submitted in the presented paper, are remarkable conforming to domains with positive Lyapunov exponents presented by Li and Moon (1990). Li and Moon have shown that for the three-well problems, two criteria in the parameter space must be met for chaos in contrast to the two-well case where only one criterion is necessary. The simulation results tell that many, but not all parameter values above these bifurcation curves do cause chaotic responses in the system. The higher driving amplitude is, the more likely chaos is found. Thus, both in the case of two- and three-well potential systems the thresholds, which are corresponding to the homoclinic and heteroclinic bifurcation curves, are undervalued.

Fig. 2 presents the chaotic domains in the amplitude–damping coefficient (γ, f) plane with fixed value of parameter $\omega = 0.73$ and initial conditions $x(0) = 0.5, \dot{x}(0) = 0.1$. As in the previous case the time period for the simulation is $\frac{100\pi}{\omega}$ of non-dimensional time units. We have decided that the half of time period corresponds to the time interval $[t_0, t_1]$ in the space where all transient processes are damping. The integration step size is $\frac{\pi}{90\omega}$. The space of parameters is uniformly sampled in rectangular $(0 < \gamma \leqslant 0.5;\ 0 < f \leqslant 0.5)$ by 50×50 nodal points. Initial conditions of the closed trajectories are distinguished by 0.5% with ratio to characteristic vibration amplitudes A_x and $A_{\dot{x}}$ correspondingly, and α parameter is equal to 1/3.

Phase planes of the initial conditions have been analyzed for fixed values of parameters $\gamma = 0.1$, $\omega = 0.714$. For small f, the steady state motion is a periodic limit cycle in one of the three attractors. Fig. 3 shows the phase planes of the initial conditions for different values of the amplitude of excitation: (a) $f = 0.04$ and (b) $f = 0.07$. Depending on the initial conditions, both chaotic and regular motions may appear. The instability peculiar to chaotic vibrations is observed close to the separatrix branches. While the f value is increasing, the domains of chaotic vibrations are fast augmenting. The time period for the simulation is taken as $\frac{100\pi}{\omega}$ non-dimensional time units and the integration step size is equal to $\frac{\pi}{90\omega}$. The space of parameters is uniformly sampled in rectangular $(-1.25 < x(0) \leqslant 1.25;\ -0.5 < \dot{x}(0) \leqslant 0.5)$ by 120×120 nodal points. Initial conditions of the closed trajectories are distinguished by 0.5% with a ratio to characteristic vibration amplitudes A_x, $A_{\dot{x}}$, correspondingly, and α parameter is equal to 1/3.

The investigation of the phase planes of the initial conditions in the case of chaotic dynamics allows observing the fractal structure of basin boundaries. The fractal basin boundaries are the subsidiary means to check a bifurcation.

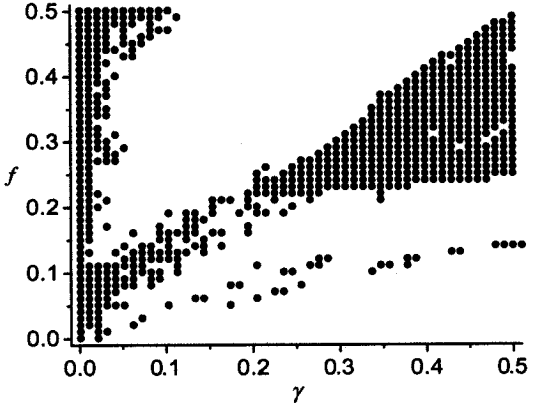

Fig. 2. Domains of chaotic behavior for the three-well potential system in the (γ, f) plane $(\omega = 0.73, x(0) = 0.5, \dot{x}(0) = 0.1)$.

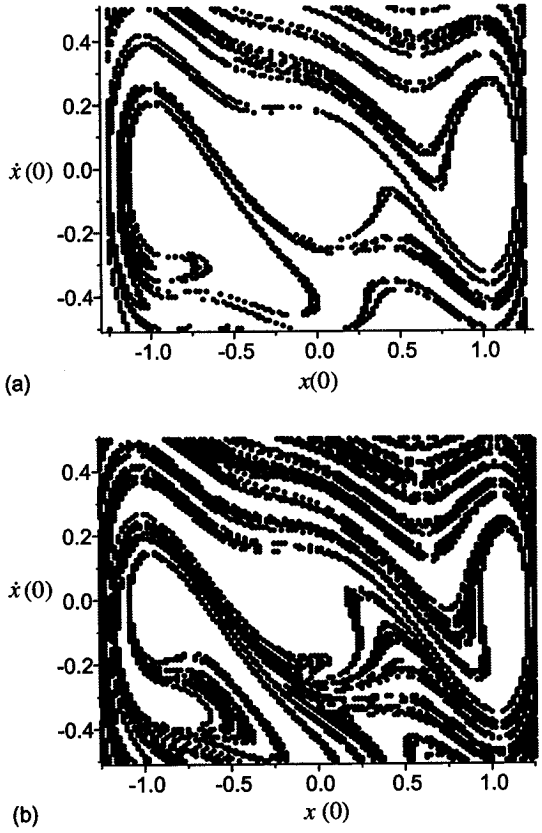

Fig. 3. The phase plane of the initial conditions for the three-well potential system for the different values of the amplitude of excitation: (a) $f = 0.04$; (b) $f = 0.07$ ($\gamma = 0.1$, $\omega = 0.714$).

4. Conclusions

The new approach based on analysis of the wandering trajectories has been applied to investigate the appearance of chaos in three-well potential oscillator. This approach is very effective, not difficult to use and it does not demand too much computer time for calculations. Result's comparison with other investigations demonstrates a very good agreement with other groups. Our approach can obtain multiply connected domains, which are peculiar to chaos and which are characteristics of many experiments based on the chaotic vibrations. The methodology described in this paper can be successfully used to predict a chaos appearance in non-linear dynamical systems with several degree of freedoms as well as in non-linear discontinuous systems.

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