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Nonlinear coupled problems in dynamics of shells

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Abstract

The coupled system of three partial differential equations governing a flexible shallow shell dynamics is analysed. No any prior assumptions about the temperature distribution through the shell thickness are applied. The efficiency of the method used here when applied to the solution of integral–differential equations with different dimensions (three-dimensional equations related to the Kirchhoff–Love model) and of different type (heat transfer equations and the hyperbolic equations of shell theory) is demonstrated. Many computational results are reported and discussed.

Keywords: Thermal loads; Temperature; Shell; Nonlinearity; Differential and integral equations

1. Introduction

Although the thermoelasticity problems arise in many practical applications like aerospace and nuclear engineering and a lot of various mathematical and numerical approaches are devoted to analysis of constructions subjected to nonuniform high speed thermal and mechanical loads, in majority of publications a coupling and interaction of temperature and deformation fields is not taken into account. Fundamental monograph about coupled thermoelasticity was published by Biot [1]. The thermal problems of theory of plates and shells have been considered in the series of monographs [2–15]. In addition we would like to mention also the following important works devoted to the subject of the paper: Marguerre [16], Malkin [17], Thurn [18], Williams [19,20], Nowacki [21], Rama Rao and Johns [22], Wilde [23,24], Zorski and Lyons [25], Kaul [26], Ryabov

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[27], Borkowski [28,29], Marszał [30], Guryanov [31], Thein Wah [32], Durgaryan and Bakhshinyan [33] and others [34–38]. Since in the reveiw work [39] already 531 references devoted to thermoelasticity problems of plates and shells including 150 papers published only in the period of three years 1965–1967 have been discussed, therefore we cite only a few papers closely related to our research.

Recent publications describing current computational methods of thermoelasticity problems are included in the monograph by Awrejcewicz and Krys'ko [40].

The presented paper summarizes experience acquired by the authors during recent scientific research in the field of thermodynamics of plates and shells and presents generalized theory and certain solutions to some thermoelasticity problems. More rigorous mathematical approach to this problem can be found also in Refs. [41,42].

2. Method of solution for nonlinear coupled problems

In this paper, a shallow shell with a constant thickness h having a shape Ω in plane and a boundary $d\Omega_1$ is considered. The shell material is assumed to be isotropic and physically nonlinear. An orthogonal system of coordinates x_1, x_2, x_3 is used. The x_1, x_2 coordinate axes coincide with the directions of the principal curvatures of the middle surface of the shell, and the x_3 coordinate is oriented towards the centre of curvature of the middle surface and is normal to the middle surface. The displacements of the points of the middle surface along x_1, x_2 and x_3 are denoted by $u_1(x_1,x_2,t)$, $u_2(x_1,x_2,t)$ and $w(x_1,x_2,t)$, respectively. The initial curvatures with respect to x_1, x_2 are denoted by k_1, k_2, respectively.

The governing equations on the basis of the kinematic Kirchhoff–Love model, can be derived using the approach presented in [43,44]. In general, a system of three differential equations with different dimensions: a three-dimensional heat transfer equation with coupling of deformation and temperature, and the two dimensional equations for the shell motion can be obtained.

Then the system of the obtained differential equations can be transformed to the following nondimensional form:

$$\begin{aligned}
&\lambda_1^2 \frac{\partial^2\theta}{\partial x_1^2} + \lambda_2^2 \frac{\partial^2\theta}{\partial x_2^2} + \frac{\partial^2\theta}{\partial x_3^2} = \dot{\theta} + \dot{\beta} e, \\
&\frac{1}{12(1-\nu^2)}\left(\lambda^{-2}\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \lambda^2 \frac{\partial^4 w}{\partial x_2^4}\right) - L(w,F) - \nabla_k^2 F + \lambda^{-1}(M_T + \Delta_T M)_{x_1x_1} \\
&\quad + \lambda(M_T + \Delta_T M)_{x_2x_2} - \lambda^{-1}(\Delta M_{11})_{x_1x_1} - \lambda(\Delta M_{22})_{x_2x_2} - 2(\Delta M_{12})_{x_1x_2} - q + æ(\ddot{w} + \varepsilon\dot{w}) = 0,
\end{aligned} \tag{1}$$

$$\begin{aligned}
&\lambda^{-2}\frac{\partial^4 F}{\partial x_1^4} + 2\frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} + \lambda^2 \frac{\partial^4 F}{\partial x_2^4} + (1-\nu)[\lambda^{-1}(N_T + \Delta_T T)_{x_1x_1} + \lambda(N_T + \Delta_T T)_{x_2x_2}] + \nabla_k^2 w + \tfrac{1}{2}L(w,w) \\
&\quad - \lambda^{-1}(\Delta T_{22} - \nu\Delta T_{11})_{x_1x_1} - \lambda(\Delta T_{11} - \nu\Delta T_{22})_{x_2x_2} + 2(1+\nu)(\Delta T_{12})_{x_1x_2} = 0.
\end{aligned}$$

In the above and later the following notation is used: $\varepsilon_{ij}(i,j=1,2)$ are the tangential deformations of the middle surface, and $æ_{ij}$ are its bending deformations; $-h/2 \leqslant x_3 \leqslant h/2$; K, G are the moduli of volume compression and shear, e_0 is the average deformation, $\gamma(e_1)$ is the shear function; $æ(e_0) \equiv 1$ is the elongation function; ν is Poisson's ratio; E is Young's modulus; $\theta(x_1,x_2,x_3,t) = T_1(x_1,x_2,x_3,t) - T_0$ is the temperature increase at the point x_1,x_2,x_3 at the time moment t, and $T_1(x_1,x_2,x_3,t)$ is the absolute temperature at this point at time t; $|\theta/T_0| \ll 1$ is assumed; α_t is the coefficient of linear thermal expansion;

$$
\begin{aligned}
T_{ii} &= \frac{Eh}{1-\nu^2}(\varepsilon_{ii}+\nu\varepsilon_{jj}) + \Delta T_{ii} - N_T - \Delta_T T,\\
M_{ii} &= \frac{Eh^3}{12(1-\nu^2)}(æ_{ii}+\nu æ_{jj}) + \Delta M_{ii} - M_T - \Delta_T M, \quad (i,j=1,2,\ i\neq j),\\
T_{12} &= \frac{Eh}{2(1+\nu)}\varepsilon_{12} + \Delta T_{12}, \quad M_{12} = \frac{Eh^3}{24(1+\nu)}æ_{12} + \Delta M_{12}, \quad N_T = \frac{E\alpha_t}{1-\nu}\int_{-h/2}^{h/2}\theta\,\mathrm{d}x_3,\\
M_T &= \frac{E\alpha_t}{1-\nu}\int_{-h/2}^{h/2}\theta x_3\,\mathrm{d}x_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta T_{ii} &= \frac{Eh}{1+\nu}[\varepsilon_{ii}p_1^1 + \varepsilon_{jj}p_1^2 + h(æ_{ii}p_2^1 + æ_{jj}p_2^2)],\\
\Delta M_{ii} &= \frac{Eh^2}{1+\nu}[\varepsilon_{ii}p_2^1 + \varepsilon_{jj}p_2^2 + h(æ_{ii}p_3^1 + æ_{jj}p_3^2)], \quad (i,j=1,2,\ i\neq j),\\
\Delta T_{12} &= \frac{Eh}{2(1+\nu)}(\varepsilon_{12}p_1^3 + hæ_{12}p_2^3), \quad \Delta M_{12} = \frac{Eh^2}{2(1+\nu)b}(\varepsilon_{12}p_2^3 + hæ_{12}p_2^3),\\
\Delta_T T &= E\alpha_T W_1, \quad \Delta_T M = E\alpha_T W_2, \quad P_k^n = \frac{1}{h^k}\int_{-h/2}^{h/2} F_n x_3^{k-1}\,\mathrm{d}x_3, \quad (n,k=1,2,3);
\end{aligned}
$$

theory of small elasto-plastic deformations yields $G\gamma(e_i) = (1/3)\sigma_i(e_i)/e_i$, $\gamma(e_i) = \sigma_i(e_i)/3Ge_i$; $\nabla^2\theta = (T_0/R_T)(\partial/\partial t)((c\theta/T_0) + \beta e)$, where c is the specific heat capacity; $e = \varepsilon_{11}^z + \varepsilon_{22}^z + \varepsilon_{33}^z$; ∇^2 is the three-dimensional Laplace operator; $\beta = k\alpha_t$; F is Airy's stress function;

$$
\begin{aligned}
T_{ii} &= \frac{\partial F}{\partial x_{jj}^2}, \quad T_{12} = \frac{\partial^2 F}{\partial x_1 \partial x_2} \quad (i,j=1,2,\ i\neq j); \quad \nabla_k^2(\cdot) = K_2\frac{\partial^2(\cdot)}{\partial x_1^2} + K_1\frac{\partial^2(\cdot)}{\partial x_2^2};\\
L((\cdot),(\cdot)) &= \frac{\partial^2(\cdot)}{\partial x_1^2}\frac{\partial^2(\cdot)}{\partial x_2^2} - 2\frac{\partial^2(\cdot)}{\partial x_1\partial x_2}\frac{\partial^2(\cdot)}{\partial x_1\partial x_2} + \frac{\partial^2(\cdot)}{\partial x_2^2}\frac{\partial^2(\cdot)}{\partial x_1^2}.
\end{aligned}
$$

Note that using the introduced information a reader is able to derive the analysed set of equations or, alternatively, more detailed description yielding the analysed equations is included in chapter 6 of the book [40].

As the initial conditions we take

$$\begin{aligned} &w\big|_{t=0} = \varphi_1, \quad \theta\big|_{t=0} = \varphi_2, \\ &w\big|_{t=0} = \Psi_1, \end{aligned} \tag{2}$$

and as the boundary conditions we take

$$\begin{aligned} &w = M_{11} = \varepsilon_{22} = \frac{\partial \varepsilon_{22}}{\partial x_1} - \frac{\partial \varepsilon_{12}}{\partial x_2} = 0, \quad x_1 = \text{const}, \\ &w = M_{22} = \varepsilon_{11} = \frac{\partial \varepsilon_{11}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_1} = 0, \quad x_2 = \text{const}, \end{aligned} \tag{3}$$

$$\frac{\partial \theta}{\partial x_3}\bigg|_{x_3=0.5} = q_T, \quad \frac{\partial \theta}{\partial x_3}\bigg|_{x_3=-0.5} = 0, \quad \frac{\partial \dot{\theta}}{\partial h}\bigg|_{\Gamma} = 0. \tag{4}$$

The boundary conditions (4) characterize a thermal shock with an intensity q_T on the surface $x_3 = 0.5$ (the other surfaces of the shell are thermally isolated). More detailed description of the boundary conditions is given in Refs. [45–48]. We take $t \in [0, t_1]$, where t_1 is the observation time of the behaviour of the shell; $\Omega_1 = (0,1) \times (0,1)$ is the space in which the independent variables x_1, x_2 vary, which is referred to as the middle surface; $\overline{\Omega}_1 = \Omega_1 \cup \partial\Omega_1$; $\partial\Omega_1$ is the edge of the middle surface; $Q_1 = \Omega_1 \times (0, t_1)$; $\Gamma = \partial\Omega_1 \times [0, t_1]$; $\Omega_2 = \Omega_1 \times (-h/2, h/2)$; h is the constant shell thickness; $\overline{\Omega}_2 = \Omega_2 \cup \partial\Omega_2$; $\partial\Omega_2$ is the surface surrounding the shell volume in three-dimensional space; $Q_2 = \Omega_2 \times (0, t_1)$; and $S = \partial\Omega_2 \times (0, t_1)$. The following nondimensional parameters have been used:

$$\begin{aligned} &\bar{x}_1 = \frac{x_1}{a}, \quad \bar{x}_2 = \frac{x_2}{b}, \quad \bar{x}_3 = \frac{x_3}{h}, \quad \bar{w} = \frac{w}{h}, \quad \lambda_1 = \frac{h}{a}, \quad \lambda_2 = \frac{h}{b}, \quad \lambda = \frac{a}{b}, \\ &\bar{k}_1 = k_1 \frac{a^2}{h}, \quad \bar{k}_2 = k_2 \frac{b^2}{h}, \quad \bar{F} = \frac{F}{Eh^3}, \quad \bar{\varepsilon} = \varepsilon \frac{h^2}{\alpha}, \quad \bar{t} = t \frac{\alpha}{h^2}, \\ &æ = \frac{a^2 b^2 \rho \alpha^2}{Eh^6}, \quad \bar{q}_T = q_T \frac{ab\alpha_T}{hk_t}, \quad \bar{\theta} = \theta \frac{\alpha_T ab}{h^2}, \\ &\Delta\overline{T}_{ij} = \Delta T_{ij} \frac{ab}{Eh^3}, \quad \Delta\overline{M}_{ij} = \Delta M_{ij} \frac{ab}{Eh^4}, \\ &\beta = \frac{T_0 E \alpha_T}{3(1-2\nu)\rho h^2}. \end{aligned} \tag{5}$$

In (1)–(4), the bars over nondimensional expressions have been omitted.

The first equation of (1) is three-dimensional and of parabolic type, whereas the second and third ones are two-dimensional. In addition, they are also integral-type equations. In order to reduce the partial differential equations to ordinary differential equations with respect to time, the finite-difference method is used to discretize the derivatives along the spatial coordinates x_1, x_2, x_3 with an $O(h^2)$ approximation. This method results in ODEs for w and θ, and a system of algebraic equations for the Airy function F.

The following difference operators have been used in relation to the spatial coordinates x_1, x_2, x_3, with a mesh of uniform spacings h_1, h_2, h_3, respectively,

$$\Lambda_i y = \frac{1}{h_i^2}[y(x_i - h_i) - 2y(x_i) + y(x_i + h)]_{i=1,2,3},$$

$$\Lambda_{ij} y = \frac{1}{4h_i h_j}\left[y(x_i + h_i, x_j + h_j) + y(x_i - h_i, x_j - h_j) - y(x_i + h_i, x_j - h_j) - y(x_i - h_i, x_j + h_j)\right],$$

$$\Lambda_i^2 y = \frac{1}{h_i^4}[y(x_i - 2h_i) - 4y(x_i - h_i) + 6y_i(x_i) - 4y(x_i + h_i) + y(x_i + 2h_i)],$$

$$\Lambda_{ij}^2 y = \frac{1}{h_i^2 h_j^2}\left[y(x_i - h_i, x_j - h_j) - 2y(x_i - h_i, x_j) + y(x_i - h_i, x_j + h_j) - 2y(x_i, x_j - h_j) + 4y(x_i, x_j) - 2y(x_i, x_j + h_j) + y(x_i + h_i, x_j - h_j) - 2y(x_i + h_i, x_j) + y(x_i + h_i, x_j + h_j)\right]. \tag{6}$$

Using the difference operators (6) and introducing a new variable $\mathrm{d}w/\mathrm{d}t = \dot{w}$, the system of ODEs is reduced to first-order ODEs with respect to time and to algebraic equations:

$$\left[1 + \frac{1+\nu}{1-\nu}\beta\right]\frac{\mathrm{d}\theta_{ijk}}{\mathrm{d}t} = (1-2\nu)\beta\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda^{-1}\Lambda_1 F_{ij} + \lambda\Lambda_2 F_{ij} - \left(\Delta T_{11} + \Delta T_{22} - 2N_T - 2\Delta_T T\right)_{ij} - \frac{x_{3k}}{1-\nu}(\lambda^{-1}\Lambda_1 w_{ij} + \lambda\Lambda_2 w_{ij})\right] \quad (i = 0,\ldots,n;\ j = 0,\ldots,m;\ k = 0,\ldots,p), \tag{7}$$

$$\frac{\mathrm{d}w_{ij}}{\mathrm{d}t} = \dot{w}_{ij},$$

$$\frac{\mathrm{d}\dot{w}_{ij}}{\mathrm{d}t} + \varepsilon\dot{w}_{ij} = (-1/æ)[A(w) + B(w,F) + C(M_T, \Delta_T M, \Delta M_{11}, \Delta M_{22}, \Delta M_{12})] + \lambda_1^2\Lambda_1\theta_{ijk} + \lambda_2^2\Lambda_2\theta_{ijk} + \Lambda_3\theta_{ijk}, \tag{8}$$

$$D(F) = E(w) + G(N_T, \Delta_T T, \Delta T_{11}, \Delta T_{22}, \Delta T_{12}). \tag{9}$$

In (8) and (9), the following notation has been used:

$$A(w) = \frac{1}{12(1-\nu^2)}\left(\lambda^{-2}\Lambda_1^2 w_{ij} + 2\Lambda_{12}^2 w_{ij} + \lambda^2\Lambda_2^2 w_{ij}\right),$$

$$B(w,F) = K_1\Lambda_2 F_{ij} + K_2\Lambda_1 F_{ij} + \Lambda_1 w_{ij}\Lambda_2 F_{ij} + \Lambda_2 w_{ij}\Lambda_1 F_{ij} - \Lambda_{12} w_{ij}\Lambda_{12} F_{ij}, C(M_T, \Delta_T M, \Delta M_{11}, \Delta M_{22}, \Delta M_{12}) = \lambda^{-1}\Lambda_1(M_T + \Delta_T M - \Delta M_{11})_{ij} + \lambda\Lambda_2(M_T + \Delta_T M - \Delta M_{22})_{ij} - \Lambda_{12}(\Delta M_{12})_{ij},$$

$$D(F) = 12(1-\nu^2)A(F),$$

$$E(w) = -K_1\Lambda_2 w_{ij} - K_2\Lambda_1 w_{ij} - \Lambda_1 w_{ij}\Lambda_2 w_{ij} + (\Lambda_{12} w_{ij})^2,$$

$$G(N_T, \Delta_T T, \Delta T_{11}, \Delta T_{22}, \Delta T_{12}) = -(1-\nu)\lambda^{-1}\Lambda_1(N_T + \Delta_T T)_{ij} + \lambda(1-\nu)\Lambda_2(N_T + \Delta_T T)_{ij}$$
$$+ \lambda^{-1}\Lambda_1(\Delta T_{22} - \nu\Delta T_{11})_{ij} + \lambda\Lambda_2(\Delta T_{11} - \nu\Delta T_{22})_{ij} - \frac{1+\nu}{2}\Lambda_{12}(\Delta T_{12})_{ij}. \quad (10)$$

The following relation has been used in the first equation of (1) in order to describe the volume extension e:

$$e = (1-2\nu)\left[\lambda^{-2}F_{x_1x_1} + \lambda F_{x_2x_2} - \Delta T_{11} - \Delta T_{22} + 2(N_T + \Delta_T T)\right]$$
$$-\frac{1-2\nu}{1-\nu}x_3(\lambda^{-1}w_{x_1x_1} + \lambda w_{x_2x_2}) + \frac{1+\nu}{1-\nu}\theta. \quad (11)$$

This leads to (7).

In the procedure for solution of the coupled problem the time derivative in the right-hand side of (7) is approximated by a one-sided finite-difference relation:

$$\frac{\mathrm{d}y_{ijk}}{\mathrm{d}t} = \frac{1}{12h_t}(3y_{ijk}^{e-4} - 16y_{ijk}^{e-3} + 36y_{ijk}^{e-2} + 25y_{ijk}^{e}) + O(h_t^4), \quad (12)$$

where h_t is a time interval.

We need to attach to the difference Eqs. (7)–(10) a group of boundary conditions, which must be formulated using a central finite-difference relation (it is necessary to find the values of the fundamental functions on nodes at the edge of the shell).

A procedure for solution of the initial-boundary value problem has been constructed in the following way.

Using the results found at the previous time step, w_{ij}, N_{Tij}, ΔT_{11ij}, $\Delta_T T_{ij}$, ΔT_{22ij}, ΔT_{12ij}, the system of linear algebraic equations (9) is solved for Airy's function F_{ij}. The values of F_{ij} obtained and values of N_{Tij}, $\Delta_T T_{ij}$, M_{Tij}, $\Delta_T M_{ij}$, ΔM_{11ij}, ΔM_{22ij}, ΔM_{12ij} found at the previous time step are substituted into the right-hand sides of (7) and (8). The ODEs are then integrated with respect to time using a method which will be explained later.

Using the temperature distribution in the shell volume found above, and integrating over the thickness with the help of Simpson's rule,

$$\int_{-1/2}^{1/2} \theta(x_1, x_2, x_3)\,dx_3 = \frac{1}{6\Delta h}[\theta_0 + \theta_{2n} + 4(\theta_1 + \cdots + \theta_{2n-1}) + 2(\theta_2 + \cdots + \theta_{2n-2})], \quad (13)$$

the temperature terms N_T and M_T are found. They are used in later calculations.

Then, for each point of the shell volume, the deformation intensity is calculated:

$$e_i = \frac{\sqrt{2}}{3}\left[(e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{22} - e_{11})^2 + \frac{3}{2}e_{12}^2\right]^{1/2}. \quad (14)$$

A dependence $\sigma_i(e_i)$ is assumed (this depends on the shell material used), and σ_1 is defined at each point of the shell volume. The shear function $\gamma(e_i)$ is calculated and then the nonlinear integral terms are found, which are used during the second step of the algorithm.

In the first time step, owing to the initial conditions (2) $N_T = M_T = 0$, $\gamma \equiv 1$, all nonlinear terms are equal to zero. Use of N_T and M_T obtained in the previous step allows us to solve the integral–differential equations (1) as a system of differential equations.

We emphasize that we need to solve the system of algebraic equations (9) for each time step, but the matrix of this system remains constant. The Gauss method can be used to solve (9), and a transformation to a triangular form of the matrix is performed only once during the computations, in the first step of the calculation. The choice of the Gauss method possesses many advantages with respect to both high accuracy and a relatively short computational time. For instance, the use of a relaxation method requires computations five time longers to solve the problem under discussion.

The use of the three-dimensional heat transfer Eq. (7) is essential, because it makes any additional hypothesis about the temperature distribution through the thickness unnecessary.

During the integration of the ODEs (7) and (8), a very careful verification of each computational step is needed. In addition, an economical choice of the integration method used for the ODEs (7) and (8) is important. Consideration of the approaches proposed in this paper has led the authors to apply a combination of the explicit and implicit Adams methods, known in the literature as the Adams–Bashford or prediction–correction method.

The methods used can be represented using the following formula:

$$P_k(EC_{k+1})^n E, \tag{15}$$

where P_k means prediction, i.e. finding a solution using the explicit Adams method; k denotes the order of accuracy; E means to computation of the right-hand sides of the ODEs; C_{k+1} means correction, i.e. improvement of the accuracy using the implicit Adams method, to an accuracy of order $(k+1)$ (here the solution obtained after the operation P_k is used as an initial approximation); and n denotes the required number of iterations.

When the operations $P_k(EC_{k+1})$ have been carried out, it is recommended to compute once more the right-hand sides of the ODEs in order to use them during the next steps of computation.

The right-hand sides of the ODEs are computed $(n+1)$ times, where n is independent of k; the Runge–Kutta method of the kth order needs computation of the right-hand sides to be performed k times.

In order to compare the accuracy and computation times of the methods, a few first-order ODEs have been solved using the Runge–Kutta method of the fourth order, and the Adams method in accordance with the schemes P_3EC_4E and P_5EC_6E. Results have been obtained for some equations with known solution, including $y = e^x \sin 10x$, and $y = \exp(2(x^{11}/11) - (x^{12}/12))$. For $x \in [0; 3]$, the following results were obtained: for $h_t = 0.1$, the Runge–Kutta method gives more accurate results; for $h_t = 0.01$, the Runge–Kutta method gives a solution similar to that obtained by the scheme P_5EC_6E; for $h_t = 0.002$, the Adams methods give solutions with better accuracy in comparison with the Runge–Kutta method. Note that a solution obtained by the scheme $P_kEC_{k+1}E$ with $k = 3$ does not differ (practically) from that obtained with $k = 5$, but a test of the scheme with $k = 3$ in comparison with the fourth order Runge–Kutta method under conditions of a shortage of computational resources achieved about 45% better accuracy. This was observed for all model solutions used.

On the basis of these obtained results, the scheme (15) with $k = 3$ was used to integrate the system (7)–(9). This scheme, with one iteration, achieved 40% of the computational time of the fourth-order Runge–Kutta method, when applied to the nonlinear coupled problem.

3. Relaxation method

Current methods applied to the solution of nonlinear problems of shells (with geometrical and physical nonlinearities) generally use a projection onto a system of nonlinear differential or nonlinear algebraic equations, with successive linearization using either Newton–Raphson or other iterative methods (for instance, the relaxation method). Nowadays there exists a wide literature devoted to such methods and their algorithms.

In 1963 Fedosev [49] proposed a dynamical approach to solve a problem related to the stability of shells. From a mathematical point of view, this method is called the "set-up" method [50]. The main idea of this method is that the solution of the nonlinear partial differential equations is reduced to a Cauchy problem of ODEs which is linear in time. This means that this method linearizes the nonlinear equations and decreases their dimension.

We discuss briefly an advantage of this method here. From a mathematical point of view, the set-up method can be treated as an iterative method to solve nonlinear algebraic equations, where each time step provides a new approximation to the exact solution. Like all iterative methods, this one is characterized by a high accuracy of computation. In addition, it does not have the common disadvantage of iterative methods of a high sensitivity to the choice of the initial approximation. Additionally, the set-up method not only gives a very simple rule for obtaining nonunique solutions of static problems, but also allows one to find the stable and unstable branches of the equilibrium position of the system under consideration and to capture all process of the jumping behaviour of a shell.

In the process of solution of homogeneous equations via traditional methods, in order to obtain a nontrivial solution one needs to introduce an artificial excitation (in the theory of shells this corresponds, for instance, to a small transverse load, a small curvature or some other initial imperfection). However, this influences (sometimes significantly) the results obtained. In the case of the set-up method, the initial conditions play the role of the initial excitations, and small changes of these conditions do not influence the static solution obtained. Another advantage of the method is related to its simple realization, because nowadays there are many effective algorithms and programs devoted to solution of the Cauchy problem.

Let us clarify the method for obtaining unstable solutions using an example of an arbitrary nonlinear algebraic equation,

$$f(x) = 0. \tag{16}$$

We construct two differential equations from for (16),

$$c\left(\frac{d^2x}{dt^2} + \varepsilon \frac{d\dot{x}}{dt}\right) = \pm f(x). \tag{17}$$

In order to obtain the complete set of roots of (16), we need to solve the two differential equations (17). However, in practice we can do this in the following way.

We take an arbitrary initial condition (the initial approximation) $x_{in} = x_0$ for (17) with the positive right-hand side, and solving the equation we obtain x_1. Then we take the initial condition $x_0 = x_1 + \delta$ and solve (17) with the negative value of the right-hand side, and so on. As an example, we consider the solution of the equation

$$f(x) = x^4 - 12x^3 + 47x^2 - 60x = 0. \tag{18}$$

This equation has four real roots $x_1 = 0$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$. We have solved this equation by the set-up (SU) and Newton (N) methods. In Table 1 the roots found by both methods, the number of iterations (I) needed to obtain the solution and the initial approximations are reported.

It can be seen that the SU method, in contrast to the Newton method, finds more exactly a stable root of the Eq. (18) close to the initial approximation.

Let us consider one more (transcendental) equation:

$$f(x) = \arctan x = 0. \tag{19}$$

This equation has a root $x = 0$. It is known that for (19) the Newton method approaches this root only for initial values such that $|x| < 1.39$, but the set-up method approaches this root for $\forall x_{in}$. For instance, taking $x_{in} = 2$, the root $x = 0$ is found after 33 iterations.

We now illustrate the high efficiency of the set-up method using a series of nonlinear problems of the theory of plates and shells, which are characterized by a wide range of properties with respect to different nonlinearity types (geometrical, physical and geometrical–physical) as well as different models (Kirchhoff–Love or Timoshenko).

We formulate now a nonlinear equation governing the dynamics of a shallow shell including transverse shear effects, in hybrid form:

$$\begin{aligned}
&\nabla_k^2 F + L(w,F) + \frac{2}{3}\left[\lambda_1\left(\frac{\partial \gamma_x}{\partial x_1} + \frac{\partial^2 w}{\partial x_1^2}\right) + \lambda_2\left(\frac{\partial \gamma_y}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2}\right)\right] + q - \frac{\partial^2 w}{\partial t^2} - \varepsilon\frac{\partial w}{\partial t} = 0,\\
&\frac{1}{5}\left[\lambda^{-2}\frac{\partial^2 \gamma_x}{\partial x_1^2} + A_{1212}\frac{\partial^2 \gamma_y}{\partial y_2^2} + (A_{1122} + A_{1212})\frac{\partial^2 \gamma_y}{\partial x_1 \partial x_2}\right] - \frac{1}{20}\left[\lambda^{-1}\frac{\partial^2 w}{\partial x_1^2} + (A_{1122} + A_{1212})\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]\\
&- 2\lambda_1\left(\gamma_x + \frac{\partial w}{\partial x_1}\right) = 0 \quad (x_1 \leftrightarrow x_2),\\
&\nabla^4 F + \nabla_x^2 w + \frac{1}{2}L(w,w) = 0.
\end{aligned} \tag{20}$$

Table 1
Comparison of SU and Newton methods

Exact values	$x_1 = 0$	$x_2 = 3$	$x_3 = 4$	$x_4 = 5$
Initial approximation (x_{in})	$x_1 = 2.5$	$x_2 = 3.5$	$x_3 = 4.5$	$x_4 = 10$
SU	$x_1 = 0$	$x_2 = 3$	$x_3 = 4$	$x_4 = 5$
N	$x_1 = 5$	$x_2 = 5$	$x_3 = 3$	$x_4 = 5$
Iteration number I(SU)	68	54	87	60
Iteration number I(N)	7	12	9	10

The symbol ($x_1 \leftrightarrow x_2$) denotes that a third equilibrium equation can be obtained from the second equation by a cyclic change of indices. The system (20) has been transformed to a nondimensional form as earlier. The Eq. (20) include several geometrical and physical–geometrical quantities; in particular, $k_{x_1} = a^2/(R_{y_1}(2h))$ and $k_{x_2} = a^2/(R_{y_2}(2h))$ (geometrical parameters characterizing the curvatures ($1/R_{x_1}$ and $1/R_{x_2}$) of the shell and its size ($a, b, 2h$)), and $\lambda_1 = G_{13}/(A_{1111}(b/2h)^2)$ and $\lambda_2 = G_{23}/(A_{1111}(a/2h)^2)$ are the physical–geometrical parameters characterizing the effect of transverse shear on the solution.

The following boundary conditions are attached to (20):

$$w = M_n = \gamma_{\bar{n}} = T_n = T_{\bar{n}}|_\Gamma = 0,$$

where $n = x$, $\bar{n} = y$ define the shell boundary; $t \in [0, T]$, where T is the observation time; $\Omega_1 = (0;1) \times (0;1)$ is the space in which the independent parameters x_1 and x_2 vary, defining the middle surface; $\bar{\Omega}_1 = \Omega_1 \cup \partial\Omega_1$, where $\partial\Omega_1$ is the contour (edge) of the middle surface; $Q_1 = \Omega_1 \times (0; T)$; and $\Gamma = \partial\Omega_1 \times [0; 1]$.The following initial conditions initial conditions are applied:

$$w|_{t=0} = \left.\frac{\partial w}{\partial t}\right|_{t=0} = 0. \tag{21}$$

The problem defined by (20) and (21) is reduced to a Cauchy problem Cauchy problem using higher approximations of the Bubnov–Galerkin method Bubnov–Galerkin method:

$$\begin{aligned} w &= \sum_{i,j=1}^{N} A_{ij}(t) \sin i\pi x_1 \sin j\pi x_2, \\ F &= \sum_{i,j=1}^{N} B_{ij}(t) \sin i\pi x_1 \sin j\pi x_2, \\ \gamma_{x_1} &= \sum_{i,j=1}^{N} C_{ij}(t) \cos i\pi x_1 \sin j\pi x_2, \\ \gamma_{x_2} &= \sum_{i,j=1}^{N} D_{ij}(t) \sin i\pi x_1 \cos j\pi x_2. \end{aligned} \tag{22}$$

As a result we obtain a system of N ODEs for $A_{ij}(t)$ and a system of algebraic equations for $B_{ij}(t)$, $C_{ij}(t)$ and $D_{ij}(t)$, which fortunately can be solved in closed form. The integrals over the middle surface $\Omega_1 = (0;1) \times (0;1)$ were computed analytically for each of the problems considered. The system of ODEs for $A_{ij}(t)$ with the initial conditions (21) was solved using the Runge–Kutta method with an automatic choice of the integration step.

Let us analyse the influence of the damping coefficient ε appearing in the first equation of the system (20) on the value of the dynamic critical load, when an impulse load of infinitesimal duration is applied (Fig. 1). We definite $\varepsilon = \varepsilon_{cr}$ as $\min \varepsilon$, when the dynamic critical load $q = q_{cr}^d$ is equal to the static critical load $q = q_{cr}^s$ (i.e. $q_{cr}^s = q_{cr}^d$). The criterion for a dynamic load given in [40] is used here. Plots of $q(w)$ and $t(w)$ are shown in the Fig. 2 for a cylindrical shell ($k_x = 0$, $k_{x_2} = 48$,

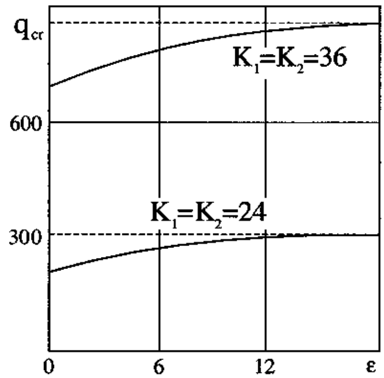

Fig. 1. Influence of damping coefficient ε on a dynamical critical load.

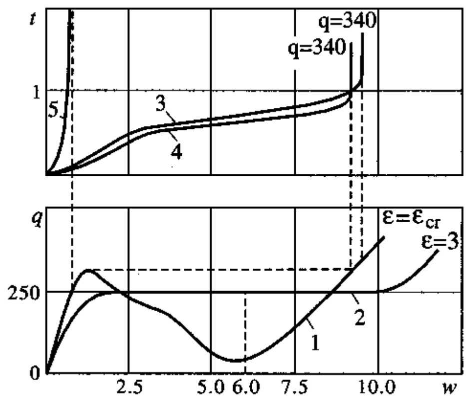

Fig. 2. Dependences $q(w)$, $t(w)$ for a cylindrical shell ($N = 5$).

$\lambda_1 = \lambda_2 = 1000$). Curve 2 corresponds to $\varepsilon = 3$, whereas curve 1 corresponds to a static solution for $q^d_{cr} < q^s_{cr}$. Both parts of the figure need to be analysed simultaneously. For $\varepsilon = \varepsilon_n$ and for a given load q, the deflection approaches a constant value with increase of time. Having a set of values $\{t(w_i), q_i)\}$ one can construct a dependence $q(w)$. Curves 3, 4, 5 for $t(w)$, were obtained for $q = 255$, 320 and 350, with $\varepsilon = \varepsilon_{cr}$. From Fig. 1 it can be seen that for $\varepsilon \to \varepsilon_{cr}$, $q^d_{cr} \to q^s_{cr}$, for an impulse load that is uniformly distributed over the shell surface and constant in time (the results were obtained from (22) with $N = 5$). A further increase of the number of terms in (22) does not change the fundamental functions or their derivatives up to second order. The efficiency and high accuracy of the Bubnov–Galerkin method when applied to both static and dynamic problems were demonstrated long ago (see, for instance [51]).

In solving nonlinear problems of composite shells exhibiting weak shear stiffness within the Timoshenko-type theory, an increase in the number of modes of equilibrium for a given load occurs, and local stability loss is observable. The stable equilibrium configuration was chosen here from the set of various possible configurations using the set-up method. Plots of $q(w)$ for $k_{x_1} = k_{x_2} = 24$, $\lambda = 1$, $\lambda_1 = \lambda_2 = 40, 20, 10, 5$ (corresponding to curves 1, 2, 3, 4, respectively) are presented in Fig. 3 (obtained from (22) with $N = 7$). The $q(w)$ curves were obtained using a static

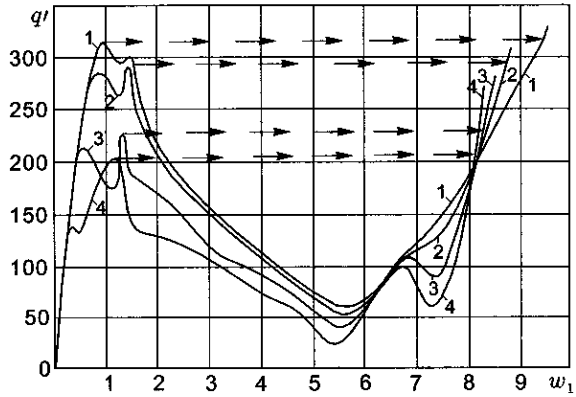

Fig. 3. Dependences $q(w)$, $t(w)$ for a cylindrical shell ($N = 7$).

method, i.e. a large set of nonlinear algebraic equations for A_{ij}, B_{ij}, C_{ij} and D_{ij} was solved using the Newton–Raphson method for a fixed value of the deflection in the centre of the shell. The arrows indicate external static loads which have been detected through the set-up method (the other equilibrium configurations are not physically realized).

In practice, one can find shallow shell structures with values of k_{x_1} and k_{x_2} even greater than 500. However, in the literature, only problems related to $k_{x_1}, k_{x_2} < 50$ have been addressed. The reason is clear. When the geometric parameters are increased, the number of possible equilibrium configurations increases greatly. This requires one to increase the number of terms in the approximating functions (and nodes) used when either variational or finite-difference methods are applied to static problems, in order to properly describe the complex surface shape. The convergence of the solution obtained for the nonlinear algebraic equations dramatically decreases, because their order increases.

The set-up method avoids the above disadvantages. We do not need to solve any system of equations at all, and in particular, not a nonlinear one. There is no problem in analysing shells with $24 \leqslant k_{x_1} = k_{x_2} \leqslant 500$ (we have taken $N = 15$ in (21)). The numerical results agree fully with experiment for $k_{x_1} = k_{x_2} = 409.1$. The difference in q^s_{cr} is 15%, and qualitatively the picture of stability loss is the same [52].

In Fig. 4, contours of equal relative deflection $w(x_1, x_2)$, $0 \leqslant x_1, x_2 \leqslant 0.5$, for $k_{x_1} = k_{x_2} = 409.1$ are drawn. Cases a–e are related to the pre-critical state, whereas f corresponds to a post-critical stability loss. With an increase of k_{x_1} and k_{x_2}, the shell loses its stability first in the corner zones, and during "jump" behaviour, holes appear on the axial curves, which is also indicated by experimental data.

We now illustrate the high efficiency of the set-up method using an example of a physically and geometrically nonlinear problem of a plate. We recall (1) without the temperature terms:

$$\frac{1}{12(1-\nu^2)}\nabla^4 w - L(w,F) - \nabla_k^2 F - \lambda^{-1}(\Delta M_{11})_{x_1x_1} - 2(\Delta M_{12})_{x_1x_2} - \lambda(\Delta M_{22})_{x_2x_2} = q - æ(\ddot{w} + \varepsilon\dot{w}),$$

$$\nabla^4 F + \nabla_k^2 w + \frac{1}{2}L(w,w) - \lambda^{-1}(\Delta T_{22} - \nu\Delta T_{11})_{x_1x_1} + 2(1+\nu)(\Delta T_{12})_{x_1x_2} - \lambda(\Delta T_{11} - \nu\Delta T_{22})_{x_2x_2} = 0. \tag{23}$$

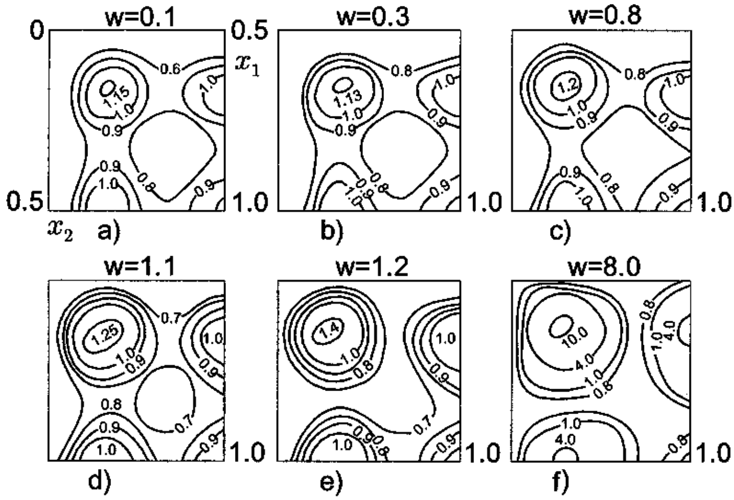

Fig. 4. Diagrams for equal relative deflections $w(x_1, x_2)$.

The construction of a solution to (23) using the set-up method has been earlier described. We consider the geometrically and physically nonlinear problem of bending of a square plate ($\lambda = 1$). We assume that a uniformly distributed load with intensity q on the surface $x_3 = -0.5$. The physical parameters of the plate material (AMC alloy) have been taken to be as follows: $E = 69$ GPa, $\mu = 0.3$, $\rho = 2800$ kg/m^3, $\sigma_i(e_i)$, and

$$\begin{aligned} &\sigma_i = E_1 e_i, \quad e_i \leqslant e_s, \\ &\sigma_i = E_1 e_s + E_2(e_i - e_s), \quad e_i > e_s. \end{aligned} \tag{24}$$

For the shear function, we obtain

$$\begin{aligned} &\gamma(e_i) = \frac{\sigma_i(e_i)}{3Ge_i}, \\ &\gamma \equiv 1, \quad e_i \leqslant e_s, \\ &\gamma = \frac{E_1}{E} + \left(1 - \frac{E_1}{E}\right)\left(\frac{e_s}{e_i}\right), \quad e_i > e_s, \\ &\frac{E_1}{E} = 0.57735, \quad e_s = 0.98 \times 10^{-3}. \end{aligned} \tag{25}$$

The geometrical parameters of the plate were taken to be as follows $a = b = 0.1$ m, $a/h = 50$. The boundary conditions were defined by $M_{11} = F = \partial F/\partial x_1 = 0$, $x_1 = \text{const}$, $\partial M_{11}/\partial x_1 + 2\partial M_{12}/\partial x_1 = 0$, $x_1 = \text{const}$ and the initial conditions were $w|_{t=0} = 0$, $\dot{w}|_{t=0} = 0$.

The results were compared with results obtained using the method of variational iterations and variable elastic parameters [40]. In the set-up method, the physical nonlinearity was taken into account through the theory of plasticity using Iliushin's method of elastic solutions. Here we

Table 2
Comparison of computational methods

q	$w(0.5;0.5)$		
	1	2	3 (%)
23.4	0.96	0.95	1.04
60.0	2.03	2.00	1.96
70.3	2.16	2.23	3.24

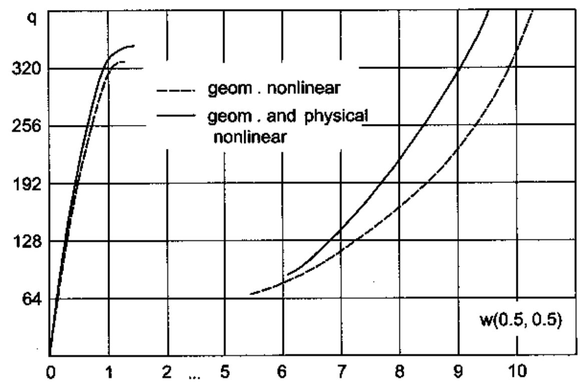

Fig. 5. Dependence $q(w(0.5;0.5))$.

remark that, from a mathematical point of view, the latter method corresponds to some variants of the simple iteration method. A numerical realization of Iliushin's method of elastic solutions possesses many advantages in comparison with the method of variable elastic parameters. In Table 2, the values of the deflection at the centre of the plate obtained by the method described in [40] (column "1") and by the set-up method (column "2") are reported, for three values of the transverse-load parameter. Column 3 reports the difference as a percentage.

First of all, it can be concluded from Table 2 that the results obtained by these two methods are similar. Some of the difference is caused by the specific structures of two algorithms. The static approach relies on a reduction from partial differential equations (PDEs) to ODEs as a first approximation and on application of the variable–elasticity method. The dynamical approach is based on the set-up method and the method of elastic solutions.

We consider now the stability of a geometrically nonlinear elastic shell. The following parameters were chosen $k_{x_1} = k_{x_2} = 24$, $E = 69$ GPa, $\mu = 0.3$, $a = b = 0.1$ m, $h = 0.83 \times 10^{-3}$ m, $\sigma_i(e_i)$ a defined by (24), $E_1/E = 0.4478$, $e_s = 1.35 \times 10^{-3}$. The boundary conditions and initial conditions are the same as in the previous case.

The dependence $q(w(0.5;0.5))$ obtained is shown in Fig. 5.

4. Numerical investigations and reliability of the results obtained

In the previous section we have shown that reliable solutions of problems of shallow-shell theory with both geometrical and geometrical–physical nonlinearities can be obtained when the method described in Section 2 is applied. In this section, we show the reliability of the solution to the three-dimensional nonstationary heat transfer equation obtained when the set-up method is used. In addition, we also discuss and illustrate (using the technique described here) a solution to a coupled thermoelastic problem of a flexible shallow shell without any prior assumptions about the temperature distribution through the thickness, i.e. using the three-dimensional heat transfer equation. As a matter of fact, we need to demonstrate the efficiency of the method when it is applied to the solution of integral–differential equations with different dimensions (three-dimensional heat transfer equations and two-dimensional equations related to the Kirchhoff–Love model) and of different types (heat transfer equations and hyperbolic equations of shell the theory).

In the equations governing the shell motion, a temperature does not appear directly, but its integral characteristics (thermal forces and moments) are used; these appear because of the reduction of the three-dimensional heat transfer equation to a two-dimensional one.

The solution to the three-dimensional heat transfer equation has been sought using the finite-difference method. A number of researchers have emphasized that neglecting the nonlinear form of the temperature variation through the shell thickness greatly influences the solution to a nonstationary thermoelastic problem of a shell or a plate. Therefore, a solution to the three-dimensional nonstationary heat transfer equation without any introduction of hypotheses about the temperature distribution through the thickness possesses important practical meaning.

Let us recall the first equation of the system (1):

$$\frac{\partial\theta}{\partial t}=\lambda_1^2\frac{\partial^2\theta}{\partial x_1^2}+\lambda_2^2\frac{\partial^2\theta}{\partial x_2^2}+\frac{\partial^2\theta}{\partial x_3^2}+w_0. \tag{26}$$

In this given equation, there is no term representing the shell geometry; however, the validity of this approach has been proved [53,54]. Using the finite-difference method along the spatial coordinates with an $O(h^2)$ approximation, we obtain the following ODEs with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\theta_{ijk}}{\partial t}=\lambda_1\Lambda_1\theta_{ijk}+\lambda_2\Lambda_2\theta_{ijk}+\Lambda_3\theta_{ijk}+w_{0ijk}. \tag{27}$$

We need to attach to the system (27) the initial and boundary conditions formulated earlier, represented in a suitable (finite-difference) form.

We consider a numerical example (second type of boundary conditions) and convective heat transfer (third type of boundary conditions). The space was divided into $(6\times 6\times 6)$, $(8\times 8\times 8)$ and $(12\times 12\times 12)$ parts. The results obtained were compared with the analytical solution proposed by Kovalenko [54], showing very good agreement.

The results of a problem related to a thermal shock on a shell surface which is thermally isolated on its other sides are given in Table 3. The results where the surface was partitioned into $(12\times 12\times 12)$ and $(8\times 8\times 8)$ parts differ from the analytical results by no more than 0.2% and 0.5%, respectively.

We consider one of the methods of reduction of the three-dimensional problem to a two-dimensional one. In the case of linear problems of thermoelasticity, application of the operator method for thin-walled structures has been proposed [55], where thin-walled conditions from the

Table 3
Comparison of results obtained for different mesh sizes

Solution method	Co-ordinates	Time		
		$t=0.2$	$t=0.4$	$t=0.6$
Exact	$z=0.5$	67.19	97.01	124.06
	$z=0$	21.06	47.66	74.26
	$z=-0.5$	8.18	31.55	57.71
$(12\times 12\times 12)$	$z=0.5$	67.04	96.92	123.97
	$z=0$	20.98	47.58	74.17
	$z=-0.5$	8.16	31.49	57.62
$(8\times 8\times 8)$	$z=0.5$	66.86	96.80	123.87
	$z=0$	20.89	47.48	74.07
	$z=-0.5$	8.14	31.41	57.53
$(4\times 4\times 4)$	$z=0.5$	65.87	96.17	123.33
	$z=0$	20.38	46.96	73.56
	$z=-0.5$	8.08	32.00	57.03

point of view of heat transfer theory have also been proposed. Let us begin with the first invariant of the deformation tensor:

$$e = e_{11} + e_{22} + e_{33}. \tag{28}$$

The quantity e_{33} can be found from the condition of a plane strain state ($\sigma_{33} = 0$), and it reads

$$e_{33} = -\frac{v}{E}(\sigma_{11} + \sigma_{22}) + \alpha_T\theta. \tag{29}$$

We shall now formulate the heat transfer equations in the absence of physical nonlinearities. Substituting σ_{11} and σ_{22} into (29), for a physically linear body, we obtain

$$e_{33} = -\frac{v}{1-v}(e_{11} + e_{22}) + \frac{1+v}{1-v}\alpha_T\theta. \tag{30}$$

Substituting (29) into (28), we obtain the first invariant of the deformation tensor:

$$\begin{aligned} e &= \frac{1-2v}{1-v}(e_{11} + e_{22}) + \frac{1+v}{1-v}\alpha_T\theta \\ &= \frac{1-2v}{1-v}\left[\frac{1-v}{Eh}(T_{11} + T_{22}) + 2\alpha_T N_T + z(æ_{11} + æ_{22})\right] + \frac{1+v}{1-v}\alpha_T\theta. \end{aligned} \tag{31}$$

Substituting (31) into the heat transfer equation, we obtain

$$K_T\frac{\partial\theta}{\partial t} - K_T\left(\frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2} + \frac{\partial^2\theta}{\partial x_3^2}\right) = \frac{E\alpha_T T_0}{1-2v}\frac{\partial}{\partial t}(e_{11} + e_{22} + e_{33}) + W_0, \tag{32}$$

and we introduce the following notation:

$$\alpha_1 = \frac{K_T}{C}, \quad \gamma_0 = \frac{\alpha_T E T_0}{C(1-2v)}, \quad \beta = \frac{\alpha_1}{1+\alpha_T\gamma_0\frac{1+v}{1-v}},$$

$$\gamma = \frac{\gamma_0(1-2v)}{(1-v)(1+\gamma_0\alpha_T\frac{1+v}{1-v})}, \quad D^2 = \nabla^2 - \frac{1}{\beta}\frac{\partial}{\partial t},$$

$$D_1^2 = \frac{1}{\beta}\frac{\partial}{\partial t}, \quad M = æ_{11} + æ_{22}, \quad N = \frac{1-v}{Eh}(T_{11}+T_{22}) + 2\alpha_T N_T,$$

$$N_T = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \theta \, dx_3. \tag{33}$$

In operator form, the heat transfer equation reads

$$\frac{\partial^2\theta}{\partial x_3^2} + D^2\theta - \gamma D_1^2(N + x_3 M) = 0. \tag{34}$$

The solution to (34) obtained using the symbolic method reads

$$\theta = \frac{\sin x_3 D}{D}\left\{\left.\frac{\partial\theta}{\partial x_3}\right|_{x_3=0} - \gamma\frac{D_1^2}{D^2}M\right\} + \cos x^3 D\left\{\theta|_{x_3=0} - \gamma\frac{D_1^2}{D^2}N\right\} + \gamma\frac{D_1^2}{D^2}(N + zM). \tag{35}$$

Integrating with respect to x_3, we obtain

$$N_T = \frac{1}{h}\int_{-h/2}^{h/2}\theta \, dx_3 = \frac{\sin\frac{h}{2}D}{\frac{h}{2}D}\left\{\theta|_{x_3=0} - \gamma\frac{D_1^2}{D^2}N\right\} + \frac{1}{\gamma}\frac{D_1^2}{D^2}N, \tag{36}$$

$$\begin{aligned} M_T &= \frac{12}{h^3}\int_{-h/2}^{h/2}\theta x_3 \, dx_3 \\ &= \frac{12}{h^2D^2}\frac{\sin\frac{h}{2}D}{\frac{h}{2}D}\left\{\left.\frac{\partial\theta}{\partial x_3}\right|_{x_3=0} - \gamma\frac{D_1^2}{D^2}M\right\} - \frac{12}{h^2D^2}\cos\frac{h}{2}D\left\{\left.\frac{\partial\theta}{\partial x_3}\right|_{x_3=0} - \gamma\frac{D_1^2}{D^2}M\right\} - \gamma\frac{D_1^2}{D^2}M. \end{aligned} \tag{37}$$

Removing $\theta|_{x_3=0}$ and $\partial\theta/\partial x_3|_{x_3=0}$ from (37) and (36) and substituting into (35), we obtain the temperature θ expressed in terms of the integral characteristics:

$$\theta(x_1,x_2,x_3,t) = \frac{\frac{h}{2}D}{\sin\frac{h}{2}D}\cos x_3 D N_T + \frac{h^3D^2}{24}\frac{\sin x_3 D}{\sin\frac{h}{2}D - \frac{h}{2}D\cot\frac{h}{2}D}M_T + \gamma\frac{D_1^2}{D^2}\left\{\frac{\frac{h}{2}D\cos zD}{\sin\frac{h}{2}D} - 1\right\}N - \frac{\gamma}{3}\frac{D_1^2}{D^2}\left\{\frac{\frac{h^3}{8}D^3\cot\frac{h}{2}D}{1-\frac{h}{2}\cot\frac{h}{2}D} - 3\right\}M. \tag{38}$$

After the boundary conditions on the surfaces

$$x_3 = \pm\frac{h}{2} : \left.\frac{\partial\theta}{\partial x_3}\right|_{x_3=-0.5} = 0 \quad \text{and} \quad \left.\frac{\partial\theta}{\partial x_3}\right|_{x_3=+0.5} = 0$$

are satisfied, the following equations are obtained:

$$-\frac{h}{2}D^2N_T + \frac{h^3D^3}{24}\frac{\cot\frac{h}{2}D}{1-\frac{h}{2}D\cot\frac{h}{2}D}M_T + \gamma\frac{h}{2}D_1^2N - \frac{\gamma}{3}\frac{D_1^2}{D^2}\left\{\frac{\frac{h^3}{8}D^3\cot\frac{h}{2}D}{1-\frac{h}{2}\cot\frac{h}{2}D} - 3\right\}M = 0, \tag{39}$$

$$\frac{h}{2}D^2N_T + \frac{h^3D^3}{24}\frac{\cot\frac{h}{2}D}{1-\frac{h}{2}D\cot\frac{h}{2}D}M_T + \gamma\frac{h}{2}D_1^2N - \frac{\gamma}{3}\frac{D_1^2}{D^2}\left\{\frac{\frac{h^3}{8}D^3\cot\frac{h}{2}D}{1-\frac{h}{2}\cot\frac{h}{2}D} - 3\right\}M = 0. \tag{40}$$

These equations yield an equation for the thermal stresses N_T,

$$D^2N_T - \gamma D_1^2 N = 0, \tag{41}$$

and an equation for the thermal moments M_T,

$$h^2D^2M_T - 12M_T - \gamma h^2 D_1^2 N = 0. \tag{42}$$

Eq. (42) has been obtained by keeping only two terms of the series of $\cot(h/2)D$, which corresponds to a cubic form of the temperature distribution (38) through the thickness.

In terms of the nondimensional parameters introduced earlier, (41) and (42) read

$$\lambda_1\lambda_2\left(\lambda^{-1}\frac{\partial^2 N_T}{\partial x_1^2} + \lambda\frac{\partial^2 N_T}{\partial x_2^2}\right) - \frac{\alpha_T^2 E T_0}{C}\left(\frac{\partial}{\partial t}\left(\lambda^{-1}\frac{\partial^2 F}{\partial x_1^2} + \lambda\frac{\partial^2 F}{\partial x_2^2}\right)\right) = \left(1+\gamma_0\alpha_T\frac{1+v}{1-v}\right)\left(1+\frac{2\gamma_0\alpha_T}{1+\gamma_0\alpha_t\frac{1+v}{1-v}}\right)\frac{\partial N_T}{\partial t}, \tag{43}$$

$$\lambda_1\lambda_2\left(\lambda^{-1}\frac{\partial^2 M_T}{\partial x_1^2} + \lambda\frac{\partial^2 M_T}{\partial x_2^2}\right) - 12M_T + \frac{\alpha_T^2 E T_0}{C(1-v)}\frac{\partial}{\partial t}\left(\lambda^{-1}\frac{\partial^2 w}{\partial x_1^2} + \lambda\frac{\partial^2 w}{\partial x_2^2}\right) = \left(1+\gamma_0\alpha_T\frac{1+v}{1-v}\right)\frac{\partial M_T}{\partial t}. \tag{44}$$

We need to attach to these equations the equations governing the motion of the shell and the continuity equation:

$$\frac{1}{12(1-\nu^2)}\nabla^2\nabla^2 w - \nabla_k^2 F - L(w,F) + \frac{a^2b^2\rho\alpha_1^2}{Eh^3}\frac{\partial^2 w}{\partial t^2} + \frac{1}{12(1-\nu^2)}\left(\lambda^{-1}\frac{\partial^2 M_T}{\partial x_1^2} + \lambda\frac{\partial^2 M_T}{\partial x_2^2}\right) = q,$$
$$\nabla^2\nabla^2 F = -\lambda^{-1}\frac{\partial^2 N_T}{\partial x_1^2} - \lambda\frac{\partial^2 N_T}{\partial x_2^2} - \nabla_k^2 w - \frac{1}{2}L(w,w). \tag{45}$$

As a result, we have obtained a system of four PDEs (43)–(45) governing the shell motion with coupling of both the deformation and the temperature fields. Unlike the integral–differential set of equations usually obtained, the system above is only differential and has one dimension. Its approximate, because in the series of $\cot(h/2)D$ only the first two terms are taken, which in practice leads to a cubic distribution of temperature through the shell thickness. However, the system remains a hybrid one, i.e. parabolic and hyperbolic.

We need to attach to the system (43)–(45) the boundary and initial conditions. In order to solve the system (similarly to the integral–differential system (1), we have applied a finite-difference method with respect to the spatial coordinates with an $O(h^2)$ approximation. This finally results in ODEs for w_{ij}, N_{Tij}, M_{Tij} and a system of algebraic equations for F_{ij}. The system of algebraic equations was (again) solved using the Gauss method for every time step, and for the system of differential equations, the Runge–Kutta method with an automatically chosen step of integration was chosen. Unlike the situation for the set-up method, where the values of the thermal loads occurring in the three-dimensional heat transfer equation must be taken from the previous step of the integration in time, we do not need to take these values from the previous step now.

In order to estimate the influence of the assumption that we have made, the stress–strain state of a shell made from AMC alloy with the boundary conditions $T_{11}(\partial w/\partial x_1) + T_{12}(\partial w/\partial x_2 = 0)$, $x_1 = \text{const}$, has been analysed.

It has been assumed also that on the surfaces $x_1 = 0$, $x_2 = 0$, $x_1 = 1$, $x_2 = 1$, the following boundary condition of the first type for temperature has been used: $\theta = 0$, i.e. $N_T = M_T = 0$. At the initial time instant $t = 0$ a heat load, extending over an infinite time, uniformly distributed

Table 4
Comparison of results obtained for different methods

Time	$t = 0.1$		$t = 0.2$		$t = 0.3$		$t = 0.4$	
Method	1	2	1	2	1	2	1	2
Function								
$w(0.5;0.5)$	0.99	0.99	1.78	1.78	0.85	0.85	0.02	0.02
$w(0.25;0.25)$	0.68	0.68	1.07	1.07	0.68	0.68	0.03	0.03
$F(0.5;0.5)$	0.38	0.38	0.63	0.63	0.33	0.33	0.00	0.00
$F(0.25;0.25)$	0.15	0.15	0.23	0.23	0.14	0.14	0.004	0.004
$w_{x_1x_2}(0.5;0.5)$	−2.34	−2.34	−12.86	−12.86	0.90	0.94	−0.75	−0.74
$w_{x_1x_2}(0.25;0.25)$	−8.94	−8.94	−10.52	−10.52	−9.78	−9.77	−0.36	−0.37
$F_{x_1x_2}(0.5;0.5)$	−4.63	−4.63	−8.46	−8.46	−3.44	−3.44	−0.03	−0.02
$F_{x_1x_2}(0.25;0.25)$	−1.83	−1.83	−2.38	−2.38	−1.93	−1.93	−0.12	−0.12
$N_T(0.5;0.5)$	0.124	0.126	0.228	0.330	0.092	0.093	0.001	0.001
$M_T(0.5;0.5)$	−0.101	−0.101	0.298	0.310	0.353	0.344	0.080	0.084

over the shell surface, was applied, with $q = 81$ and $k_{x_1} = k_{x_2} = 24$. The results obtained are presented in Table 4.

Method 1 in Table 4 corresponds to the solution of the system (43)–(45). Method 2 corresponds to the solution of the integral–differential equations. The results obtained indicate the very high efficiency of the approaches presented here for the solution of coupled thermoelastic problems of flexible shells subjected to transverse impulse-type loads.

5. Conclusions

In Section 2, a method of solution for nonlinear coupled problems is addressed. The finite-difference method is used to discretize the derivatives in order to obtain a system of ordinary differential and algebraic equations. A procedure for solution of the initial-boundary value problem is described. In Section 3, a relaxation method which reduces the problem to a Cauchy problem of ordinary differential equations is presented. Some advantages and disadvantages of the method are discussed. In addition, the high efficiency of the method is illustrated using examples of problems of plates and shells with physical and physical–geometrical nonlinearities. In Section 4, numerical investigations are described, and the reliability of the solution of the three-dimensional nonstationary heat transfer equation obtained using the relaxation method (later refered to as the set-up method) is shown. In addition, a solution to a coupled thermoelastic problem of a flexible shallow shell without any prior assumptions about the temperature distribution through the thickness is given.

The use of the three-dimensional heat transfer equation is addressed. The efficiency of the method used here when applied to the solution of integral–differential equations with different dimensions (three-dimensional equations related to the Kirchhoff–Love model) and of different type (heat transfer equations and the hyperbolic equations of shell theory) is demonstrated.

Since many of the detailed comments and conclusions are already included in the text, they are not repeated now.

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