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# A direct numerical method for quantifying regular and chaotic orbits

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**Abstract**

Both a theoretical argument and a numerical algorithm to identify periodic and chaotic orbits are presented and discussed. Reliability of the approach is verified using the Duffing oscillator through the standard computation of Lyapunov exponents. Advantages of the proposed approach are given.


## 1. Introduction

After the identification of chaotic deterministic oscillations in simple three-dimensional physical systems, many problems devoted to measurement, identification and quantitification of chaotic orbits have been developed. The classical approach for studying non-linear dynamical systems is associated with the investigation of periodic or quasi-periodic orbits. Hence, a natural way of analyzing dynamical systems is to take advantage of the already developed tools based on periodic and quasi-periodic orbits. In the case of an isolated periodic orbit, even in a strongly non-linear regime, one may linearize locally the flow to obtain the associated eigenvalues of the monodromy matrix and, in turn, the characteristic multipliers (exponents). Therefore the stability in the sense of Lyapunov and the potential local bifurcations may be predicted with a high accuracy [1,2]. In the case of quasi-periodic orbits, one may use a similar approach by approximating the quasi-periodicity by periodicity with a rather long period, which in a mathematical sense corresponds to approximating the irrational numbers by rational ones [3,4]. Another approach to deal with quasi-periodicity is based on the application of the finite difference method to solve a partial differential equation system in order to calculate numerically a torus [5–7]. It seems that in the case of quasi-periodic orbits stability analysis, the method based on the Lyapunov exponents does not seem to be appropriate [8].

This paper is motivated by the latter observation. Although the concept of Lyapunov exponents has been extended directly to quantify chaotic orbits and a suitable numerical technique for their calculation has been proposed by Wolf et al. [9], we are revisiting the problem and proposing a theoretical argument for a direct numerical method for quantifying regular and chaotic orbits. We also present a numerical algorithm to identify both periodic and chaotic orbits. According to Wolf's algorithm, the calculation of the Lyapunov exponent $\lambda$, as a measure of trajectory divergence, is based on a choice of a base trajectory $\mathbf{x}^*(t, \mathbf{x}^{(0)})$. For each time step $t_k$, the ODE system

$$\dot{\mathbf{x}} = f(t, \mathbf{x}) \tag{1}$$

where $\mathbf{x} \in R^n$ is the state vector, $f(t, \mathbf{x})$ is defined in $R \times R^n$. The vector function $f(t, \mathbf{x})$ describes the time derivative of the state vector (1), and it is integrated by taking $\mathbf{x}^*(t_k)$ and a neighboring point $\mathbf{x}^*(t_k) + \eta$ as the initial conditions.

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Thus, in order to find $\lambda$, Eq. (1) and the corresponding variational equation $\dot{\eta} = \mathbf{A} \cdot \eta$, in which $\mathbf{A}$ is the matrix of partial derivatives $\nabla f(\mathbf{x}^*(t_k))$, are solved $N$ times ($N$—number of time steps), with the requirement that the trajectories should not separate to much. The averaging of the local separations over long times results in a reliable value of $\lambda$. In this paper's approach, as it will be argued next, it is enough to solve the equation set (1) only twice instead of $N$ times for each selected trajectory. This essentially simplifies the procedure of investigation and decreases the computational time. This is important, especially when discontinuous systems are analysed, since the evaluation of solutions for such systems could involve intensive computer calculations.

## 2. Analysis of the wandering trajectories

The chaotic behavior of non-linear deterministic dynamical systems assumes the wandering of trajectories about various unstable equilibrium states. By analyzing such trajectories, one may also characterize the chaotic vibration regions in control parameter space.

A dynamical system may be expressed as a set of ordinary differential equations such as (1). It is assumed, that $f(t, \mathbf{x})$ is smooth enough to guarantee existence and uniqueness of a solution of the set (1). However, the right-hand side can be discontinuous while the solution of the set of differential equations (1) remains continuous (this problem will be considered elsewhere). The continuous dependence of a solution of (1) on the initial conditions $\mathbf{x}^{(0)} = \mathbf{x}(t_0)$ is used here with: for initial condition $\mathbf{x}^{(0)}$, $\tilde{\mathbf{x}}^{(0)} \in R^n$, for every number $T > 0$, no matter how large, and for every preassigned arbitrary small $\varepsilon > 0$, there is a positive number $\delta > 0$ such that if the distance $\rho$ between $\mathbf{x}^{(0)}$ and $\tilde{\mathbf{x}}^{(0)}$ obeys $\rho(\mathbf{x}^{(0)}, \tilde{\mathbf{x}}^{(0)}) < \delta$ for $|t| \leqslant T$, it takes place the inequality

$$\rho(\mathbf{x}(t), \tilde{\mathbf{x}}(t)) < \varepsilon.$$

That is, if the initial points are chosen close enough, then during the preassigned arbitrary large time interval $-T \leqslant t \leqslant T$, the distance between corresponding trajectory points are less than a given positive number $\varepsilon$.

To analyze trajectories of the set (1), we introduce the characteristic vibration amplitudes $A_i$ of components of the motion $x_i(t)$ $(i = 1, 2 \ldots n)$:

$$A_i = \frac{1}{2}\left| \max_{t_1 \leqslant t \leqslant T} x_i(t) - \min_{t_1 \leqslant t \leqslant T} x_i(t) \right| \quad (i = 1, 2 \ldots n).$$

Here $[t_1, T] \subset [t_0, T]$, and $[t_0, T]$ is the time interval in which the trajectory is considered. The interval $[t_0, t_1]$ is the time interval in which all transient processes are damped. The characteristic vibration amplitudes $A_i$ can be calculated during the integration of the trajectory.

The embedding theorem: If $S_\varepsilon(\mathbf{x}) = \{\tilde{\mathbf{x}} \in R'' : \rho(\mathbf{x}, \tilde{\mathbf{x}}) < \varepsilon\}$ is the hyper-sphere with center in point $\mathbf{x}$ and with radius $\varepsilon$ and $P_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i}(\mathbf{x}) = \{\tilde{\mathbf{x}} \in R'' : |x_i - \tilde{x}_i| < \varepsilon_i \forall i = \overline{1, n}\}$ is the $n$-dimensional parallelepiped, then for any $\varepsilon < 0$ there is parallelepiped $P_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i}(\mathbf{x})$ such that $P_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i}(\mathbf{x}) \subset S_\varepsilon(\mathbf{x})$ . And conversely, for any parallelepiped $P_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i}(\mathbf{x})$ it is possible to indicate $\varepsilon < 0$ such that $S_\varepsilon(\mathbf{x}) \subset P_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i}(\mathbf{x})$.

Let us choose in the parallelepiped $P_{\delta_1, \delta_2, \ldots, \delta_i}(\mathbf{x}^{(0)})$ an initial point $\tilde{\mathbf{x}}^{(0)}$, neighbouring to $\mathbf{x}^{(0)}$ such that $|x_i^{(0)} - \tilde{x}_i^{(0)}| < \delta_i$, where $\delta_i$ are small in comparison with $A_i (i = 1, 2 \ldots n)$. In the case of regular motion, it is expected that the $\varepsilon_i$ in the inequality $|x_i(t) - \tilde{x}_i(t)| < \varepsilon_i$ are also small in comparison with $A_i (i = 1, 2 \ldots n)$. The wandering orbits attempt to fill some bounded domain of the phase space. They are characterized by unpredictability and sensitive dependence on the initial conditions. The neighboring trajectories at the instant $t_0$ diverge exponentially on the average afterwards. Hence, the absolute values of differences $|x_i(t) - \tilde{x}_i(t)|$ for some instant $t_1$ can take values in the interval $[0, 2A_i]$. By analyzing the equilibrium states of (1), it is easy to choose an $\alpha$ parameter, $0 < \alpha < 1$, such that from the truth of the statement

$$\exists t^* \in [t_1, T] : |x_i(t^*) - \tilde{x}_i(t^*)| > \alpha A_i \quad (i = 1, 2, \ldots, n). \tag{2}$$

It follows that there is a time interval for which the representative points of the neighbouring trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ move about various equilibrium states, but in principle, these trajectories are sensitive to changes in the initial conditions. Thus, these trajectories are wandering. Indeed, as it has already been mentioned, all trajectories are in the close bounded domain of these space $R^n$. We choose the divergence measure for the trajectories, which is *inadmissible* for the case of 'regularity' of the motion. When the characteristic vibration amplitudes $A_i$ are found, the divergence measures $\alpha A_i$ of the observable trajectories in the directions of the generalized coordinates $x_i$, $i = 1, 2, \ldots, n$ are determined by $\alpha$.

Let us briefly discuss the choice of the parameter $\alpha$. Note that this choice is non-unique and the $\alpha$ parameter can take various values on the interval $(0, 2)$. There are values of the parameter $\alpha$, which in priori correspond to the inadmissible

divergence measures $\alpha A_i (i-1,2,\ldots,n)$ of the trajectories in the sense of regularity. For example, $\alpha \in \{\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4}\}$ but other choices are possible. If the representative points of the observable trajectories move chaotically, then for another choice $\alpha$ from the set of a priori 'appropriate' $\alpha$ the divergence of the trajectories will be recorded at another time $t^*$. As numerical experiments show, the resulting domains of chaotic behaviour with various a priori appropriate values of $\alpha$ parameter are practically congruent. Therefore, in this work, pictures for different values of $\alpha$ are not presented.

A similar non-unique choice of parameters occurs when applying another criteria for the chaotic oscillations. For instance, according to a standard procedure for the calculation of the Lyapunov exponents $d(t) = d_0 2^{\lambda t}$. Here $\lambda$ is Lyapunov exponent, $d_0$ is the initial distance measure between the starting points, and $d(t)$ is the distance between trajectories at instant $t$. The base 2 is chosen for a convenience. In all other respects, the parameter $\alpha > 1$ in the relation $d(t) = d_0 \alpha^{\lambda t}$ is arbitrary. That is, the parameter $\alpha$ can take values, for example, $\alpha \in \{2,3,4,5\}$ but other choices are possible. In general, the specificity of numerical approaches is like that, all parameters have to be concrete and most of them should be non-unique.

The parameter $\alpha$ might have another physical interpretation. Assume that for the non-linear dynamical system under investigation, it is possible to identify the singular points (equilibria). In the case, for instance, of a two-well potential systems we have two nodes and one saddle. An external periodic excitation applied to such one-degree-of-freedom system may cause a chaotic response. The unpredictable switches between the two potential wells can be the source of chaos. A phase point may wander between all three singular points. Consider two neighboring nodes. In this case, a choice of $\alpha$, due to formula $\alpha A_i \sim (1/2)d$, is related to the distance $d$ between the two nodes separated by the saddle. However, many of the non-linear dynamical systems do not have analytical solutions, and sometimes it is laborious to find the singular points. This situation occurs often in non-smooth dynamical systems. In this case, the it is recommended to take the $\alpha$ parameter from a priori appropriate values.

In fact, our approach has been successfully applied in the case of smooth and non-smooth systems. By varying parameters and using condition (2), it is possible to find domains of chaotic motion (including transient and alternating chaos) and domains of regular motion.

**Remark.** All inequalities (2) do not have to be checked for the case when the equations of the motion under investigation can be transformed to a normal form. It means that the inequalities related to velocities $x_j = \dot{x}_i$ may be canceled. In another words, solutions related to regular motion with respect to $x_i$ are also regular in relation to $x_j = \dot{x}_i$. Here $i,j \in \{\overline{1,n}\}$.

## 3. Investigation of the appearance of chaos in smooth dynamical systems using the Duffing equation example

Let us consider the non-autonomous Duffing equation:

$$\ddot{x} + \gamma\dot{x} - \frac{1}{2}x(1-x^2) = f\cos\omega t. \tag{3}$$

For this system, the condition (2) has the form:

$$\exists t^* \in [t_1, T] : |x(t^*) - \tilde{x}(t^*)| > \alpha A.$$

Using this condition, different planes of parameters of the Duffing equation are investigated. The dynamics of this equation is determined by three parameters $f$, $\omega$, $\gamma$, and by the initial conditions $x(0)$ and $\dot{x}(0)$. In Fig. 1 (a), dots represent the domains of chaotic behavior in the amplitude—frequency of excitation $(\omega, f)$ plane for fixed value of the parameter $\gamma = 0.15$ and the initial conditions $x(0) = 0.1$, $\dot{x}(0) = 0.01$. The length of the simulation is $100\pi/\omega$ in non-dimensional time units. We choose half this time to correspond to the time interval $[t_0, t_1]$ in which all transient processes are damped. The integration step size is $\pi/30\omega$. The space of parameters $(0 < \omega \leqslant 1.15;\ 0 < f \leqslant 0.55)$ is uniformly sampled in a rectangular grid using $40 \times 40$ nodal points. Initial conditions of neighboring trajectories differ from each other by an amount of 0.5% with respect to a characteristic vibration amplitude $A$, that is $|x(0) - \tilde{x}(0)| \approx 0.005A$, and $\alpha = 1/3$.

The domains of chaotic behavior agree well with the smooth threshold which corresponds to the homoclinic trajectory criterion [10]. The domains also agree remarkably well with the results of the investigations based on the calculation of the Lyapunov exponents, which was carried out using the Wolf's algorithm [9–11].

Fig. 1 (b) on the other hand, presents the chaotic domains for the Eq. (3) in the amplitude—damping coefficient $(\gamma, f)$ plane with fixed value of the parameter $\omega = 1.7$ and initial conditions $x(0) = 0.1$, $\dot{x}(0) = 0.01$. The length of the simulation is $100\pi/\omega$ in non-dimensional time units. We consider half this time to correspond to the time interval $[t_0, t_1]$

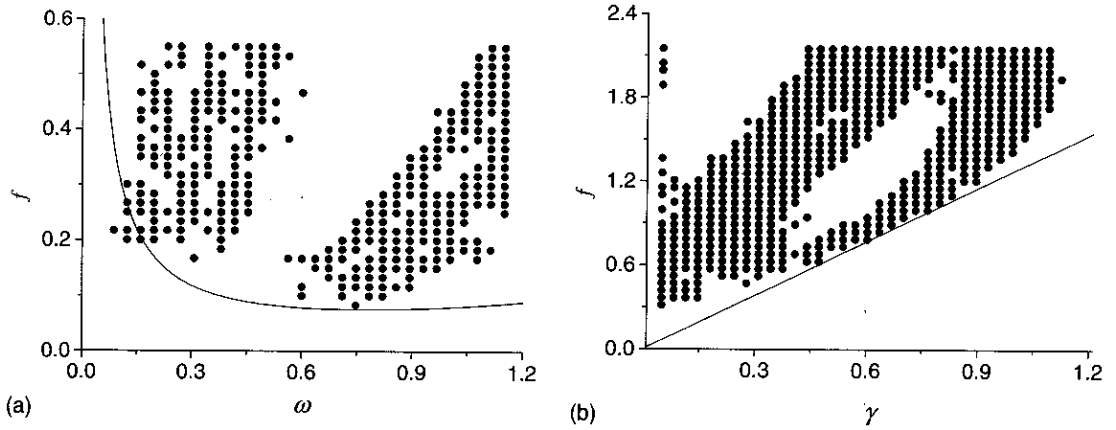


Fig. 1. Domains of chaotic behavior for the Duffing equation ($\omega = 1.7$, $x(0) = 0.1$, $\dot{x}(0) = 0.01$): (a) in the $(\omega, f)$ plane ($\gamma = 0.15$, $x(0) = 0.1$, $\dot{x}(0) = 0.01$); (b) in the $(\gamma, f)$ plane. The smooth threshold corresponds to the homoclinic trajectory criterion.

in which all the transient processes are damped. The integration step is equal to $\pi/20\omega$, the space of parameters is uniformly sampled in the rectangle $(0 < \gamma \leqslant 1.35;\ 0 < f \leqslant 2.15)$ using $40 \times 40$ nodal points. Initial conditions of neighboring trajectories differ by an 0.5% with respect to the characteristic vibration amplitude $A$, that is $|x(0) - \tilde{x}(0)| \approx 0.005A$, and $\alpha = 1/3$.

The resulting chaotic domains agree with the smooth threshold corresponding to the homoclinic trajectory criterion [10]:

$$f > \frac{4}{3}\gamma \frac{ch(\pi\omega)}{\sqrt{2}\pi\omega}.$$

(In the case of Eq. (3), a change of variables was introduced). The real boundary of chaotic oscillations in the plane $(\gamma, f)$ is not linear and qualitatively corresponds to that found by our numerical method. For both cases (a) and (b), the chaotic domains are multiply connected.

Phase planes of the initial conditions have been analyzed for Eq. (3) for fixed values of the parameters $\gamma = 0.15$, $\omega = 0.8$. Fig. 2 shows the space of initial conditions for different values of the amplitude of excitation: (a) $f = 0.06$ and (b) $f = 0.1$. Depending on the initial conditions, both chaotic and regular motion might appear. The instability, peculiar to chaotic vibrations, is observed close to the separatrix branches. The domains of chaotic vibrations increase rapidly with $f$ values. In this simulation, its length is $100\pi/\omega$ in non-dimensional time units. The integration step size is

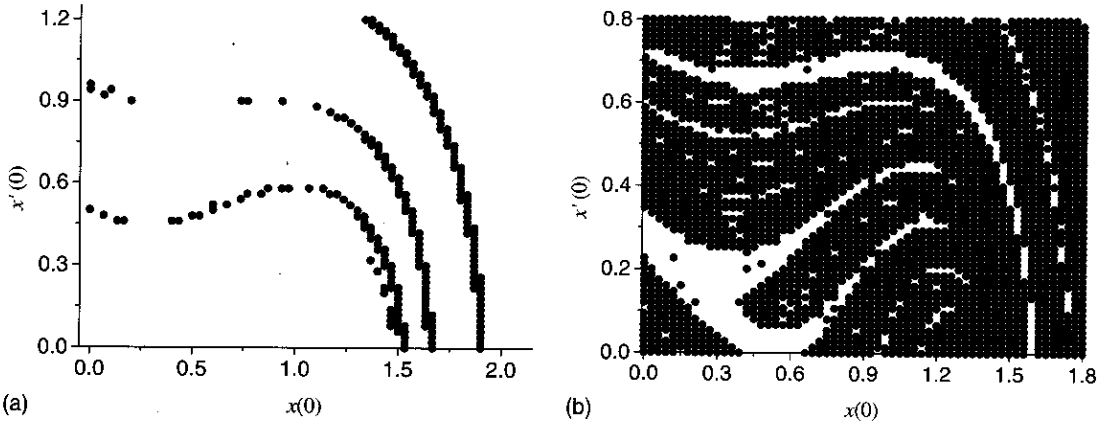


Fig. 2. The initial conditions phase plane for the Duffing equation for different values of the amplitude of excitation ($\gamma = 0.15$, $\omega = 0.8$): (a) $f = 0.06$; (b) $f = 0.1$.

$\pi/20\omega$. The space of parameters is uniformly sampled in the rectangular region $[0 < x(0) \leqslant 1.8;\ 0 < \dot{x}(0) \leqslant 0.8]$ using a $40 \times 40$ grid. Initial conditions of neighboring trajectories are distinguished by 0.5% with respect to the characteristic vibration amplitude $A$, that is, $|x(0) - \tilde{x}(0)| \approx 0.005A$, and $\alpha = 1/3$.

## 4. Conclusions

We presented a method for computation of chaotic regions in parameter space that does not use explicitly the standard Lyapunov exponent approach. The validity of our approach and accuracy of the method has been verified using Duffing equation and comparing our results with the classical one using the Wolf et al. algorithm for Lyapunov exponents. Our results agree very well with those obtained by the method of Wolf's et al., but our approach requires tremendously less computational time, since the ODEs are solved only two times, instead of $2N$ times, where $N$ denotes number of time steps. In addition, an arbitrary choice of the parameter does not require for the neighboring trajectories to diverge in an exponential way. In another words, our approach simplifies the procedure of investigating dynamical systems in comparison with that of Wolf et al.

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