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ASYMPTOTIC APPROACHES TO STRONGLY NON-LINEAR DYNAMICAL SYSTEMS

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Introduction of small parameters to the real physical problems very often is a delicate and non-trivial problem and may be produced in the various ways. Here we analyze non-linear dynamical equation

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + \varepsilon x^n = 0 \tag{1}$$

for $n \to \infty$.

We propose new formula in the following form

$$x = x_0 \sqrt[n]{1 + \varepsilon n \frac{x_1}{x_0}},$$

and this approach is compared with the usual one. We also obtained asymptotic solutions for Eq. (1) for $\gamma = \omega^2 = 0$, $n \to \infty$; $\gamma = 0$, $n \to \infty$. We show that cases $\omega^2 = 0$, $n \to \infty$ and $\gamma \ll 1$, $n \to \infty$ may be reduced to the previous ones.

Keywords: Non-linear dynamics; Strong non-linearity; Asymptotic approach; Uncomplete Beta-functions

1. INTRODUCTION

Asymptotical approaches to analyses of strongly non-linear dynamical systems still need to be developed. A special attention of a research is focused to an analysis of the low dimensional systems, which is motivated by the following observations:

1. The low dimensional systems possess very complex behaviour [1–3].
2. The fundamental behaviour of high order dimensional systems can be adequately modelled by the systems of low dimensions [3].

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3. A concept of the non-linear normal modes very often allows to reduce a high dimensional system to that with a few degree of freedom [2–4,8,12].
4. Recent results show that the approximate analytical solutions of strongly non-linear systems can be obtained with an emphasis paid to strong non-linearity [2,5–8]. As a zero order approximation a so called vibroimpact system is used, and thereafter either a method of iteration [2,5] or asymptotic techniques [6–8] are applied. However, during this procedure the following drawback occurs: the non-smooth solutions with a constant period of oscillations appear.

Contrary to that approach, in this article, a construction of smooth solutions is proposed using an asymptotical approach.

2. MODIFIED QUASILINEAR APPROACH

Consider the following non-linear equation

$$\ddot{x} + \gamma\dot{x} + \omega^2 x + \varepsilon x^n = 0, \quad n = 3, 5, 7, \ldots \tag{1}$$

For $|\varepsilon| \ll 1$ and relatively small values of n ($n=3$, 5) a solution to Eq. (1) can be obtained using either the standard Lindstedt–Poincaré or averaging methods [1,8–10]. For the large values of n the standard approach seems not to lead to correct results. Let us suppose that we are going to find a solution to Eq. (1) in the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \tag{2}$$

then the non-linear term is approximated by the formulae

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^n = x_0^n + \varepsilon n x_0^{n-1} x_1 + \cdots$$

Thus, a role of the real "small" parameter plays εn instead of ε. It seems that this changing is not so important for $n=3$ and $n=5$; however, it is expected to play a crucial role for large values of n (especially for $n \to \infty$).

In this contribution the following formula is introduced:

$$x = x_0 \sqrt[n]{1 + \varepsilon n \frac{x_1}{x_0} + \cdots}. \tag{3}$$

For small n values the expressions (2) and (3) are equivalent. For $n \to \infty$, a high order of singularity caused by the n-power root seems to play an important role.

A typical for quasilinear approach term x_1 can be obtained from the equation

$$\ddot{x} + \gamma\dot{x}_1 + \omega^2 x_1 = -x_0^n.$$

In other words, known quasilinear solutions [1,8–10] can be transformed to the formula (3). We suspect that the formula (3) may be efficiently used in the theory of quantum oscillators, where a standard quasilinear asymptotic series, even with a use of the Padé approximants, not always allow to get the satisfactory results, as it is pointed in [11].

We are going to discuss a problem of suitability of formulas (2) and (3) using the numerical simulations, which are presented in Fig. 1.

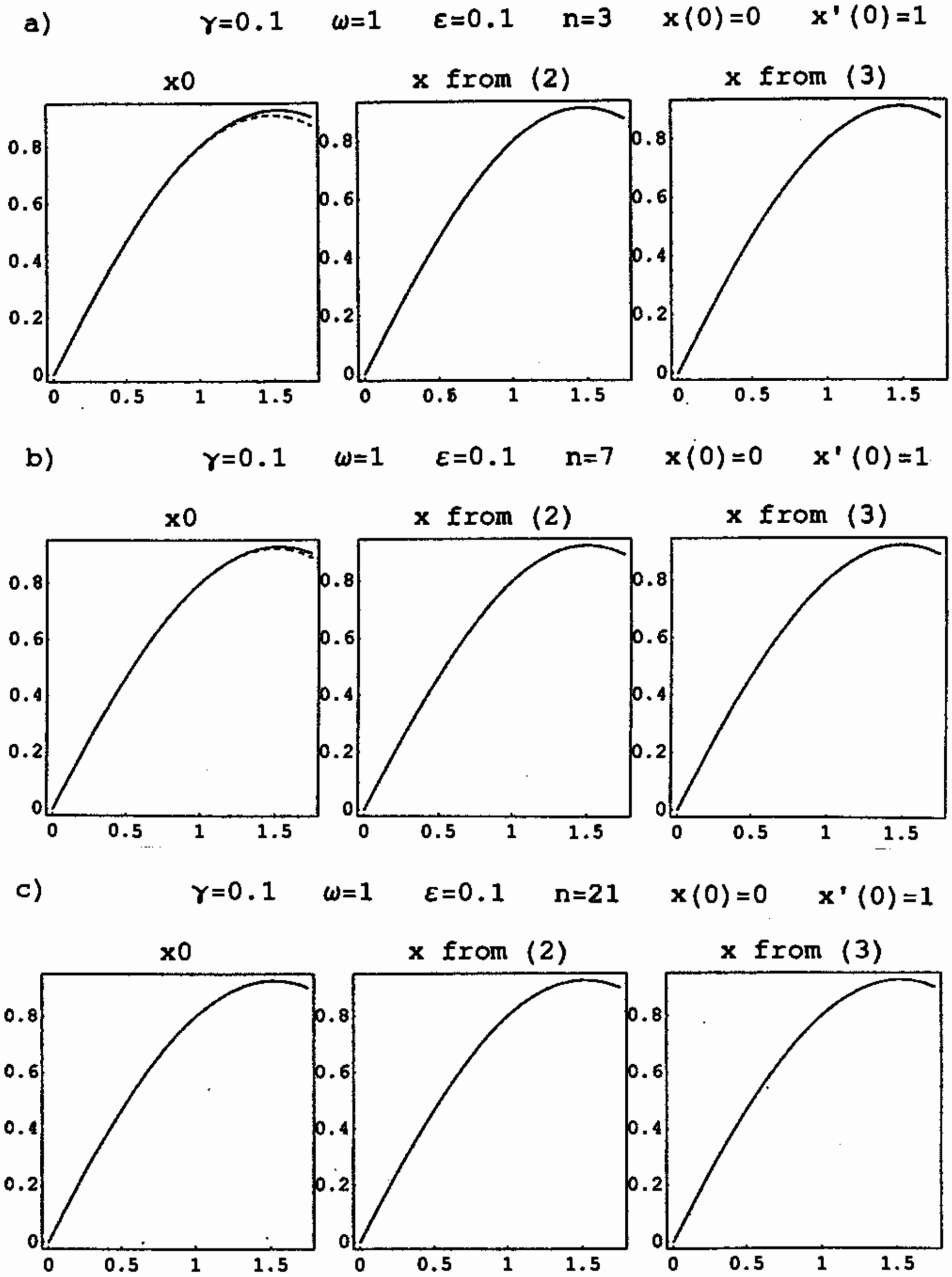

FIGURE 1 Numerical solutions of x_0, x from Eq. (2), and x from Eq. (3) for $\gamma=0.1$, $\omega=1$, $\varepsilon=0.1$, $x(0)=0$, $x'(0)=1$ and the different values of n: (a) 3; (b) 7; (c) 21.

It is seen that both formulas give a good approximation to numerical solution of Eq. (1) (dashed line). Already $x_0(t)$ gives a rather good approximation which is remarkable. In addition, an increase of n causes an improve of convergencies.

A similar behaviour has been observed for smaller values of γ and ε, which are not reported here. A decrease of γ and ε causes an improve of approximations for both cases.

Similar computations have been carried out for $\varepsilon = 1$ (the other parameters and initial conditions are unchanged) and they are reported in Fig. 2. Again similar

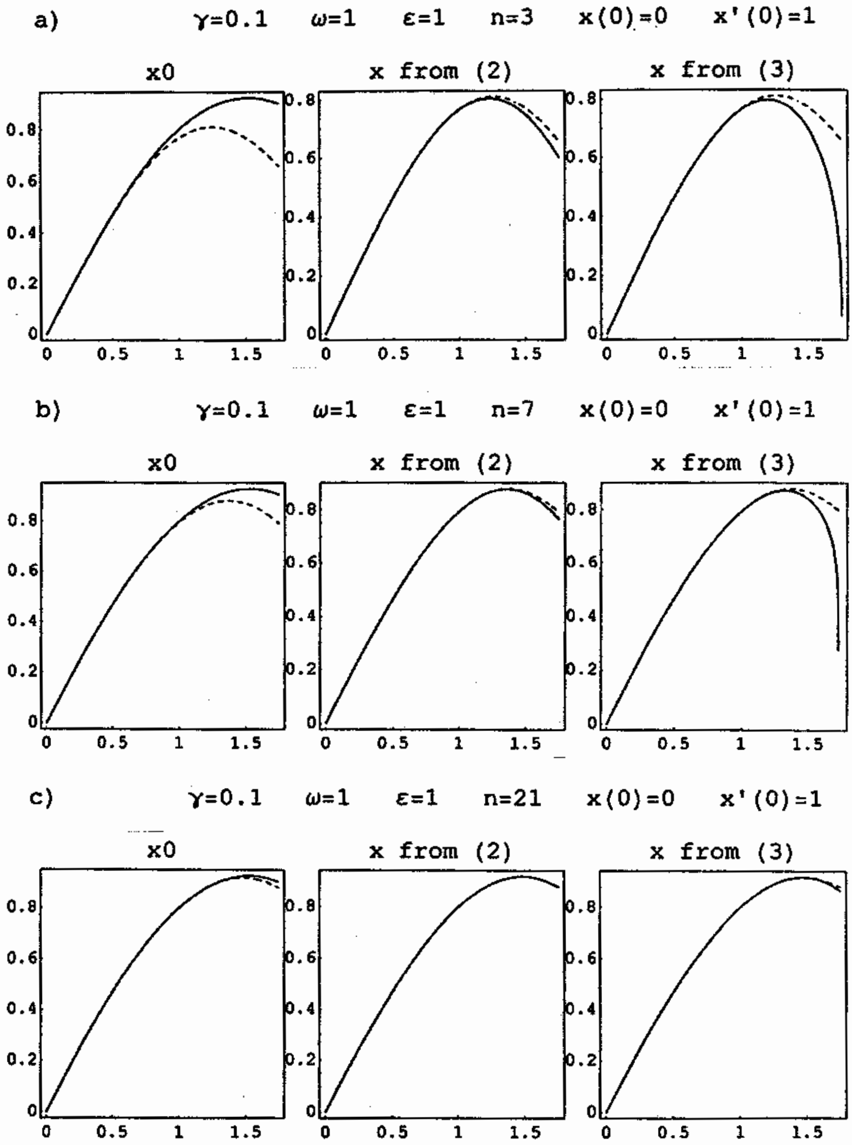

FIGURE 2 Numerical solutions of x_0, x from Eq. (2), and x from Eq. (3) for $\gamma = 0.1$, $\omega = 1$, $\varepsilon = 1$, $x(0) = 0$, $x'(0) = 1$ and the different values of n: (a) 3; (b) 7; (c) 21.

conclusions can be carried out. In general a convergence is lower than in a case reported in Fig. 1.

Therefore we have suspected that a large difference should be exhibited for relatively large initial conditions. The results of computations are presented in Fig. 3. Again a rather good approximation occurs within a 1/4 part of oscillation.

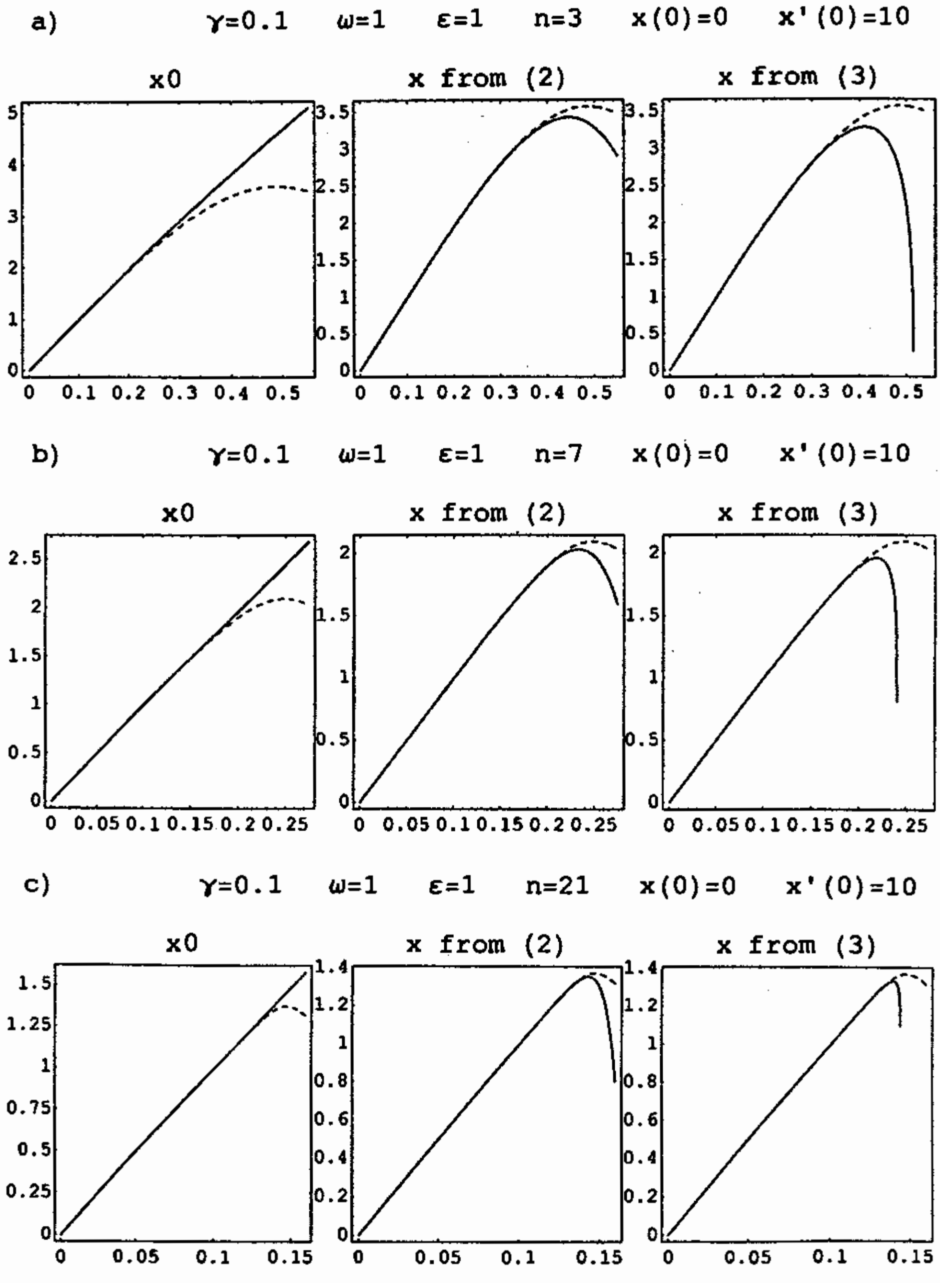

FIGURE 3 Numerical solutions of x_0, x from Eq. (2), and x from Eq. (3) for $\gamma = 0.1$, $\omega = 1$, $\varepsilon = 1$, $x(0) = 0$, $x'(0) = 10$ and the different values of n: (a) 3; (b) 7; (c) 21.

3. STRONGLY NON-LINEAR EQUATION WITHOUT DAMPING AND LINEAR TERMS

Taking $\gamma = \omega^2 = 0$ from (1) one obtains the following, non-linear equation

$$\ddot{x} + \varepsilon x^n = 0, \quad n = 3, 5, 7, \ldots \tag{4}$$

which attracted an attention of many researchers [2,5–8]. Among others, for large values of n it can model the vibroimpact processes [19,20]. In particular, this equation can be integrated using the special functions (cam, sam or the Ateb-functions) introduced by Rosenberg [12]. From a mathematical point of view the mentioned functions belong to a class of inversion of the uncomplete Beta-functions [13].

The Eq. (4) is principally non-quasilinear, and therefore a use of ε parameter for the asymptotic integration is difficult ($\varepsilon = 1$ will be taken in further considerations). In addition, the parameter n occurs which can be used into two ways. For small n the approximation $n = 1 + \delta$ is used and then, which is widely used in physics the so called small δ approach is applied [14,15]. For $n \to \infty$, a corresponding asymptotics can be formulated using the parameter n^{-1}. In the investigations reported in [2,5–8] a corresponding limiting case can be realized explicitly in an equation. It results in a change of the original system to the vibroimpact one. The last circumstance results in a non-smooth solution and an unchangability of a period. Below, a limiting transition in a solution is introduced which exhibits many advantages.

Let us consider the equation

$$\ddot{x} + x^n = 0, \tag{5}$$

with the attached initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = 1, \tag{6}$$

and with $n \to \infty$.

We are going to define a periodic solution to the Cauchy problem (5), (6). The first integral related to the problem defined by (5), (6) is as follows

$$p^2 = 1 - \frac{2x^{n+1}}{n+1}. \tag{7}$$

A change of variables $x = \sqrt[n+1]{0.5(n+1)}\xi$ and an integration allows for transformation of (7) to get the form

$$\sqrt[n+1]{\frac{2}{n+1}}\, t = \int_0^{0 \le \xi \le 1} \frac{d\xi}{\sqrt{1 - \xi^{n+1}}}. \tag{8}$$

On the other hand, the implicit solution (8), after a change of the variables $\xi = \sin^{2/(n+1)}$, yields the expression including explicitly the small parameter $\varepsilon = 2/(n+1)\theta$ of the form

$$\varepsilon^{\varepsilon/2} t = \varepsilon \int_0^{0 \le \theta \le \pi/2} \sin^{-1+\varepsilon}\theta \, d\theta. \tag{9}$$

First we consider the following expression (see [9])

$$\sin^{-1+\varepsilon}\theta = \theta^{-1+\varepsilon}\left(\frac{\theta}{\sin\theta}\right)^{1-\varepsilon} = \theta^{-1+\varepsilon}\left[\frac{\theta}{\sin\theta} - \varepsilon\ln\frac{\theta}{\sin\theta} + \cdots\right]. \tag{10}$$

Using a series approximation

$$\frac{\theta}{\sin\theta} = 1 + \frac{\theta^2}{3!} + \cdots$$

and leaving the terms with ε order in the right hand side of Eq. (10), one obtains

$$\sin^{-1+\varepsilon}\theta = \theta^{-1+\varepsilon} + \frac{\theta^2}{3!} + \cdots + o(\varepsilon).$$

With a use of "Mathematica" we got the following approximation

$$\begin{aligned}\frac{\Theta}{\sin\Theta} = 1 &+ \frac{\Theta^2}{6} + \frac{7}{36}\Theta^4 + \frac{127}{604800}\Theta^8 + \frac{73}{3421440}\Theta^{10}\\ &+ \frac{1414477}{653837184000}\Theta^{12} + \frac{8191}{37362124800}\Theta^{14}\\ &+ \frac{16931177}{762187345920000}\Theta^{16} + \frac{574961557}{2554547108585472000}\Theta^{18}\\ &+ \frac{91546277357}{401428831349145600000}\Theta^{20} + o[\Theta]^{21},\end{aligned} \tag{11}$$

and the results of approximation are presented in Fig. 4.

It is clear, that an essential part during an integration procedure is introduced by the first two terms of the series (11). Therefore, in the first approximation we take

$$\varepsilon^{\varepsilon/2} \approx \theta^{\varepsilon}.$$

It is easy to find that

$$\theta \approx \varepsilon^{1/2} t^{\varepsilon-1},$$

which reads in the initial variables

$$x \approx \sqrt[n+1]{\frac{n+1}{2}}\sin^{2/(n+1)}\left(\sqrt{\frac{2}{n+1}}t^{(n+1)/2}\right). \tag{12}$$

The found solution (12) can be calculated only on a 1/4 part of the period T, and then it can be periodically extended. The period of solution (12) reads

$$T = 4\left(\frac{\pi}{2}\sqrt{\frac{n+1}{2}}^{2/(n+1)}\right). \tag{13}$$

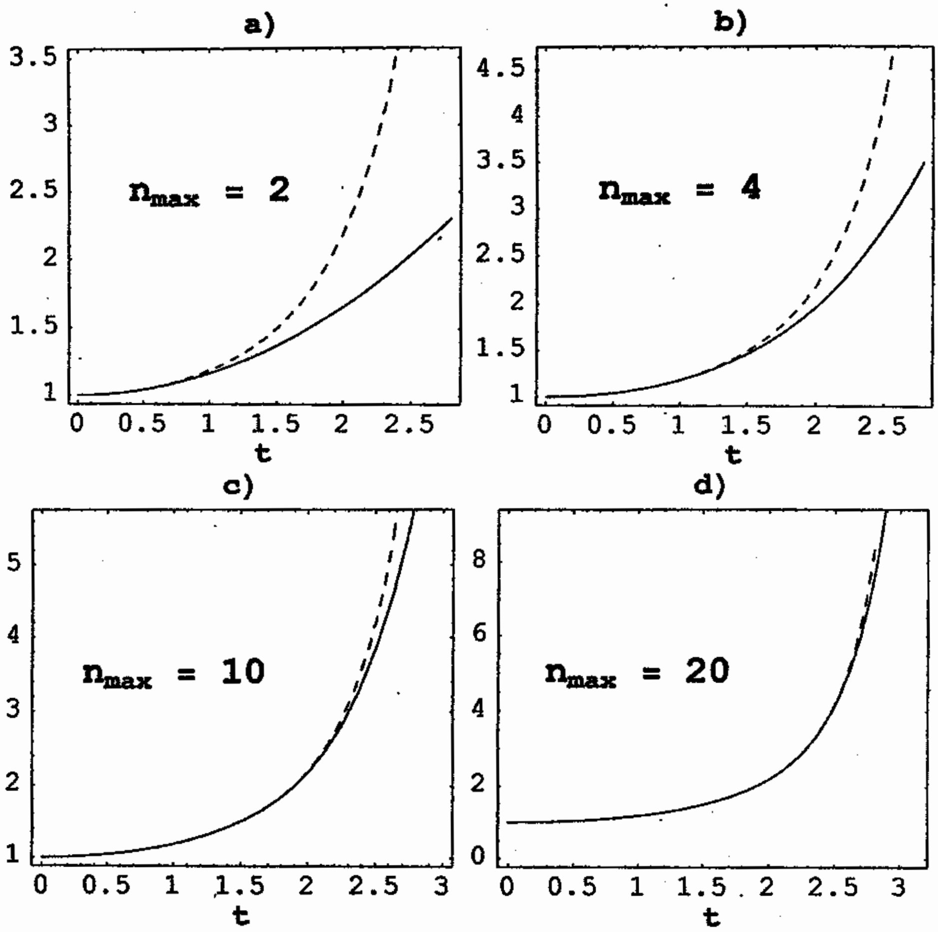

FIGURE 4 Approximations of $\Theta/\sin\Theta$ using the first n terms: (a) $n=2$; (b) $n=3$; (c) $n=6$; (d) $n=11$.

It should be emphasised that for $\varepsilon=1$ the exact solutions are obtained: $x=\sin t$, $T=2\pi$. When $n\to\infty$, one gets $T\to 4$, which should be expected.

When the solution (12) is expanded into the series in relation to t and taking into account only the first term the non-smooth solution obtained by Pilipchuk occurs [2,5].

Now we are going to estimate an accuracy of the solution (13). To this aim the following expression is used [14]:

$$\frac{2}{n+1}\int_0^{\pi/2}\sin^{-1+2/(n+1)}\theta\,d\theta=\frac{1}{n+1}B\left(\frac{1}{n+1},\frac{1}{2}\right)=A_1, \tag{14}$$

where: $B(\ldots,\ldots)$ is the uncomplete Beta-function.

An approximate value of the integral in the left hand side of expression (14) has the form

$$A_2=\left(\frac{\pi}{2}\right)^{2/(n+1)}.$$

TABLE I Estimation of error between A_1 and A_2

n	A_1	A_2	*Error%*
1	$\pi/2$	$\pi/2$	0%
3	1.30	1.25	5%
5	1.20	1.16	3%
...	...	...	...
∞	1	1	~0%

The numerical comparison between the values of A_1 and A_2 and accuracy estimation is reported in Table I.

The obtained results lead to the following conclusion. Already the first approximation of the asymptotics for $n \to \infty$ gives reasonable accuracy for practical applications (even for the small values of n).

It should be pointed out that the expressions (12), (13) are essentially generalizing an inversion of the uncompleted Beta-function for $n=1$ (sinus) and for $n=\infty$ (linear function).

In [16] the following approximations to the uncompleted Beta-functions are given

$$B(\nu, 0.5, x) \approx \frac{x^\nu}{\nu} \quad \text{for small } x,$$

and

$$B(\nu, 0.5, x) \approx \frac{\sqrt{\pi}\Gamma(\nu)}{\Gamma(0.5+\nu)} \quad \text{for } x \text{ close to } 1.$$

It is not difficult to observe (using asymptotic $\Gamma(\nu) \sim \nu^{-1}$ for $\nu \to 0$ [16]) that for small ν from the above formula the following uniform approximation of the Beta-function occurs:

$$B(\nu, 0.5, x) \sim \frac{x^\nu}{\nu} \quad \text{for } \nu \to 0,$$

which proves a validity of our results.

If for Eq. (5) the condition of the form (see [12])

$$x(0) = 0; \quad x = 1 \quad \Rightarrow \dot{x} = 0,$$

is given, then the asymptotics for $n \to \infty$ in the first order of approximation is defined by the formulas:

$$x \approx \sin^{2/(n+1)}\left(\frac{2t^2}{n+1}\right)^{n+1}, \qquad T = 4\left(\frac{\pi}{2}\right)^{2/(n+1)}\left(\frac{n+1}{2}\right)^2.$$

4. STRONGLY NON-LINEAR SYSTEM WITH DAMPING

Consider now a case $\omega^2 = 0$, and the values of damping coefficient γ can be arbitrarily taken:

$$\ddot{x} + \gamma\dot{x} + \varepsilon x^n = 0. \tag{15}$$

As it is known, linear equation with damping can be always reduced to the linear one without damping using a change of variables [17]. However, in a non-linear case a similar reduction can be carried out only for strictly defined values of parameters [18]. Fortunately, for large values of n Eq. (15) can be asymptotically reduced to Eq. (5). For this purpose we first introduce the following change of the independent variable t:

$$\tau = \exp(-\gamma t). \tag{16}$$

Therefore, instead of Eq. (15) one gets

$$\gamma^2 \exp(2\gamma t)\frac{d^2x(\tau)}{d\tau^2} + \varepsilon x^n(\tau) = 0. \tag{17}$$

Applying again a change of variables

$$x = \exp\left(\frac{2\gamma y}{n}\right)\left(\frac{\gamma^2}{\varepsilon}\right)^{1/n}, \tag{18}$$

the Eq. (17) is reduced to Eq. (5) with an order of accuracy of $1/n$. We acknowledge that the changes of variables (17), (18) results from that obtained by Gendelman and Manevitch exact solution to Eq. (1) for restricted values of the original problem parameters [18].

5. SYSTEM WITHOUT DAMPING

We take $\gamma = 0$ and, without loss of generality, we take $\omega^2 = \varepsilon = 1$. Therefore the Eq. (15) reads

$$\ddot{x} + x + x^n = 0. \tag{19}$$

Taking into account the initial conditions (6), a solution to (19) has the following implicit form

$$t = \int_0^x \frac{dx}{\sqrt{1 - x^2 - (2/(n+1))x^{n+1}}}. \tag{20}$$

The following change of variables

$$x^2 + \frac{2}{n+1}x^{n+1} = \sin^2\theta \tag{21}$$

is introduced.

A solution to Eq. (21) with regard to x^2 has the form

$$x = \sin\theta \sqrt[n-1]{1 + x_1}, \tag{22}$$

and it yields:

$$x_1 \approx \frac{-\sin^{n-1}\theta}{1 + \sin^{n-1}\theta}.$$

Finally, the expression (20) can be transformed to the form

$$t = \int_0^{0\le\theta\le\pi/2} \left[\left(1 - \frac{\sin^{n-1}\theta}{1 + \sin^{n-1}\theta}\right)^{1/(n-1)} - \frac{\sin^{n-1}\theta}{1 + \sin^{n-1}\theta}\right] d\theta, \tag{23}$$

with the $1/n$ order accuracy.

The expression standing by the integral (23) can be changed by the following one

$$\left(1 + \frac{\sin^n\theta}{1 + \sin^n\theta}\right)^{1/n} - \frac{\sin^n\theta}{1 + \sin^n\theta} = \frac{1}{1 + \sin^n\theta} + o(1/n).$$

Consider the following approximation

$$\sin^n\theta \sim \begin{cases} \theta^n, & 0 \le \theta < 1/\sqrt[n]{n} \\ 1, & \theta = \pi/2. \end{cases}$$

Using two-point Padé approximants [8], one obtains

$$\sin^n\theta \sim \frac{\theta^n}{1 + \theta^n} \quad \text{for } n \to \infty.$$

Therefore

$$\int_0^\theta \frac{1 + \theta^n}{1 + 2\theta^n} d\theta = \theta - \frac{1}{2n}\ln(1 + 2\theta^n),$$

and the implicit solution has the form

$$t = \theta - \frac{1}{2n}\ln(1 + 2\theta^n).$$

Furthermore, the function θ being sought can be presented in the form

$$\theta = t\sqrt[n]{1 + 0.5\theta_1/t}.$$

The θ_1 is given by the transcendental equation

$$\theta_1 - \ln[1 + 2t^n + t^{n-1}\theta_1] = 0,$$

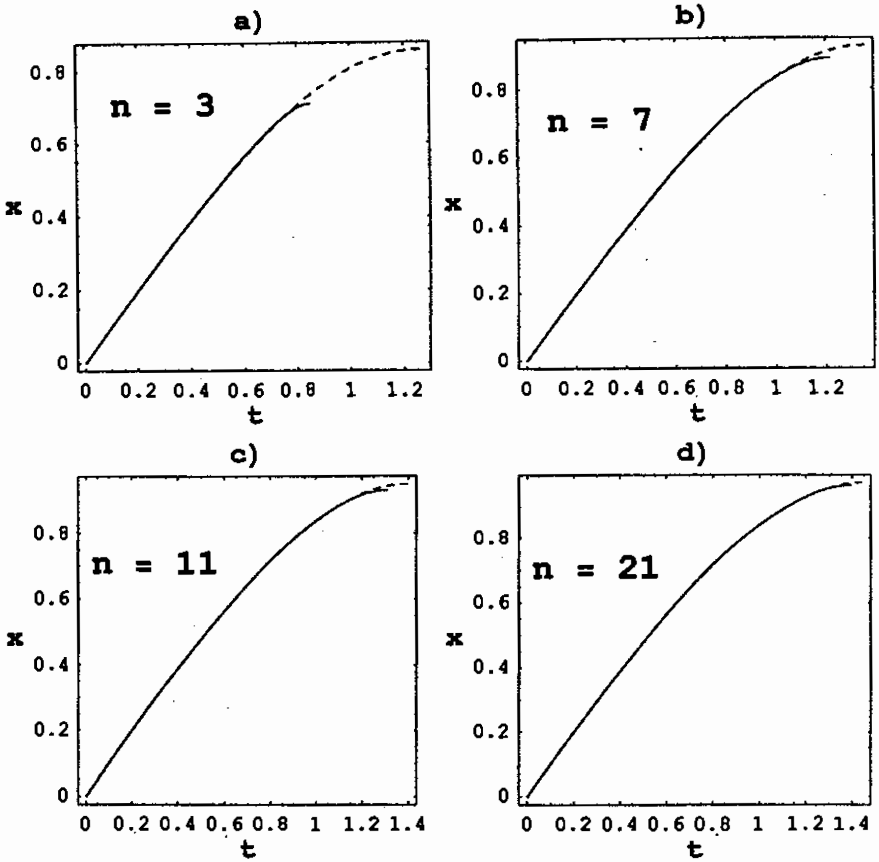

FIGURE 5 A comparison between solutions governed by (19) (dashed line) and (22), (23) for different values of n: (a) $n=3$; (b) $n=7$; (c) $n=11$; (d) $n=21$.

solutions of which can be obtained numerically for each t.

The obtained Eqs. (22) and (23) define a being sought asymptotics. In Fig. 5 a comparison between the numerical solution governed by Eq. (19), and that defined by (22) and (23) is exhibited. It is seen that an accuracy increases with an increase of n and for $n=21$ a very high accuracy is obtained.

6. SYSTEM WITH SMALL DAMPING

In this case the following change of variables is introduced:

$$\tau = \exp(-\gamma t), \qquad x = \exp\left(\frac{2\gamma y}{n}\right)\varepsilon^{-1/n}.$$

The Eq. (1) is reduced to the following one

$$\gamma^2 \frac{d^2x(\tau)}{d\tau^2} + \frac{\omega^2 x(\tau)}{\tau^2} + x^n(\tau) = 0.$$

For integration of the above equation a method of two scales [9,10] or the equivalent methods of averaging, WKB methods [10], asymptotic methods of integration of equation with slowly variable parameters [1,9], and so on, can be applied.

Introduction of a new "fast" variable $\tau_1 = \gamma^{-1}\tau$ yields the equation with formally constant (frozen) coefficients of the form

$$\frac{d^2x(\tau_1)}{d\tau_1^2} + c^2\omega^2 x(\tau_1) + x^n(\tau_1) = 0 \text{ where } c^2 = \tau^{-2}.$$

In order to integrate the above equation the earlier described method can be used.

7. CONCLUDING REMARKS

To conclude, we mentioned the possible directions of investigations which should be rather oriented to solve the particular problems of non-linear dynamical systems.

In order to improve accuracy of the obtained results the higher order approximations should be obtained.

It seems to be interesting to apply the construeted simple analytical solutions to analysis of transition to chaos [21] as well as other strongly non-linear peculiarities of behaviour of low dimensional dynamical systems [22].

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