Systems Analysis Modelling Simulation
Vol. 43, No. 3, March 2003, pp. 269–286

COUPLED THERMOELASTICITY PROBLEMS OF SHALLOW SHELLS

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(Received 22 May 2000)

The derived differential equations govern a coupled linear thermoelastic behaviour in the frame of the Kirchhoff–Love model with the neglected effect of shell element rotary inertia. Then an abstract Cauchy problem for a coupled system by two differential equations in the Hilbert space is defined. It generalizes a series of coupled thermoelastic problems including those considered previously (Kirchhoff–Love and Timoshenko-type shells). All considerations are valid for a wide class of initial and boundary value problems.

Keywords: Duhamel–Neuman law; Heat transfer; Thermoelasticity; Shell

1. INTRODUCTION

It is well known that heat transfer and/or mechanical load over the surface of a shell causes instability effects, excessive wear and even cracking [1–8]. Therefore, a stress–strain state of a shell with variable thickness by means of the Mindlin model is considered. The shell is deformated due to surface and mass loading and both heat sources and heat exchange with a surrounding medium. Then using some assumptions the heat transfer equation is formulated, and the initial and boundary conditions are defined. The heat transfer conditions represented by a temperature field on the shell surface, heat flow density and heat transfer though the shell's surface are additionally attaehed. Next an abstract, coupled problem is considered, which is the main topic of the paper. First, its relation to the shell behavior is illustrated and then existence and uniqueness of solutions are proved.

2. FUNDAMENTAL ASSUMPTIONS AND HYPOTHESES

We consider a shallow shell with variable thickness with the surface Ω_1 and (as a three-dimensional body) the volume Ω_2. We assume that the shell material is isotropic,

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ISSN 0232-9298 print: ISSN 1029-4902 online © 2003 Taylor & Francis Ltd
DOI: 10.1080/0232929031000150238

homogeneous and elastic. We introduce the orthogonal co-ordinate system x, y, z in a typical way (see [1], p. 16). The co-ordinate lines x, y are shifted and they overlap with curvatures of the averaged surface. The z axis has normal direction to the averaged surface turned into a shell curvature center.

The displacement vector components of a point (x, y) of the averaged shell surface in the time t is denoted by $u(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$; the rotation angles of the normal to the averaged surface are denoted by $\psi_x(x,y,t)$, $\psi_y(x,y,t)$ in the planes xz and yz, correspondingly; the initial shell curvatures along x, y by k_x and k_y, correspondingly; the variable shell thickness is denoted by $h(x,y)$.

We investigate stress–strain state of the shell using the kinematic Timoshenko type model with effects of rotary element inertia [1] (it is referred to also as the Mindlin model). It is assumed that shell fibers normal to the averaged surface are not curved before deformation, but they are not perpendicular to the averaged surface as well. Deformation of the normal fibers to the averaged surface is defined by the condition of a plane strain state. We assume that on the equidistant shell surface, the following relations are satisfied (see [1], p. 34):

$$\varepsilon_{11}^z = \varepsilon_{11} = ze_{11}, \quad \varepsilon_{22}^z = \varepsilon_{22} + ze_{22}, \quad \varepsilon_{12}^z = \varepsilon_{12} + ze_{12}, \tag{1}$$

$$\varepsilon_{13}^z = \varepsilon_{13}, \quad \varepsilon_{23}^z = \varepsilon_{23}, \tag{2}$$

where $|z| \leq \frac{1}{2}h(x,y), \varepsilon_{ij}\,(i,j=1,2)$ – tangential deformations of the averaged surface, $\varepsilon_{13}, \varepsilon_{23}$ – shear deformations, $e_{ij}\,(i_{ij}=1{,}2)$ – bending deformations.

As a result the following equations are obtained:

$$\varepsilon_{11} = \frac{\partial u}{\partial x} - k_x w, \quad \varepsilon_{22} = \frac{\partial v}{\partial y} - k_y w, \quad \varepsilon_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{3}$$

$$e_{11} = \frac{\partial \Psi_x}{\partial x}, \quad e_{22} = \frac{\partial \Psi_y}{\partial y}, \quad e_{12} = \frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}, \tag{4}$$

$$\varepsilon_{13} = \Psi_x + \frac{\partial w}{\partial x}, \quad \varepsilon_{23} = \Psi_y + \frac{\partial w}{\partial y}. \tag{5}$$

It should be emphasized that deformations $_{11}, _{22}, _{12}, _{13}, _{23}$ are small in comparison to unity, and therefore the derivatives of deformations can be negligible in comparison with the deformations mentioned.

Suppose that in a non-deformable and non-strained (initial) state, the shell has the temperature T_0. The surface and mass loading, internal heat sources and heat exchange with a surrounding medium causes a shell deformation, which results in a change of its temperature. We denote a shell temperature increase in comparison with the initial state in the point (x, y, z) and in the time t by $\theta(x,y,z,t)$. We also assume that $|\theta/T_0| \ll 1$, i.e. that a temperature change is small enough and it had no significant influence on elastic and thermodynamic properties of the shell material. The following isothermal constants are used: E, Young modulus; ν, Poisson ratio; α_T, linear heat expansion coefficient; ρ, density; λ_q, thermal conductivity; c_ε, heat volume capacity (for the constant deformation tensor).

In addition, assuming that the shell is in the condition of the local quasi-equilibrium (see [3], p. 24) and using the thermodynamic relations, we get the following Duhamel–Neuman law for small deformations and for an increase in temperature (see [8], p. 79):

$$\varepsilon_{11}^{z} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{22} + \alpha_T\theta, \quad \varepsilon_{22}^{z} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11} + \alpha_T\theta, \tag{6}$$

$$\varepsilon_{33}^{z} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) + \alpha_T\theta, \tag{7}$$

$$\varepsilon_{12}^{z} = \frac{2(1+\nu)}{E}\sigma_{12}, \quad \varepsilon_{13}^{z} = \frac{2(1+\nu)}{E}\sigma_{13}, \quad \varepsilon_{23}^{z} = \frac{2(1+\nu)}{E}\sigma_{23}, \tag{8}$$

which defines a link between the stress tensor components σ_{ij} and the deformation tensor ε_{ij}^{z} for the condition of the flat strain-state ($\sigma_{33}=0$). From (6) and (8) we get

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11}^{z} + \nu\varepsilon_{22}^{z}) - \frac{E}{1-\nu}\alpha_T\theta, \quad (1 \leftrightarrow 2), \tag{9}$$

$$\sigma_{12} = \frac{E}{2(1+\nu)}\varepsilon_{12}^{z}, \quad \sigma_{13} = \frac{E}{2(1+\nu)}\varepsilon_{13}^{z}, \quad \sigma_{23} = \frac{E}{2(1+\nu)}\varepsilon_{23}^{z}. \tag{10}$$

Integrating (9) and (10) with respect to z and taking into account (1)–(5), we find the corresponding stresses and transversal forces on the averaged surface:

$$\begin{aligned} N_x = \int_{-h/2}^{h/2} \sigma_{11} d_z = \frac{Eh}{2(1+\nu^2)}\left[\frac{\partial u}{\partial x} - k_x w + \nu\left(\frac{\partial v}{\partial y} - k_x w\right)\right] \\ - \frac{E\alpha_T}{1-\nu}\int_{-h/2}^{h/2} \theta\, dz, \quad (x \leftrightarrow y,\, u \leftrightarrow v,\, 1 \leftrightarrow 2). \end{aligned} \tag{11}$$

$$S = \int_{-h/2}^{h/2} \sigma_{12}\, dz = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{12}$$

$$Q_x = k^2 \int_{-h/2}^{h/2} \sigma_{13}\, dz = \frac{k^2 Eh}{2(1+\nu)}\left(\Psi_x + \frac{\partial w}{\partial x}\right) \quad (x \leftrightarrow y, 1 \leftrightarrow 2), \tag{13}$$

where $1/k^2 = (1/h)\int_{-h/2}^{h/2} f^2(z/h)\, dz = \int_{-h/2}^{h/2} f^2(\xi)\, d\xi$, $f(z/h)$–function characterizing a way of tangential distribution along thickness (see [1], pp. 35–37). The stresses are integrated (taking into account (1), (3) and (4)) with respect to the shell thickness and we obtain the moments

$$\begin{aligned} M_x = \int_{-h/2}^{h/2} \sigma_{11} z\, dz = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial \Psi_x}{\partial x} + \nu\frac{\partial \Psi_y}{\partial y}\right) \\ - \frac{E\alpha_T}{1-\nu}\int_{-h/2}^{h/2} \theta z\, dz, \quad (x \leftrightarrow y, 1 \leftrightarrow 2), \end{aligned} \tag{14}$$

$$H = \int_{-h/2}^{h/2} \sigma_{12} z \, dz = \frac{Eh^3}{24(1+\nu)} \left(\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x} \right). \tag{15}$$

3. DIFFERENTIAL EQUATIONS

Using the entropy balance equation and taking into account a link between the heat flow vector components and the thermodynamic forces due to the Fourier rule, we get the heat transfer equation (see [8], p. 87):

$$c_\varepsilon \frac{\partial \theta}{\partial t} - \lambda_q \nabla^2 \theta = -\frac{E\alpha_T T_0}{1-2\nu} \frac{\partial}{\partial t} \underline{(\varepsilon_{11}^z + \varepsilon_{22}^z + \varepsilon_{33}^z)} + q_2, \tag{16}$$

where the underlined term links a temperature increase with a speed of the volume change.

In this equation, $q_2(x, y, z, t)$ denotes the power capacity of the internal heat sources.

In order to obtain the equations governing the shell vibrations, the inertial forces of the shell element in the x, y and z directions, as well as the rotary inertia related to x and y are included. The following differential equations are obtained (in relation to displacements) [1]:

$$\frac{\partial N_x}{\partial x} + \frac{\partial S}{\partial y} + p_1 - \rho h \frac{\partial^2 u}{\partial t^2} = 0 \quad (x \leftrightarrow y,\ u \leftrightarrow v,\ 1 \leftrightarrow 2), \tag{17}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + k_x N_x + k_y N_y + q_1 - \rho h \frac{\partial^2 w}{\partial t^2} = 0, \tag{18}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} - Q_x - \rho \underline{\frac{h^3}{12} \frac{\partial^2 \Psi_x}{\partial t^2}} = 0, \quad (x \leftrightarrow y), \tag{19}$$

where p_1, p_2, q_1 are the external load intensities related to the axes x, y and z,

Substituting (11)–(15) into Eqs. (17)–(19) and attaching the heat transfer equation (16) transformed according to (1), (3), (4), (7), (9), the following full systems of differential equations governing the thermoelastic behavior are obtained:

$$\begin{aligned} &\rho h \frac{\partial^2 u}{\partial t^2} - \frac{E}{1-\nu^2} \frac{\partial}{\partial x} \left[h \left[\frac{\partial v}{\partial x} - k_x \omega + \nu \left(\frac{\partial v}{\partial y} - k_y w \right) \right] \right] - \frac{E}{2(1+\nu)} \frac{\partial}{\partial y} \left[h \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\ &\quad + \frac{E\alpha_T}{1-\nu} \frac{\partial}{\partial x} \int_{-h/2}^{h/2} \theta \, dz = p_1, \quad (x \leftrightarrow y, u \leftrightarrow v, p_1 \leftrightarrow p_2), \end{aligned} \tag{20}$$

$$\begin{aligned} &\rho h \frac{\partial^2 w}{\partial t^2} - \frac{k^2 E}{2(1+\nu)} \left[\frac{\partial}{\partial x} \left[h \left(\psi_x + \frac{\partial w}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[h \left(\psi_y + \frac{\partial w}{\partial y} \right) \right] \right] \\ &\quad - \frac{E}{(1-\nu^2)} \left[(k_x + \nu k_y) \left(\frac{\partial u}{\partial x} - k_x w \right) + (k_y + \nu k_x) \left(\frac{\partial v}{\partial y} - k_y w \right) \right] \\ &\quad + \frac{E\alpha_T}{1-\nu} (k_x + k_y) \int_{-h/2}^{h/2} \theta \, dz = q_1, \end{aligned} \tag{21}$$

$$\rho\frac{h^3}{12}\frac{\partial^2\psi_x}{\partial t^2}-\frac{\partial}{\partial x}\left[D\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right)\right]-\frac{1-\nu}{2}\frac{\partial}{\partial y}\left[D\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right]-\frac{k^2Eh}{2(1+\nu)}\left(\psi_x+\frac{\partial w}{\partial x}\right)$$
$$+\frac{E\alpha_T}{1-\nu}\frac{\partial}{\partial x}\int_{-h/2}^{h/2}\theta\,dz=0,\quad (x\leftrightarrow y), \tag{22}$$

$$c_\varepsilon(1+\varepsilon)\frac{\partial\theta}{\partial t}-\lambda_q\nabla^2\theta+\frac{E\alpha_T T_0}{1-\nu}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-(k_x+k_y)w+z\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}\right)\right]=q_2. \tag{23}$$

$$D=\frac{Eh^3}{12(1-\nu^2)},\quad \varepsilon=\frac{E\alpha_T^2T_0(1+\nu)}{c_\varepsilon(1-\nu)(1-2\nu)}. \tag{24}$$

Equations (20)–(23) represent the full system of equations of the coupled dynamical linear thermoelasticity problem. Note that no constraints are applied to the temperature field distribution along the shell thickness and therefore the equations with different dimensions are obtained. The θ function appearing in the heat transfer equation (parabolic type) depends on x, y, z and time t. The functions u, v, w, ψ_x, ψ_y in the equations governing a shell element motion (hyperbolic type) depend on two co-ordinates x, y and time t.

The obtained Timoshenko type-model yields the equations governing the thermoelastic behavior of the shallow shell in the frame of the Kirchhoff–Love model. After omitting the effect of transversal shear deformation, we get [1]:

$$\psi_x=-\frac{\partial w}{\partial x},\quad \psi_y=-\frac{\partial w}{\partial y}. \tag{25}$$

In addition, the underlined term, whieb governs the rotary inertia of the shell elements, is also neglected. Therefore, we obtain

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{E\alpha_T}{1-\nu}\int_{-h/2}^{h/2}\theta z\,dz,\quad (x\leftrightarrow y), \tag{26}$$

$$H=-D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}, \tag{27}$$

$$\theta_x=\frac{\partial M_x}{\partial x}+\frac{\partial H}{\partial y},\quad (x\leftrightarrow y). \tag{28}$$

Substituting (28) into (18) and attaching Eq. (17), we get the following system of equations governing a motion of the shell element

$$\frac{\partial N_x}{\partial x}+\frac{\partial S}{\partial y}+p_1-\rho h\frac{\partial^2 u}{\partial t^2}=0,\quad (x\leftrightarrow y, u\leftrightarrow v, 1\leftrightarrow 2),$$
$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 H}{\partial x\,\partial y}+\frac{\partial^2 M_y}{\partial y}+k_xN_x+K_yN_y+q_1-\rho h\frac{\partial^2 w}{\partial t^2}=0. \tag{29}$$

Furthermore, we substitute (11), (12), (26), (27) into (29) and attach the general heat transfer equation (23), previously transformed according to (25). As a result we get:

$$\rho h \frac{\partial^2 u}{\partial t^2} - \frac{E}{1-\nu^2}\frac{\partial}{\partial x}\left[h\left[\frac{\partial u}{\partial x} - k_x w + \nu\left(\frac{\partial v}{\partial y} - k_y w\right)\right]\right] - \frac{E}{2(1+\nu)}\frac{\partial}{\partial y}\left[h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$
$$+ \frac{E\alpha_T}{1-\nu}\frac{\partial}{\partial x}\int_{-h/2}^{h/2}\theta\, dz = p_1, \quad (x \leftrightarrow y, u \leftrightarrow v, p_1 \leftrightarrow p_2), \tag{30}$$

$$\rho h \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[h\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial^2 y}\right)\right] + 2(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\left(D\frac{\partial^2 w}{\partial x\,\partial y}\right)$$
$$+ \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right] - \frac{E}{1-\nu^2}\left[(k_x + \nu k_y)\left(\frac{\partial u}{\partial x} - k_x w\right)\right.$$
$$\left. + (k_y + \nu k_x)\left(\frac{\partial v}{\partial y} - k_y w\right)\right] + \frac{E\alpha_T}{1-\nu}\left[\nabla^2\int_{-h/2}^{h/2}\theta z\, dz + (k_x + k_y)\int_{-h/2}^{h/2}\theta\, dz\right] = q_1, \tag{31}$$

$$c_\varepsilon(1+\varepsilon)\frac{\partial\theta}{\partial t} - \lambda_q\nabla^2\theta + \frac{E\alpha_T T_0}{1-\nu}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - (k_x + k_y)w - z\nabla^2 w\right] = q_2. \tag{32}$$

The system of differential equations (30)–(32) governs a coupled dynamical problem of the linear themoelasticity within the frame of the Kirchhoff–Love model (with the neglected effect of shell element rotary inertia).

4. BOUNDARY AND INITIAL CONDITIONS

Now an initial state of the shell, as well as the conditions responsible for both mechanical and heating processes between the shell and surrounding medium, will be defined. The initial and boundary conditions result from a stationary condition of the corresponding Hamilton functionals.

As the initial conditions attached to Eqs. (30)–(32) we take (for $t=0$) the initial displacements distribution u, v and w, the velocities distribution $(\partial u/\partial t), (\partial v/\partial t), (\partial w/\partial t)$, and the temperature increase θ (the last one is equivalent to the temperature field applied in the initial time moment), of the form:

$$u|_{t=0} = u_0(x,y), \quad v|_{t=0} = v_0(x,y), \quad w|_{t=0} = w_0(x,y), \tag{33}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x,y), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = v_1(x,y), \quad \left.\frac{\partial w}{\partial t}\right|_{t=0} = w_1(x,y), \tag{34}$$

$$\theta|_{t=0} = \theta_0(x,y,z), \tag{35}$$

where $(x,y) \in \Omega_1$, $(x,y,z) \in \Omega_2$, $u_0, v_0, w_0, u_1, v_1, w_1, \theta_0$ – given functions.

In spite of the initial conditions (33)–(35) attached to the differential equations (20)–(23), a distribution (for $t=0$) of the rotary angles ψ_x, ψ_y and the angular velocities $(\partial\psi_x/\partial t)$, $(\partial\psi_y/\partial t)$ should be defined:

$$\psi_x|_{t=0} = \psi_{x0}(x,y), \quad \psi_y|_{t=0} = \psi_{y0}(x,y), \tag{36}$$

$$\left.\frac{\partial\psi_x}{\partial t}\right|_{t=0} = \psi_{x_1}(x,y), \quad \left.\frac{\partial\psi_y}{\partial t}\right|_{t=0} = \psi_{y_1}(x,y), \tag{37}$$

where $(x,y) \in \Omega_1, \psi_{x_0}, \psi_{xy_0}, \psi_{x_1}, \psi_{y_1}$ – given functions.

One of the known heat transfer conditions [4] can be applied in each point of the shell surface $\partial\Omega_2$:

– temperature field on the shell surface

$$\theta = \theta^0(x,y,z,t), \tag{38}$$

– heat flow density

$$\frac{\partial\theta}{\partial n} = \theta^0(x,y,z,t), \tag{39}$$

– heat transfer through the shell surface

$$\frac{\partial\theta}{\partial n} + \alpha\theta = \theta_n^\alpha(x,y,z,t), \quad \alpha = \text{const} > 0, \tag{40}$$

where $(x,y,z) \in \partial\Omega_2, t \in [0,T], \theta^0, \theta_n^0, \theta_n^\alpha$ – given functions; T–observation time of the shell behavior.

For the Kirchhoff–Love model (Eqs. (30)–(32)) conditions on the shell edge $\partial\Omega_1$ have the following form (four boundary conditions are attached to each edge point):

$$u = u^0(x,y,t) \quad \text{or} \quad N_x n_x + S n_y = A^0(x,y,t), \tag{41}$$

$$v = v^0(x,y,t) \quad \text{or} \quad N_y n_y + S n_x = B^0(x,y,t), \tag{42}$$

$$w = w^0(x,y,t) \quad \text{or} \quad \left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y}\right) n_x + \left(\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x}\right) n_y + \frac{\partial}{\partial S}[(M_y n_y + H n_x) n_x - (M_x n_x + H n_y) n_y] = C^0(x,y,t), \tag{43}$$

$$\frac{\partial w}{\partial n} = w_n^0(x,y,t) \quad \text{or} \quad M_x n_x^2 + 2H n_x n_y + M_y n_y^2 = D^0(x,y,t), \tag{44}$$

where $(x,y) \in \partial\Omega_2, t \in [0,T], n_x, n_y$ are components of the external normal to the contour $\partial\Omega_1, \partial/\partial n, \partial/\partial s$ – differential operators in the normal and tangential direction to $\partial\Omega_1, u^0, v^0, w^0, w_n^0, A^0, B^0, C^0, D^0$ –given functions.

In the case of the Timoshenko-type model (Eqs. (20)–(23)) five mechanical conditions are attached to each point of the contour $\partial\Omega_1$, i.e. those given by (41), (42) and additional three:

$$w = w^0(x,y,t) \quad \text{or} \quad Q_x n_x + Q_x n_x = E^0(x,y,t), \tag{45}$$

$$\psi_x = \psi_x^0(x,y,t) \quad \text{or} \quad M_x n_x + H n_y = F^0(x,y,t) \tag{46}$$

$$\psi_y = \psi_y^0(x,y,t) \quad \text{or} \quad M_y n_y + H n_x = G^0(x,y,t), \tag{47}$$

where $(x,y) \in \partial\Omega_2, t \in [0,T], w^0, \psi_x^0, \psi_y^0, E^0, F^0, G^0$ – given functions.

In each pair of the alternative boundary conditions (41)–(47), the first boundary eondition belongs to the main one (i.e. it is *a priori* given during the search for a stationary point of the Hamilton functional). The second boundary condition, called an essential one, results indirectly from a stationary condition of the Hamilton functional. In the case of a rectangular shell, the given conditions are reported in many monographs [1,2,5].

The following mechanical boundary conditions are attached (for rectangular shells):

– a simple support on elastic non-stretched (non-compressed) ribs in the tangential plane of a rib

$$w = \frac{\partial u}{\partial x} = v = \frac{\partial \psi_x}{\partial x} = \psi_y = 0, \quad x = \text{eonst}, \tag{48}$$

$$w = \frac{\partial^2 w}{\partial x^2} = \frac{\partial u}{\partial x} = v = 0, \quad x = \text{const}, \tag{49}$$

– a loosely clamped edge

$$w = u = v = \frac{\partial \psi_x}{\partial x} = \psi_y = 0, \quad x = \text{const}, \tag{50}$$

$$w = \frac{\partial^2 w}{\partial x^2} = u = v = 0, \quad x = \text{const}, \tag{51}$$

– a elamped edge

$$w = u = v = \psi_x = \psi_y = 0, \quad x = \text{const}, \tag{52}$$

$$w = \frac{\partial w}{\partial x} = u = v = 0, \quad x = \text{const}, \; x \leftrightarrow y, u \leftrightarrow v. \tag{53}$$

Conditions (48), (50), (52) correspond to the Timoshenko-type model, whereas conditions (49), (51), (53) do correspond to the Kirchhoff–Love model. Each of the mentioned groups of boundary eonditions (when a homogeneous boundary condition (38) is satisfied on the lateral shell surface) become a special case of general conditions (41)–(44) (Kirchhoff–Love shell) or (41), (42), (45)–(47) (Timoshenko-type model). For instance, in the group of conditions (48), the conditions $w=v=\psi_y=0$ are equivalent to homogeneous main conditions (45), (42), (47), whereas the conditions $\partial u/\partial x=\partial\psi_x/\partial x=0$ are equivalent to homogeneous essential conditions (41), (46), and so on.

5. ABSTRACT COUPLED PROBLEM

An abstract Cauchy problem for a system coupled by two differential equations in the Hilbert space will be stated. It generalizes a series of coupled thermoelastic problems. It also includes the above discussed thermoelastic problems of shallow shells for Kirchhoff–Love and Timoshenko-type shells. First, a necessary notation is introduced.

Let H be a certain Hilbert space. Let $L_p(0,T;H)$ be a space of the functions being measured of the time interval $[0,T]$, having the values on H and summed in $[0,T]$ with the pth degree of a norm (sufficiently bounded for $p=\infty$), where the norm for $p<\infty$ is defined in the following way

$$\|u\|_{L_p(0,T;H)}=\left[\int_0^T \|u(t)\|_H^p\,dt\right]^{1/p},$$

whereas for $p=\infty$, by relation

$$\|u\|_{L_\infty(0,T;H)}=\sup_{0\le t\le T} ess\|u(t)\|_H.$$

The space of the measured functions $u(t)$ is denoted by $W_p^k(0,T,H)$ and their values belonging to H have (in the interval $[0,T]$) the generalized derivatives (in the Sobolev sense) $u^{(j)}(t)$ up to kth order, which are summed with the pth degree of a norm (sufficiently bounded for $p=\infty$). The space $W_p^k(0,T;H)$ for $p<\infty$ is attached to the norm

$$\|u\|_{W_p^k(0,T;H)}=\left[\int_0^T\sum_{j=0}^k \|u^{(j)}(t)\|_H^p\,dt\right]^{1/p},$$

whereas for $p=\infty$ to the norm

$$\|u\|_{W_\infty^k(0,T;H)}=\sup_{0\le t\le T} ess\sum_{j=0}^k\|u(t)\|_H.$$

For $1\le p\le\infty$ each of the spaces $L_p(0,T;H), W_p^k(0,T;H)(k=1,2,\ldots)$, is a Banach space. The spaces $L_2(0,T;H), W_p^k(0,T;H)$ are Hilbert spaces and a scalar product is defined as follows

$$(u, v)_{L_2(0,T;H)} = \int_0^T (u(t), v(t))_H \, dt,$$

$$(u, v)_{W_2^k(0,T;H)} = \int_0^T \sum_{j=0}^{k} (u^{(j)}(t), v^{(j)}(t))_H \, dt.$$

The Sobolev space of the measured functions $u(x)$ is denoted by $W_2^k(\Omega)$ and the functions are defined in the Ω space of the n-dimensional arithmetic space R^n and they have all the generalized derivatives $D^\alpha u(x)$ up to the kth order, which are summed with second power in Ω. $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes an integer multi-index of differentiation, $|\alpha| = \alpha_1, \ldots, \alpha_n, D^\alpha = D_1^{\alpha_1}, \ldots, D_n^{\alpha n}, D_i \partial/\partial_{xi}$ –a Sobolev general differentiation operator. The space $W_2^k(\Omega)$is a Hilbert space and a scalar product (in this space) is defined by

$$(u, v)_{w_2^k(\Omega)} = \int \sum_{\Omega |a| \leq k} D^\alpha u D^\alpha v \, d\Omega.$$

For $k=0$ the space $W_2^k(\Omega)$ overlaps the Lebesque space $L_2(\Omega)$.

Let $H_1, \ldots, H_m$ be Hilbert subspaces. $H_1 \times \cdots \times H_m$ denotes their Descartes product, i.e. the Hilbert space is composed of all possible ordered elements $\bar{u} = \{u_i\}_{i=1}^m$, where $u_i \in H_i$ with the attached scalar product

$$(\bar{u}, \bar{v})_{H_1 \times \cdots \times H_m} = (u_1, v_1)_{H_1} + \cdots + (u_m, v_m)_{H_m}.$$

Let $u(x, t)$ be a certain measured function, defined for $x \in \Omega \subset R^n, t \in [0, T]$ having its values in R^m and the following property: For all $t \in [0, T]$, the function "$x \rightarrow u(x, t)$" belongs to a certain Hilbert space of the type $W_2^{k_1}(\Omega) \times \cdots \times W_2^{k_m}(\Omega)$. Each of these functions is identified with the mapping $u(t)$, which maps an arbitrary number $t \in [0, T]$ into the function "$x \rightarrow u(x, t)$" as an element of H. Therefore, the following notation is defined $\|u(t)\|_H, (u(t), v(t))_H, \|u(t)\|_{W_p^k(0,T;H)}, (u, v)_{W_p^k(0,T;H)},$ where u, v are the functions depending on x, t.

In the case of continuous and continuously differentiable functions, we use the usual notations: $C(\bar{\Omega}), C^k(\bar{\Omega}), C([0, T]; H)$.

Let A be a self-conjugated positively defined and (possibly) unbounded operator acting in the Hilbert space H. According to the definition of $D(A)$, we identify the Hilbert space with the same elements and with the scalar product $(u, v)_A = (Au, Av)_H$. Therefore, the notation $L_p(0, T; D(A)), W_p^k(0, T; D(A))$ is defined.

The notation introduce above is valid throughout the whole further work.

Let H_1, H_2 be two subspaces of the Hilbert space. We define the Cauchy problem for two differential equations

$$I_1 w''(t) + Lw(t) + M\theta(t) = g_1(t), \tag{54}$$

$$I_1 \theta'(t) + K\theta(t) + Nw'(t) = g_2(t), \tag{55}$$

$$w(0) = w_0, \quad w'(0) = w_I, \quad \theta(0) = \theta_0, \tag{56}$$

where w, q are the sought functions mapping the interval of time $[0, T]$ to H_1 and H_2, correspondingly; g_1, g_2 – given functions satisfying the conditions

$$g_1 \in L_2(0, T; H_1), \quad g_2 \in L_2(0, T; H_2); \tag{57}$$

I_1, I_2 – bounded self-conjugated, positively defined operators acting on H_1 and H_2, correspondingly; L, K – bounded, self-conjugated positively defined operators acting on H_1 and H_2, correspondingly and defined on continuous manifolds $D(L) \subset H_1$, $D(K) \subset H_2$, where the inverse operators L^{-1}, K^{-1} are fully continuous; M, N – linear unbounded operators acting from H_2 to H_1, and from H_1 to H_2 and linked by the relation $N \subset -M^*$, i.e. for two arbitrary elements $w \in D(N)$, $\theta \in D(M)$, the following "link condition" holds

$$(M\theta, w)_{H_1} + (Mw, \theta)_{H_2} = 0, \tag{58}$$

$D(K) \subset D(M)$, $D(L^{1/2}) \subset D(N)$, the bounded operators $MK^{-1}, NL^{-1/2}$ are similar to the operators acting from $H_2 \to H_1$ and $H_1 \to H_2$. The operator $L^{-1/2}MK^{-1/2}$ can be extended to the bounded operator from H_2 to H_1; w_0, w_1, θ_0 – given elements satisfying the conditions

$$w_0 \in D(L^{1/2}), \quad w_1 \in H_1, \quad \theta_0 \in H_2. \tag{59}$$

DEFINITION 1.1. *The ordered pair of the functions $\{w, q\}$ is called a generalized solution of the Cauchy problem* (54)–(56), *if* $w \in W^1_\infty(0, T; H) \cap L_\infty(0, T; D(L^{1/2}))$, $w(0) = w_0$, $\theta \in L_\infty(0, T; H_2) \cap L_2(0, T; D(K^{1/2}))$ *and for the arbitrary functions* v, η, *satisfying the conditions* $v \in W^1_1(0, T; H) \cap L_1(0, T; D(L^{1/2}))$, $v(T) = 0$, $\eta \in W^1_1(0, T; H_2) \cap L_2(0, T; D(K^{1/2}))$, $\eta(T) = 0$ *the following relations hold*

$$\int_0^T [-(I_1 w'(t), v'(t))_{H_1} + (L^{1/2} w(t), L^{1/2} v(t))_{H_1}$$
$$+ (L^{-1/2} M\theta(t), L^{1/2} v(t)_{H_1}]\, dt - (I_1 w_1, v(0))_{H_1} = \int_0^T (g_1(t), v(t))_{H_1}\, dt, \tag{60}$$

$$\int_0^T [-(I_2\theta(t), \eta'(t))_{H_2} + (K^{1/2}\theta(t), K^{1/2}\eta(t))_{H_2} - (Nw(t)\, \eta'(t))_{H_2}]\, dt$$
$$- (I_2\theta_0, \eta(0))_{H_2} - (Nw_0\eta(0))_{H_2} = \int_0^T (g_2(t), \eta(t))_{H_2}\, dt. \tag{61}$$

DEFINITION 1.2. *The ordered pair of function $\{w, q\}$ is called a "classical" solution to the Cauchy problem* (54)–(56), *if*

$$w \in W^2_\infty(0, T; H_1) \cap W^1_\infty(0, T; D(L^{1/2})) \cap L_\infty(0, T; D(L)),$$
$$\theta \in W^1_\infty(0, T; H_2) \cap W^1_2(0, T; D(K^{1/2})) \cap L_\infty(0, T; D(K)),$$

and the functions w, θ satisfy the conditions (54), (55) *almost everywhere in the interval* $[0, T]$ *and also satisfy the relations* (56).

REMARK 1.1. *In the Definitions* 1.1, 1.2 *maximal smoothness of the functions* w, θ *is required, which can be achieved using formulas of the corresponding Theorems* 1.1, 1.2. *Integral relations (60), (61) are obtained due to a generalized approach of the generalized solutions to the problems of mathematical physics (see, for instance, [6]). First, the initial differential equations (54), (55) are multiplied by the functions* $v(t), \eta(t)$ *and then they are integrated by parts using the initial conditions (56). It implies (among others) that a "classical" solution to the problem (54)–(56) is its generalized solution at the same time.*

Let us check if the formally stated Cauchy problem defined by (54)–(56) is the essential generalization of thc given coupled thermoelastic problems for shallow shells (in a case of homogeneous boundary conditions). We consider first an initial-boundary problem for the system of differential equations (20)–(23) with the initial conditions (33)–(37), with the homogeneous heat boundary conditions (38) (main condition) or (39) (essential condition). Let $\Omega_1 \subset R^2$ be the bounded space with piece-wise smooth boundary of C^1. A splitting of either the boundary Ω_1 (mechanical conditions) or Ω_2 (heat conditions) into two non-crossing parts with the main and essential boundary conditions corresponds to each of the alternative boundary conditions. A measure of the boundary part related to the main condition is defined strongly positively. The main thermal condition is applied on a whole lateral shell surface. Therefore, all mechanical boundary conditions can be applied only via the displacements u, v, w and angles of rotation ψ_x, ψ_y (the function θ does not influence these boundary conditions). We denote the space of the vector-functions $\bar{w} = (x,y)$ of the form

$$\bar{w} = \{u, v, w, \psi_x, \psi_y\} \tag{62}$$

by $\dot{W}_2^2(\Omega))^5$.

All their components belong to the space $W_2^2(\Omega)$ and satisfy both main and essential mechanical boundary eonditions of the considered problem. $(\dot{W}_2^1(\Omega_1))^5$ is the space of vector functions (62) with all components from $W_2^1(\Omega)$ and satisfying main mechanical conditions of the considered problem. $\dot{W}_2^2(\Omega_1)$ is the space of functions $\theta \in W_2^2(\Omega_1)$ satisfying both main and essential heat boundary conditions. $\dot{W}_2^1(\Omega_1)$ is the spaee of the functions $\theta \in W_2^1(\Omega_1)$ satisfying the main heat condition.

The following restrictions are applied to the given functions and constants appearing in (20)–(23), (33)–(37):

$$h \in C^1(\bar{\Omega}_1); \quad h(x,y) > 0, \quad (x,y) \in \bar{\Omega}_1; \tag{63}$$

$$\rho, E, \alpha_T, c_\varepsilon, \lambda_q, T_0, T = \text{const}, \quad k = \text{const} \neq 0, \tag{64}$$

$$\begin{gathered} k_x, k_y = \text{const}, \quad 0 < \nu = \text{const} < \tfrac{1}{2}; \\ p_1, p_2, q_1 \in L_2(\Omega_1 \times (0,T)), \quad q_2 \in L_2(\Omega_2 \times (0,T)); \end{gathered} \tag{65}$$

$$\bar{w}_0 = \{u_0, v_0, w_0, \psi_{x0}, \psi_{y0}\} \in (\dot{W}_2^1(\Omega_1))^5; \tag{66}$$

$$\bar{w}_1 = \{u_1, v_1, w_1, \psi_{x1}, \psi_{yI}\} \in (L_2(\Omega_1))^5; \tag{67}$$

$$\theta_0 \in L_2(\Omega_2). \tag{68}$$

Assuming that the following relations hold

$$H_1 = (L_2(\Omega_1))^5, \quad H_2 = L_2(\Omega_2), \tag{69}$$

$$I_1\bar{w} = \{\rho hu, \rho hv, \rho hw, \tfrac{1}{2}\rho h^3\psi_x, \tfrac{1}{2}\rho h^3\psi_y\}, \tag{70}$$

$$I_2\theta = \frac{c_3}{T_0}(1+\varepsilon)\theta, \tag{71}$$

$$\begin{aligned} L\bar{w} = \Bigg\{ & \ell_x\bar{w}, \ell_y\bar{w}, -\frac{k^2E}{2(1+\nu)}\left\{\frac{\partial}{\partial x}\left[h\left(\psi_x + \frac{\partial w}{\partial x}\right)\right]\right. \\ & \left.+\frac{\partial}{\partial y}\left[h\left(\psi_y + \frac{\partial w}{\partial y}\right)\right]\right\} - \frac{Eh}{1-\nu^2}\left[(k_x + \nu k_y)\left(\frac{\partial u}{\partial x} - k_x w\right)\right. \\ & \left.+(k_y + \nu k_x)\left(\frac{\partial v}{\partial y} - k_y w\right)\right], \tau_x\bar{w}, \ \tau_y\bar{w}\Bigg\}, \quad D(L) = (\dot{W}_2^2(\Omega_1))^5, \end{aligned} \tag{72}$$

$$\begin{aligned} \ell_x\bar{w} = & -\frac{E}{1-\nu^2}\frac{\partial}{\partial x}\left\{h\left[\frac{\partial u}{\partial x} - k_x w + \nu\left(\frac{\partial v}{\partial y} - k_y w\right)\right]\right\} \\ & -\frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left[h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \quad (x \leftrightarrow y, u \leftrightarrow v), \end{aligned} \tag{73}$$

$$\begin{aligned} \tau_x\bar{w} = & -\frac{\partial}{\partial x}\left[D\left(\frac{\partial\psi_x}{\partial x} + \nu\frac{\partial\psi_y}{\partial y}\right)\right] - \frac{1-\nu}{2}\frac{\partial}{\partial y}\left[D\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)\right] \\ & + \frac{k^2Eh}{2(1+\nu)}\left(\psi_x + \frac{\partial w}{\partial x}\right), \quad (x \leftrightarrow y), \end{aligned} \tag{74}$$

$$K\theta = -\frac{\lambda_q}{T_0}\nabla^2\theta, \quad D(K) = \dot{W}_2^2(\Omega_2), \tag{75}$$

$$M\theta = \left\{m_x\theta, m_y\theta, \frac{E\alpha_T}{1-\nu}(k_x + k_y)\int_{-h/2}^{h/2}\theta\,dz, \tilde{m}_x\theta, \tilde{m}_y\theta\right\}, \quad D(M) = \dot{W}_2^1(\Omega_2), \tag{76}$$

$$m_x\theta = \frac{E\alpha_T}{1-\nu}\frac{\partial}{\partial x}\int_{-h/2}^{h/2}\theta\,dz, \quad \tilde{m}_x\theta = \frac{E\alpha_T}{1-\nu}\frac{\partial}{\partial x}\int_{-h/2}^{h/2}\theta z\,dz \quad (x \leftrightarrow y), \tag{77}$$

$$N\bar{w} = \frac{E\alpha_T}{1-\nu}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - (k_x + k_y)w + z\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\right)\right], \quad D(N) = (\dot{W}_2^1(\Omega_1))^5, \tag{78}$$

$$\bar{w}_0 = \{u_0, v_0, w_0, \psi_{x0}, \psi_{y0}\}, \quad \bar{w}_1 = \{u_1, v_1, w_1, \psi_{x1}, \psi_{y1}\}, \tag{79}$$

$$q_1 = \{p_1, p_2, q_1, 0, 0\}, \quad q_2 = \frac{1}{T_0} q_2, \tag{80}$$

the considered initial and boundary value problem is related to that of (54)–(56).

Instead of the sought functions $w(t)$ and initial values w_0, w_1, the vector function $\bar{w} = \{u, v, w, \psi_x, \psi_y\}$ and the given vector functions $\bar{w}_0, \bar{w}_1$ defined by (79) are sought. Note, that in our case (with an accuracy of the equivalent scalar product) we have

$$D(L^{1/2}) = (\dot{W}_2^1(\Omega_1))^5, \quad D(K^{1/2}) = (\dot{W}_2^1(\Omega_2))^5. \tag{81}$$

Consider further the initial-boundary value problem is related to that of (30)–(32). With the initial conditions (33)–(35) and the homogeneous boundary conditions of the form (41)–(44), (38) (or (39)). Suppose that the introduced assumptions about smoothness of the boundary Ω_1 and about splitting properties Ω_1, Ω_2 corresponding to each pair of the alternative boundary conditions are still valid. The space of vector-function $\bar{w}(x, y)$ of the form

$$\bar{w} = \{u, v, w\} \tag{82}$$

with the components from $W_2^1(\Omega_1)$, $W_2^2(\Omega_1)$ and $W_2^4(\Omega_1)$, which satisfy both the main and essential mechanical boundary conditions of the considered problem is denoted by $(\dot{W}_2^2(\Omega_1))^2 \times \dot{W}_2^4(\Omega_1)$. By $\dot{W}_2^1(\Omega_1))^2 \times \dot{W}_2^2(\Omega_1)$ we denote the space of vector-functions (82) with components from $W_2^1(\Omega_1), W_2^1(\Omega_1), W_2^2(\Omega_1)$, which satisfy the main mechanical boundary conditions of the considered problem. The following constraints are attached to the constants and functions occurring in (30)–(35) (in addition to (64), (65) and (68)):

$$h \in C^2(\bar{\Omega}_1); \quad h(x, y) > 0, \quad (x, y) \in \bar{\Omega}_1; \tag{83}$$

6. ON EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this clause the theorems about existence and uniqueness of general and "classical" solutions to the abstract coupled problem (54)–(56) are outlined, as well as the corresponding theorems for coupled thermoelastic problems of shallow shells are formulated. The known compact method (see [7], pp. 10–11) is applied while proving the theorem. The emphasis is focused on *a priori* estimation of exact and approximate solutions of the considered problem (54)–(56).

Let $\{v_i\}_{i=1}^{\infty}, \{\eta_j\}_{j=1}^{\infty}$ be certain linearly independent systems of elements from $D(L)$ and $D(K)$ of full values in $D(L^{1/2})$ and $D(K^{1/2})$, correspondingly; $i(n), j(n)$ – two subseries of the natural series. We denote linear shells of the finite elements $\{v_i\}_{i=1}^{\infty}$ and $\{\eta_j\}_{j=1}^{\infty}$ by H_1^n and H_2^n. PG is the operator of the orthogonal projection of the Hilbert space $H(= H_1$ or $H_2)$ on the finite dimensional subspace $G(= H_1^n$ or $H_2^n)$.

The initial problem (54)–(56) will be solved with the Bubnov–Galerkin method (BGM):

$$\begin{aligned} &P_{H_1^n}(I_1 w_n''(t) + Lw_n(t) + M\theta_n(t)) = P_{H_1^n} g_1(t),\\ &P_{H_2^n}(I_2\theta_n'(t) + K\theta_n(t) + Nw_n(t)) = P_{H_2^n} g_2(t),\\ &w_n(t) \in H_1^n \quad \theta_n(t) \in H_2^n, \quad \forall t \in [0,T];\\ &w_n(0) = w_{0n}, \quad w_n' = w_{1n}, \quad \theta_n(0) = \theta_{0n}, \end{aligned} \tag{95}$$

where $\{w_{0n}\}_{n=1}^{\infty}, \{w_{1n}\}_{n=1}^{\infty}, \{\theta_{0n}\}_{n=1}^{\infty}$ are arbitrary series of the elements from H_1^n, H_1^n, H_2^n, correspondingly, which satisfy the conditions

$$w_{0n} \to w_0, \text{ in } D(L^{1/2}), \quad w_{1n} \to w_1, \text{ in } H_1, \quad \theta_{0n} \to \theta_0, \text{ in } H_2. \tag{96}$$

The exact solution $\{w_n, \theta_n\}$ of the problem (95) is called the approximate solution of the problem (54)–(56) obtaincd by means of the BGM. Because problem (95) is equivalent to the Cauchy problem the system of $2i(n)+j(n)$ creates the differential equations with $2i(n)+j(n)$ unknown functions. Therefore, its solution exists and it is unique for an arbitrary natural n.

THEOREM 1.1. *If the conditions* (57)–(59) *are satisfied, then a general solution* $\{w, \theta\}$ *to problem* (54)–(56) *exists and is unique.*

DEFINITION 1.3. *The self-conjugated positively defined operators A and A_0 acting in the same Hilbert space H are similar if $D(A) = D(A_0)$.*

It is known that if the operators A and A_0 are similar, then $D(A^{1/2}) = D(A_0^{1/2})$, and all operators AA_0^{-1},$A^{-1}A_0$,$A^{1/2}A_0^{1/2}$,$A^{-1/2}A_0^{1/2}$ are bounded in H together with their increase operators.

Let L_0, K_0 be certain self-conjugated, positively defined and similar operators forming an acute angle with the operators L and K, correspondingly; $\{v_i\}_{i=1}^{\infty}$ – basis in H_1 composed of the eigenelements of the operator L_0; $\{\eta_j\}_{j=1}^{\infty}$ – basis in H_2, composed of eigenelements of the K_0 operator. The eigenvalues of the operators L_0 and K_0 are denoted by λ_i and e_j, correspondingly: $L_0 v_i = \lambda_i v_i, L_0\eta_j = e_j\eta_j, i, j = 1, 2, \ldots$

Note, that because the operators L^{-1}, K^{-1} are fully continuous the same property holds for the operators L_0^{-1}, K_0^{-1}, which implies that

$$\lambda_i \to \infty, \quad e_j \to \infty, \qquad (i, j \to \infty) \tag{97}$$

As the basis of the BGM method, the following series of elements are used: $\{v_i\}_{i=1}^{\infty}, \{\eta_j\}_{j=1}^{\infty}$. As the initial data of the problem (95) the projection of the initial data of (56) to the corresponding finite dimensional spaces is used:

$$w_{0n} = P_{H_1^n} w_0, \quad w_{1n} = P_{H_1^n} w_1, \quad \theta_{0n} = P_{H_2^n}\theta_0. \tag{98}$$

More strong conditions are imposed (instead of (57) and (59)) on the given functions g_1, g_2 and the elements w_0, w_1, θ_0:

$$g_1 \in W_2^1(0, T, H_1), \quad g_2 \in W_2^1(0, T, H_2), \tag{99}$$

$$w_0 \in D(L), \quad w_1 \in D(L^{1/2}), \quad \theta_0 \in D(K). \tag{100}$$

THEOREM 1.2. *The "classical" solution to problem* (54)–(56) *exists and is unique when conditions* (58), (99), (100) *are satisfied.*

Finally, we formulate theorems on existence and uniqueness of solutions of the coupled thermoelastic problem of the shallow shell which follow from Theorems 1.1 and 1.2.

THEOREM 1.3. *If conditions* (63)–(68) *are satisfied, then the initial boundary problem for differential equations* (20)–(23) *with initial conditions* (33)–(37) *and the homogeneous boundary conditions has a single (general) solution* $\{u, v, w, \psi_x, \psi_y\}$. *It fulfills the following conditions:*

$$\{u, v, w, \psi_x, \psi_y\} \in W^1_\infty(0, T; L_2(\Omega_1))^5) \cap L_\infty(0, T; (W^1_2(\Omega_1))^5),$$
$$\theta \in L_\infty(0, T; L_2(\Omega_2))^5 \cap L_2(0, T; (W^1_2(\Omega_2))).$$

THEOREM 1.4. *Let conditions* (63), (64), *as well as the conditions*

$$p_1, p_2, q_1 \in W^1_2(0, T; L_2(\Omega_1)), \quad q_2 \in W^1_2(0, T; L_2(\Omega_2)),$$
$$\bar{w}_0 = \{u_0, v_0, w_0, \psi_{x0}, \psi_{y0}\} \in \dot{W}^2_2(\Omega_1)^5,$$
$$\bar{w}_1 = \{u_1, v_1, w_1, \psi_{x1}, \psi_{y1}\} \in \dot{W}^1_2(\Omega_1)^5, \quad \theta_0 \in \dot{W}^2_2(\Omega_2)$$

be satisfied. Then the initial-boundary problem for differential equations (20)–(23) *with initial conditions* (33)–(37) *and the homogeneous boundary conditions have a single ("classical") solution* $\{u, v, w, \psi_x, \psi_y\}$, *which satisfied the conditions*

$$\bar{w}_0 = \{u_0, v_0, w_0\} \in (\dot{W}^1_2(\Omega_1))^2 \times \dot{W}^2_2(\Omega_1); \tag{84}$$

$$\bar{w}_1 = \{u_1, v_1, w_1\} \in (L_2(\Omega_1))^3. \tag{85}$$

The considered initial and boundary value problem is reduced to that defined by (54)–(56), when the following relations hold:

$$H_1 = (L_2(\Omega_1))^3, \quad H_2 = L_2(\Omega_2), \tag{86}$$

$$I_1\bar{w} = \{\rho h u, \rho h v, \rho h w\}, \quad I_2\theta + \frac{c_\varepsilon}{T_0}(1+\varepsilon)\theta, \tag{87}$$

$$\begin{aligned} L\bar{w} = \Bigg\{ &\ell_x\bar{w}, \ell_y\bar{w}, \frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right] + 2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right) \\ &+ \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right] - \frac{Eh}{1-\nu^2}\left[(k_x + \nu k_y)\left(\frac{\partial u}{\partial x} - k_x w\right)\right. \\ &\left.\left. + (k_y + \nu k_x)\left(\frac{\partial v}{\partial y} - k_y w\right)\right]\right\}, \quad D(L) = (\dot{W}^2_2(\Omega_1))^2 \times \dot{W}^4_2(\Omega_1), \end{aligned} \tag{88}$$

$$K\theta = -\frac{\lambda_q}{T_o}\nabla^2\theta, \quad D(K) = \dot{W}_2^2(\Omega_2) \tag{89}$$

$$M\theta = \left\{m_x\theta, m_y\theta, \frac{E\alpha_T}{1-\nu}\left[\nabla^2\int_{-h/2}^{h/2}\theta\,dz + (k_x+k_y)\times\int_{-h/2}^{h/2}\theta\,dz\right]\right\}, \quad D(M) = \dot{W}_2^2(\Omega_2) \tag{90}$$

$$N\bar{w} = \frac{E\alpha_T}{1-\nu}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-(k_x+k_y)w+z\nabla^2 w\right], \quad D(N) = (\dot{W}_2^1(\Omega_1))^2\times\dot{W}_2^2(\Omega_1), \tag{91}$$

$$\bar{w}_0 = \{u_0, v_0, w_0\}, \quad \bar{w}_1 = \{u_1, v_1, w_1\}, \tag{92}$$

$$q_1 = \{p_1, p_2, q_1\}, \quad q_2 = \frac{1}{T_0}q_2. \tag{93}$$

The operators ℓ_x, ℓ_y, m_x, m_y occurring in (88), (90) are defined by relations (73), (77). In the role of the function being sought $w(t)$ and the initial conditions w_0, w_1, the vector function $\bar{w} = \{u, v, w\}$ and the given vector functions $\bar{w}_0, \bar{w}_1$ defined by (92) appear. Note, that in the considered case, the relation holds

$$D(L^{1/2}) = (\dot{W}_2^1(\Omega_1))^2\times\dot{W}_2^2(\Omega_1), \quad D(K^{1/2}) = \dot{W}_2^1(\Omega_2). \tag{94}$$

REMARK 1.2. *The operators L, K for the considered coupled thermoelasticproblems can be defined as self-conjugated extensions (according to Fridrichs [9]) of the initial differential operators given either by relations (72), (75) or (88), (89) on sets of sufficiently smooth functions satisfying the homogeneous boundary conditions.*

$$\begin{aligned}
&\{u, v, w, \psi_x, \psi_y\} \in W_\infty^2(0, T;\ L_2(\Omega_1))^5 \cap, \\
&\cap L_\infty^1(0, T;\ (\dot{W}_2^1(\Omega_1))^5 \cap L_\infty(0, T;\ (\dot{W}_2^2(\Omega_1))^5, \\
&\theta \in W_\infty^1(0, T;\ L_2(\Omega_2)) \cap W_2^1(0, T;\ \dot{W}_2^1(\Omega_2)) \cap \\
&\cap L_\infty(0, T;\ \dot{W}_2^2(\Omega_2)).
\end{aligned}$$

THEOREM 1.5. *If conditions (64), (65), (68), (83)–(85) are satisfied, then the initial boundary problem for differential equations (30)–(32) with the attached initial (33)–(35) and homogeneous boundary conditions has a single (general) solution $\{u, v, w, \theta\}$, which satisfies the conditions:*

$$\begin{aligned}
&\{u, v, w\} \in W_\infty^1(0, T;\ L_2(\Omega_1))^3 \cap, \\
&\cap L_\infty(0, T;\ (\dot{W}_2^1(\Omega_2))^2\times\dot{W}_2^2(\Omega_1), \\
&\theta \in L_\infty(0, T;\ L_2(\Omega_2)) \cap L_2(0, T;\ \dot{W}_2^1(\Omega_2)).
\end{aligned}$$

THEOREM 1.6. *Let conditions* (64), (83), *as well as the given below conditions*

$$p_1, p_2, q_1 \in W_2^1(0,T;\ L_2(\Omega_1)), \quad q_2 \in W_2^1(0,T;\ L_2(\Omega_2)),$$

$$\bar{w}_0 = \{u_0, v_0, w_0\} \in (\dot{W}_2^2(\Omega_1))^2 \times \dot{W}_2^4(\Omega_1),$$

$$\bar{w}_1 = \{u_1, v_1, w_1\} \in (\dot{W}_2^1(\Omega_1))^2 \times \dot{W}_2^2(\Omega_1), \quad \theta_0 \in \dot{W}_2^2(\Omega_2)$$

be satisfied. Then, the initial-boundary problem for differential equations (30)–(32) *with the initial* (33)–(35) *and homogeneous boundary conditions has a single ("classical") solution* $\{u, v, w, \theta\}$, *which fulfills the following conditions:*

$$\{u, v, w\} \in W_\infty^2(0,T;\ L_2(\Omega_1))^3 \cap,$$

$$\cap\, W_\infty^1(0,T;\ (\dot{W}_2^1(\Omega_1))^2 \times \dot{W}_2^2(\Omega_2) \cap,$$

$$\cap\, L_\infty(0,T;\ (\dot{W}_2^2(\Omega_1))^2 \times \dot{W}_2^4(\Omega_1),$$

$$\theta \in W_\infty^1(0,T;\ L_2(\Omega_2)) \cap W_2^1(0,T;\ \dot{W}_2^1(\Omega_2))\cap$$

$$\cap\, L_\infty(0,T;\ \dot{W}_2^2(\Omega_1))$$

The obtained results follow from Theorems 1.1, 1.2 when either notation (69)–(81) (for differential equations (20)–(23) with initial conditions (33)–(37)) or notations (86)–(94) (for differential equations (30)–(32) with initial conditions (33)–(35)) are used.

Acknowledgement

This work has been supported by the Polish State Committee for Scientific Research (grant No 8T11F01416).

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