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Quasi-Fractional Approximation of Solution to Non-Autonomous Duffing's Equation

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Abstract

A regular perturbation procedure and quasi-fractional approximants are used for analytical construction of a homoclinic orbit for a Duffing's equation.

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1. Introduction

Many papers and books, see Drazin (1992), Guckenheimer and Holmes (1983), Holmes (1980), Jordan and Smith (1999), Melnikov (1963), Nayfeh and Balachandran (1995), Sanders (1982), Smith and Yorke (1992), Wiggins (1990), Wiggins and Holmes (1987), Smith (1998), Vakakis and Azeez (1998), Vakakis (1994), Mikhlin (1985), Mikhlin (1995), Mikhlin (2000), Gelfreich and Lazutkin (2001) and references therein, are devoted to analytical construction of homoclinic orbits. During investigation of the splitting of a separatrix usually Melnikov's technique is used (Melnikov, 1963). In order to construct a solution for a driven oscillator various averaging procedures (Sanders, 1982, Gelfreich and Lazutkin, 2001), regular perturbation approaches with certain modifications (Drazin, 1992, Guckenheimer and Holmes, 1983, Holmes, 1980, Jordan and Smith, 1999, Nayfeh and Balachandran, 1995, Smith and Yorke, 1992, Wiggins, 1990, Wiggins and Holmes, 1987, Vakakis, 1994), multi-scale techniques (Smith, 1998), as well as Padé or quasi-fractional approximants (Vakakis, 1994, Mikhlin, 1985, Mikhlin, 1995, Mikhlin, 2000) are applied.

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While applying the mentioned methods, a fundamental problem related to non-homogeneous asymptotic series appears resulting finally in the occurrence of terms either similar to singular ones or exponentially small ones.

In this paper we discuss the mentioned drawbacks and methods to avoid them that are based on making use of analytical approaches. It is to be accomplished on an example of Duffing's equation with negative stiffness

$$\ddot{x} - x + x^3 = \varepsilon F \sin(\omega\eta) \tag{1}$$

where: $\eta = t - t_0$, t denotes time, whereas t_0 denotes the initial time moment.

The existence of a homoclinic orbit and its splitting was rigorously mathematically proved by Gelfreich and Lazutkin (2001).

2. Vakakis' solution

Vakakis (1994) obtained an analytical series for a separatrix to equation (1) in relation to ε. Now, some of the key points of the Vakakis approach will be presented.

The solution being sought can be presented in the form (for both $t \to \pm\infty$ we have an analogous solution)

$$x = \sqrt{2}\sec h\eta + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \tag{2}$$

After splitting with respect to ε, the following recurrent system of equations is obtained

$$\varphi(x_1) \equiv \ddot{x}_1 + 6\sec h^2(\eta x_1) = F\sin(\omega\eta) \tag{3}$$

$$\varphi(x_2) = -3\sqrt{2}\sec h(\eta x_1^{\ 2}) \tag{4}$$

$$\varphi(x_3) = 6\sqrt{2}\sec h\eta x_1 x_2 - x_1^{\ 3} \tag{5}$$

. . . .

The solution to the first order approximation equations has the form

$$x_1 = -F\left[\left(\int_0^\eta x_1^{(2)}(\eta)\sin(\omega\eta)d\eta\right)x_1^{(1)}(\eta) - \left(\int_\eta^\infty x_1^{(1)}(\eta)\sin(\omega\eta)d\eta\right)x_1^{(2)}(\eta)\right] \tag{6}$$

where

$$x_1^{(1)}(\eta) = \sin\eta/\cos^2\eta \tag{7}$$

$$x_1^{(2)} = x_1^{(1)}(\eta)\left[\frac{3}{2}\eta + \frac{1}{4}\sinh(2\eta) - cth\,\eta\right] \tag{8}$$

It is not difficult to check that expression (6) approaches a periodic solution for $t \to \infty$.

For rapid changes of the excitation ($\omega = \omega_0 / \varepsilon,\ \ \omega_0 \sim \text{const}$), solution (6) can be simplified to the form

$$x = \frac{\varepsilon^2 F}{\omega_0^2}\cos(\omega_0 t_0 / \varepsilon) x_1^{(1)}(\eta) - \frac{\varepsilon^3 F}{\omega_0}\sin(\omega_0 (t + t_0)/\varepsilon) + 0(\varepsilon^4) \tag{9}$$

3. Analysis of the solution

Solution (6), (7), (8) can be presented in the form

$$\begin{aligned} x_1 &= \sum_{j=1}^{\infty} \exp(-j\eta)\left[C_j^{(1)}\sin(\omega\eta) + C_j^{(2)}\cos(\omega\eta)\right] + \\ &+ \underline{t\sum_{j=1}^{\infty} \exp(-j\eta)\left[D_j^{(1)}\sin(\omega\eta) + D_j^{(2)}\cos(\omega\eta)\right]} \equiv x_{11} + x_{12} \end{aligned} \tag{10}$$

where: $C_j^{(1)}$, $C_j^{(2)}$, $D_j^{(1)}$, $D_j^{(2)}$ are certain coefficients.

The underlined terms are referred to as the so called "singular terms". As has been mentioned by Smith (1998), who gave some examples of such terms: "Generally, regular perturbations lead to the appearance of what might be called "secular term" such as te^{-t} in the language of periodic solutions. Such terms are not present in the exact solution. The secular terms control the slope of the separatices in the neighborhood of the saddle point."

The mentioned drawbacks can be omitted using the Poincaré-Lindstedt-Lighthill method, see Nayfeh and Balachandran (1995). The general idea of the mentioned approaches is as follows.

If, in result of application of the singular perturbation method, the following solution is obtained

$z \approx A\exp(-\alpha t) + \underline{B\varepsilon t\exp(-\alpha t)}$,

then it can be presented in the following asymptotically equivalent form

$z \approx A\exp[(-\alpha + B\varepsilon / A)t]$,

without the earlier underlined singular term.

In our case, in order to omit non-homogeneity, the following can be applied

$$\sqrt{2}\sec h\eta = \sum_{j=1}^{\infty} A_j \exp(-j\eta) \tag{11}$$

where: A_j are some coefficients.

In the next step, the solution of zero order approximation (11) and the underlined term of the first order approximation to the solution to (10), x_{12} can be substituted by the following asymptotically equivalent expression:

$$x_{01} = \sum_{j=1}^{\infty} A_j \exp\left[\left(-j + \varepsilon\left(D_j^{(1)} \sin(\omega\eta) + D_j^{(2)} \cos(\omega\eta)\right)\right)\eta\right] \tag{12}$$

where: $D_j^{(1)} = \widetilde{D}_j^{(1)} / A_j$, $D_j^{(2)} = D_j^{(2)} / A_j$.

In result of the introduced changes, solution (12) does not include the singular terms, and the general first order approximation solution can be presented in the following form

$x \approx x_{01} + \varepsilon x_{11}$.

It should be emphasized that a similar approach can be used also during construction of separatices of autonomous dynamical systems being analysed, see Smith (1998) for instance.

4. Quasifractional approximants

The solution of zero order approximation $\sqrt{2}\sec h(t)$ decreases in time for $|t| \to \infty$ in a manner governed by the decaying term $\exp(-|t|)$. The applicability domain of the solution is defined by $|t| < \ln\varepsilon^{-1}$. If $|t| > \ln\varepsilon^{-1}$, the following equation is obtained in the first approximation

$\ddot{\widetilde{y}}_0 - \widetilde{y}_0 = \varepsilon F \sin \omega t,$

with the corresponding solution

$$\widetilde{y}_0 = \frac{-\varepsilon F}{\omega^2 + 1} \sin \omega t.$$

Therefore, a typical situation occurs in the two-scale approach. Namely, for different interval changes of the independent variable there exist different analytical expressions, see for instance Smith (1998). A uniform solution can be obtained by asymptotical matching of the series (Smith, 1998). Contrary to this approach, we use a procedure of two point quasi-fractional approximants, proposed by Martin and Baker (1991), see also Andrianov and Awrejcewicz

(2001), Awrejcewicz et al. (1998). Now, we briefly define quasi-fractional approximants. Let us suppose that we have a perturbation approach in powers of ε for $\varepsilon \to 0$ and asymptotic expansions $F(\varepsilon)$ containing, for example, logarithm for $\varepsilon \to \infty$. By definition, a quasi-fractional approximant is a ratio R with unknown coefficients: a_i, b_i, containing both powers of ε and $F(\varepsilon)$. The coefficients a, b are chosen in such a way that (a) the expansion of R in powers of ε match the corresponding perturbation expansion; and (b) the asymptotic behaviour of R for $\varepsilon \to \infty$ coincides with $F(\varepsilon)$.

According to the last method we find the following solution for $t \to +\infty$ (a similar solution is obtained for $t \to -\infty$)

$$y \approx \frac{2\sqrt{2}e^{-t} - \varepsilon\alpha \sin \omega t}{1 + e^{-2t}} \tag{13}$$

where $\alpha = \dfrac{F}{\omega^2 + 1}$.

The obtained solution, (13), is valid for arbitrary values of ω. An account of higher approximations is related to the construction of the successive approximations in the intervals: I ($|t| < \ln \varepsilon^{-1}$) and II ($|t| > \ln \varepsilon^{-1}$) and their quasi-fractional matching.

5. Concluding remarks

The presented approach leads to the obtaining of uniformly suitable analytical series with respect to ε for separatices of the Duffing equation with negative stiffness.

In the next step, following this research direction, one may develop a construction of a solution for relatively large values of ε.

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