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Stick-Slip Chaotic Oscillations in a Quasi-Autonomous Mechanical System

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Abstract

The results of the investigations of occurrence of a chaotic stick-slip and slip-slip motion in a very weakly forced oscillator, using both the Melnikov's technique and an approach based on the analysis of the wandering trajectories, are compared. A good agreement of the analytical chaotic threshold and numerical simulation are demonstrated.

Keywords: chaos, Melnlkov's method, Poincaré maps, Lyapunov exponents.

1 Introduction

It is well known that chaotic dynamical systems may be quantified using Lyapunov exponents in both smooth [1, 2] and non-smooth [3, 4] cases. In [5] two key pointS of investigation were addressed by the authors. Namely, a new numerical method to trace regular and chaotic domains of any nonlinear system governed by ordinary differential equations was proposed. The approach introduced is tested using Duffing and Lorenz type oscillators, as well as a non-smooth two degrees-of-freedom mechanical system with friction.

It has been shown, among other results, that our method is much simpler and faster then the well known Wolf's algorithm.

In the paper [6] the Melnikov's method was applied to a non-smooth one degree-of-freedom very weakly forced (quasi-autonomous) oscillator to predict onset of the stick-slip and slip-slip chaos. It was shown using an analytical approach that it is possible to predict stick-slip chaos using extremely small external forcing. The critical chaotic threshold points, where infinitely small external periodic perturbations applied to an autonomous system yields chaos, has been found. Numerical simulation and Poincaré maps have confirmed the validity of the approach. The analytical results obtained allow one to analyze and control the stick-slip chaotic dynamics.

In the present paper we have obtained domains (depending on the parameters) of the stick-slip chaotic dynamics of the oscillator referred to above using a technique based on the analysis of wandering trajectories developed in [5]. The analytical and numerical results are compared and discussed. We succeeded in obtaining a more precise estimate then the analytical prediction proposed in reference [6].

2. Stick-Slip Oscillator with Periodic

Consider a mechanical system which consist of a mass m riding on a driving belt (as shown in Fig. 1). The belt is moving with a constant velocity v_*. The mass m is attached to inertial space by a Duffing type spring, where k_1 and k_2 are stiffness coefficients. A friction force θ, which depends on the relative velocity, acts between the mass m and belt. Additionally, the mass m is forced by a small periodic external excitation $\Gamma\cos\omega t$. Γ and ω are the amplitude and frequency of excitation, respectively. The one degree-of-freedom stick-slip oscillations are governed by the following second-order differential equation

$$m\ddot{x} - k_1 x + k_2 x^3 = \varepsilon[\Gamma\cos\omega t - \theta(\dot{x} - v_*)],$$

where $\varepsilon > 0$ is the perturbation parameter,

$$\theta(\dot{x} - v_*) = \theta_0 \operatorname{sign}(\dot{x} - v_*) - A(\dot{x} - v_*) + B(\dot{x} - v_*)^3,$$

and it corresponds to the ratio of the friction characteristic to the relative velocity. This friction model is presented in Fig. 2. θ_0, A, and B are friction coefficients.

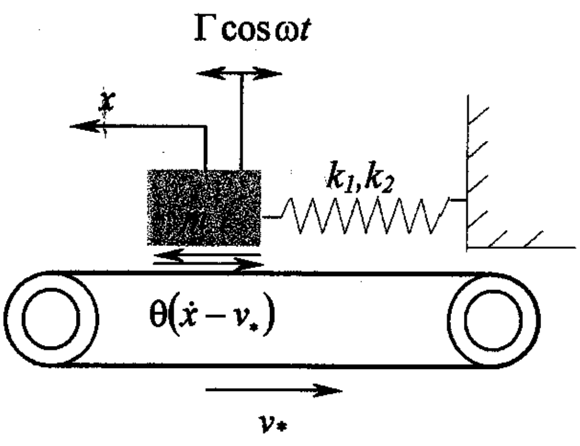

Figure 1. The one degree-of-freedom mechanical system with stick-slip oscillations.

It is possible to rewrite this equation in the dimensionless form

$$\ddot{x} - ax + bx^3 = \varepsilon[\gamma\cos\omega t - T(\dot{x} - v_*)] \qquad (1)$$

where:

$$T(\dot{x} - v_*) = T_0 \operatorname{sign}(\dot{x} - v_*) - \alpha(\dot{x} - v_*) + \beta(\dot{x} - v_*)^3,$$

and $a = k_1/m$, $b = k_2/m$, $\gamma = \Gamma/m$, $T = \theta/m$, $T_0 = \theta_0/m$, $\alpha = A/m$, $\beta = B/m$.

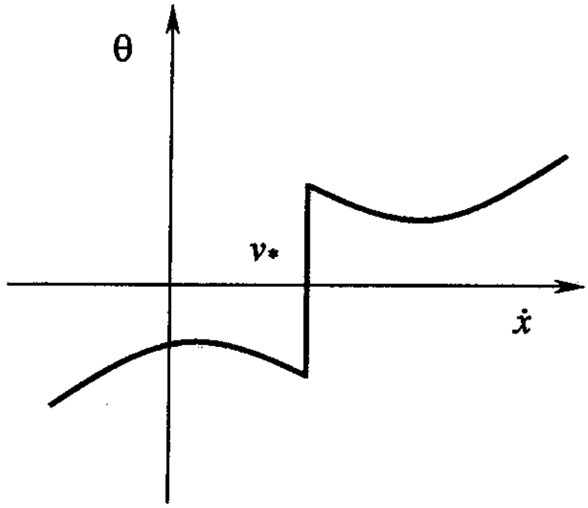

Figure 2. Friction model.

In the paper [6] the Melnikov function for mechanical system (1) has been obtained. Then after computing the corresponding integrals, the Melnikov criterion has been applied to obtain the following inequality:

$$\pi\gamma\omega\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right) > \frac{16}{35}\beta\frac{a^4}{b\sqrt{2ab}} - \frac{4}{3}\left(\alpha - 3\beta v_*^2\right)\frac{a^2}{\sqrt{2ab}} + \\ + \begin{cases} 2T_0 a\left[\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{b}{2a^2}v_*^2}} - \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{b}{2a^2}v_*^2}}\right] & \text{for } v_* < \frac{a}{\sqrt{2b}}, \\ 0 & \text{for } v_* \geq \frac{a}{\sqrt{2b}}. \end{cases} \qquad (2)$$

This inequality gives the possibility of chaotic threshold estimation in the forced stick-slip oscillator (1).

3. Analysis of the Wandering Trajectories

The chaotic behavior of nonlinear deterministic systems supposes that the trajectories of motion wander around the various equilibrium states. They are characterized by unpredictability and sensitive dependence on the initial conditions.

various equilibrium states. They are characterized by unpredictability and sensitive dependence on the initial conditions. By analyzing trajectories of motion of these systems, it is possible to find the regions of chaotic vibrations in control parameters space.

Let a dynamical system be expressed as the following set of ordinary differential equations

$$\dot{\mathbf{x}} = f(t,\mathbf{x}), \qquad (3)$$

where $\mathbf{x} \in R^n$ is the state vector, $f(t,\mathbf{x})$ is defined in $R \times R^n$ and describes the time derivative of the state vector. It is supposed, that $f(t,\mathbf{x})$ is smooth enough to guarantee existence and uniqueness of a solution of the equations (3). The right-hand side can be discontinuous while the solution of the set of differential equations (3) remains continuous. The continuous dependence property on the initial conditions $\mathbf{x}^{(0)} = \mathbf{x}(t_0)$ of a solution of the equations (3) will be used: for every initial conditions $\mathbf{x}^{(0)}$, $\tilde{\mathbf{x}}^{(0)} \in R^n$, for every number $T > 0$, no matter how large, and for every preassigned arbitrary small $\varepsilon > 0$ it is possible to find a positive number $\delta > 0$ such that if the distance ρ between $\mathbf{x}^{(0)}$ and $\tilde{\mathbf{x}}^{(0)}$ $\rho(\mathbf{x}^{(0)}, \tilde{\mathbf{x}}^{(0)}) < \delta$ and $|t| \le T$, the following inequality is satisfied

$$\rho(\mathbf{x}(t), \tilde{\mathbf{x}}(t)) < \varepsilon .$$

That is if the initial points are chosen close enough, than during the preassigned arbitrary large time interval $-T \le t \le T$ the distance between simultaneous positions of moving points will be less given positive number ε.

The metric ρ on R^n can be determined in various ways, for example

$$\rho_1(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\sum_{i=1}^{n}(x_i - \tilde{x}_i)^2}, \quad \rho_2(\mathbf{x},\tilde{\mathbf{x}}) = \sum_{i=1}^{n}|x_i - \tilde{x}_i|$$

or $\rho_3(\mathbf{x},\tilde{\mathbf{x}}) = \max_{1 \le i \le n}|x_i - \tilde{x}_i|$, where

$$\mathbf{x} = (x_1, x_2, ..., x_n) \in R^n, \quad \tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n) \in R^n .$$

It is assumed that the trajectories $\mathbf{x}(t)$ remain in a closed bounded domain of the space R^n, i.e.,

$$\exists C_i \in R : \max_t |x_i(t)| \le C_i .$$

To analyze trajectories of the set (3), we introduce the characteristic vibration amplitudes A_i of components of the motion $x_i(t)$ $(i=1,2 \dots n)$:

$$A_i = \frac{1}{2}\left|\max_{t_1 \le t \le T} x_i(t) - \min_{t_1 \le t \le T} x_i(t)\right|, \qquad (i=1,2 \dots n).$$

Here $[t_1, T] \subset [t_0, T]$ and $[t_0, T]$ is the time interval, in which the trajectory is considered. The interval $[t_0, t_1]$ is the time interval, in which all transient processes are damped. The characteristic vibration amplitudes A_i can be calculated simultaneously with the integration of the trajectory.

From the embedding theorem if $S_\varepsilon(\mathbf{x}) = \{\tilde{\mathbf{x}} \in R^n : \rho(\mathbf{x}, \tilde{\mathbf{x}}) < \varepsilon\}$ is the hyper-sphere with centre at the point $\mathbf{x}$ and with radius ε and $P_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}(\mathbf{x}) = \{\tilde{\mathbf{x}} \in R^n : |x_i - \tilde{x}_i| < \varepsilon_i\}$ is the n-dimensional parallelepiped then for any $\varepsilon > 0$ there is parallelepiped $P_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}(\mathbf{x})$ such that $P_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}(\mathbf{x}) \subset S_\varepsilon(\mathbf{x})$. And conversely, for any parallelepiped $P_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}(\mathbf{x})$ it is possible to indicate $\varepsilon > 0$ such that $S_\varepsilon(\mathbf{x}) \subset P_{\varepsilon_1,\varepsilon_2,...,\varepsilon_n}(\mathbf{x})$.

Let us choose in the parallelepiped $P_{\delta_1,\delta_2,...,\delta_n}(\mathbf{x}^{(0)})$ two neighboring initial points $\mathbf{x}^{(0)}$ and $\tilde{\mathbf{x}}^{(0)}$, such that $|x_i^{(0)} - \tilde{x}_i^{(0)}| < \delta_i$, where δ_i is small in comparison with A_i $(i=1,2 \dots n)$. In the case of regular motion, it is expected that the ε_i in the inequality $|x_i(t) - \tilde{x}_i(t)| < \varepsilon_i$ is also small in comparison with A_i $(i=1,2 \dots n)$. The wandering orbits attempt to fill some bounded domain of the phase space. The neighboring trajectories at the instant t_0 diverge exponentially on the average afterwards. Hence, the absolute values of differences $|x_i(t)-\tilde{x}_i(t)|$ for some instant t_1 can take values in the interval $[0, 2A_i]$. By analyzing the equilibrium states of (3) it is easy to choose an α parameter, $0<\alpha<1$, such that from the truth of the statement

$$\exists t^* \in [t_1, T]: \left|x_i(t^*)-\tilde{x}_i(t^*)\right| > \alpha A_i \quad (i=1,\ 2,\ \ldots\ n). \qquad (4)$$

it follows that there is a time interval or set of time intervals, for which the affixes of the closed at the initial instant trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ move around various equilibrium states or these trajectories are sensitive to changing of the initial conditions. Thus, these trajectories are wandering.

Indeed, as it has already been mentioned, all trajectories are in the closed bounded domain of the space R^n. We choose the measure of divergence of the trajectories, which is *inadmissible* for the case of 'regularity' of the motion. When the characteristic vibration amplitudes A_i are found, the divergence measures αA_i of the observable trajectories in the directions of the generalized coordinates x_i $(i=1,\ 2,\ \ldots\ n)$ are determined by α.

Let us briefly discuss the choice of the α parameter. Note that this choice is non-unique and the α parameter can take various values of the interval (0, 2). There are values of the α parameter, which a priori correspond to the inadmissible divergence measures αA_i $(i=1,\ 2,\ \ldots\ n)$ of the trajectories in the sense of 'regularity'. For example, $\alpha \in \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$. Other choices are possible. If the representative points of the observable trajectories move chaotically, then for another choice α from the set of a priori 'appropriate' α, the divergence of the trajectories will be recorded at another time t^*. As numerical experiments show, the domains of chaotic behaviour obtained with various a priori 'appropriate' values of the parameter α are practically congruent. Therefore, in this work figures for different values of α are not presented.

A similar non-unique choice of parameters occurs when applying another criteria for the chaotic oscillations. For instance, according to the procedure for calculation of the Lyapunov exponents $d(t)=d_0 2^{\lambda t}$. Here λ is a Lyapunov exponent, d_0 is the initial distance measure between the starting points, $d(t)$ is the distance between trajectories at instant t. The base 2 is chosen for a convenience. In all other respects the parameter $\alpha > 1$ in the relation $d(t)=d_0\alpha^{\lambda t}$ is arbitrary. That is α can takes values, for example, $\alpha \in \{2,3,4,5\}$ or other choices are possible. In general, the specificity required by numerical methods is such that, all parameters have to be concrete and most of them can be non-unique.

The parameter α might have another physical interpretation. Assume that for the nonlinear dynamical system under investigation, it is possible to identify the singular points (equilibria). In the case, for instance, of two-well potential systems we have two nodes and one saddle. An external periodic excitation applied to such one-degree-of-freedom system may cause a chaotic response. Chaos is characterized by unpredictable switches between the two potential wells. A phase point may wander between all three singular points. Consider two neighboring nodes. As a result of a switch, representative points close at the initial instant of the phase trajectories, are in motion about different equilibrium states afterward. Hence a choice of α, in the relation $\alpha A_i \cong \frac{1}{2} d$, is related to the distance d between the two nodes separated by a saddle. However, many nonlinear dynamical systems do not have analytical solutions, and sometimes it is laborious to find the singular it is recommended the parameter α be taken from a priori 'appropriate' values.

Our approach has been successfully applied in the case of smooth and non-smooth systems. By varying parameters and using condition (4), it is possible to find domains of chaotic motion (including transient and alternating chaos) and domains of regular motion.

Remark. All inequalities (4) do not have to be checked for the case, when the equations of motion under investigation can be transformed to a normal form. It means that the inequalities related to velocities $x_j = \dot{x}_i$ may be canceled. In another words, solutions related to regular motion with respect to x_i are also regular in relation to $x_j = \dot{x}_i$. Here $i, j \in \{\overline{1,n}\}$.

4. Comparison of Analytical and Numerical Results

The mechanical system (1) was investigated using both Melnikov's technique [6] and approach based on the wandering trajectories analysis.

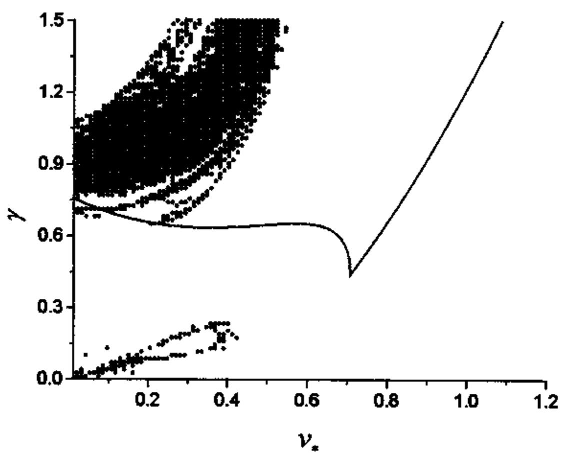

Figure 3. Domains of stick-slip chaos in the (v_*, γ) plane ($a = b = 1$, $\alpha = \beta = T_0$=0.3, ω=2, $x(0) = 1$, $v(0) = 0.4$). The smooth chaotic threshold is obtained using Melnikov's technique.

The results obtained for $a = b = 1$, $\alpha = \beta = T_0$=0.3, ω=2, $x(0) = 1$, $v(0) = 0.4$ are shown in the (v_*, γ) plane in Fig. 3.

The $\gamma(v_*)$ red curve presented in Fig. 3 is plotted using an analytical prediction (2) and separates the graph into two parts. Above this curve, chaos can appear, because near the line, the stable and unstable manifolds intersect transversally.

The domains of the stick-slip chaotic dynamics of the system (1), obtained on the basis of the wandering trajectories analysis are marked by blue dots. The condition (4) reads

$$\exists t^* \in [t_1, T]: \left| x(t^*) - \tilde{x}(t^*) \right| > \alpha A,$$

where: $A = \frac{1}{2}\left| \max_{t_1 \le t \le T} x(t) - \min_{t_1 \le t \le T} x(t) \right|$.

The time period for the simulation is 300 non-dimensional time units. During computations, a half period corresponds to the time interval $[t_0, t_1]$, where transitional processes are damped. The integration step size is equal to 3×10^{-3} in non-dimensional time units. The plane of parameters (v_*, γ) is uniformly sampled in rectangle parallelepiped ($0 < v_* \le 1.1$; $0 < \gamma \le 1.5$) by 100×100 nodal points. Initial conditions of the closed trajectories are distinguished by 0.5 percent with ratio to characteristic vibration amplitudes A and α parameter is equal to 1/3.

The results obtained show a good agreement between the analytical chaotic threshold and numerical simulation. According to the approach applied in this paper, chaotic motion of the oscillator is observed before the cusp at $v_* = \frac{a}{\sqrt{2b}} = \frac{\sqrt{2}}{2} \approx 0.71$.

5. Conclusions

Systems with friction are nonsmooth and cause some difficulties in both the theoretical

Systems with friction are nonsmooth and cause some difficulties in both the theoretical and numerical analyses. We have obtained a more precise definition to the domains of the stick-slip and slip-slip chaotic dynamics of a one degree-of-freedom very weakly forced oscillator in the (ν_*, γ) plane using a new approach based on analysis of the wandering trajectories. A comparison with analytical prediction, obtained using Melnikov's technique [6], demonstrates a good agreement with the results presented.

The standard numerical methods, in particular the direct calculations of Lyapunov exponents, are widely used in the literature, but they are time-consuming. Our approach is effective, convenient to use, requires much less computational time in comparison with other approaches, and can be applied to an investigation of a wide class of problems. According to this approach the characteristic vibration amplitudes A_i's are produced simultaneously to the integration of the trajectory. Thus, it is sufficient to integrate two equations only for each selected trajectory.

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