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Numerical Analysis of Self-Excited by Friction Chaotic Oscillations in Two-Degrees-of-Freedom System Using Exact Hénon Method

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Abstract

In this paper the algorithm for numerical integration of the ordinary differential equations including discontinuous term describing friction is proposed. The introduced algorithm is based on the Hénon method, which is extremely useful to locate and track the stick to slip and slip to stick transitions. This numerical technique further referred as the 'exact' one is used to investigate and to estimate a validity of various approximations to frictional behaviour widely used in many references (the Stribeck-curve, Coulomb type and experimental friction models).

A simple two-degrees-of-freedom mechanical system has been applied for estimation of validity of the different friction models as well as to take a proper value of the control parameter ε used for smooth friction approximation by arctan function. The relative errors have been calculated as well as the differences in the phase spaces have been tracked.

In addition, some interesting examples of stick-slip regular and chaotic dynamics exhibited by the investigated two-degrees-of-freedom system have been illustrated and discussed.

1. Introduction

In spite of that the investigations of mechanical systems with friction have very long tradition, there remain still some problems which are not satisfactory solved. On the other hand, both experimental and numerical results indicate validity of stick-slip behaviour in many mechanical devices modelled by discrete multibody systems (Awrejcewicz, 1996; Brandl and Pfeiffer, 1999; Van de Vrande et al., 1999). Since the first numerical analysis of stick-slip chaotic oscillations excited by friction has been covered out in references (Awrejcewicz and Delfs, 1990a, 1990b), they appear many other dynamics caused by friction. However, there is observed a lack of an appropriate numerical approach to deal with stick-slip dynamics, which corresponds to a sudden change of a system dimension. One of the attempts to solve this problem has been proposed in reference (Van de Vrande et al., 1999), where the Hénon method has been applied (Hénon, 1982). Our research follows these references and is confirmation of the earlier works (Awrejcewicz and Olejnik, 2001a, 2001b).

2. Analysed system and friction

The analysed self-excited system with two-degrees-of-freedom is shown in Fig. 1.

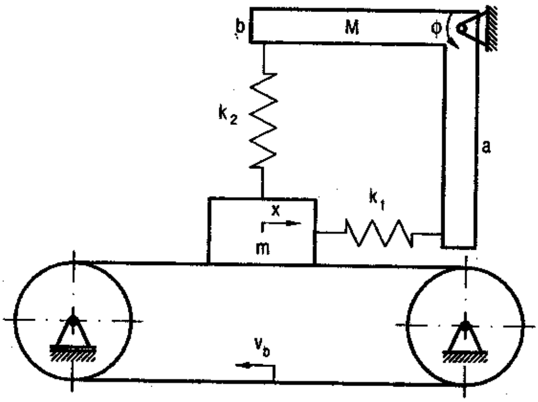

Fig. 1. The considered system

Two bodies oscillate along the co-ordinates x and φ. Friction force F is occurring between mass m and the belt that moves with a constant velocity. As usually, the constant stiffness coefficients are denoted by k_i, $i=1,2$. The friction static forces F_s is defined by

$$F_s = \mu_0 F_N \tag{1}$$

where: $F_N = mg$ is the pressing force generated by the mass m, and μ_0 is the value of the static friction coefficient.
The dynamic friction force is governed by the equations

$$F = -\mu F_N \operatorname{sgn} \upsilon_{rel} = -\frac{\mu F_s}{\mu_0} \operatorname{sgn} \upsilon_{rel} \tag{2}$$

where $\upsilon_{rel} = \dot{x} - v_d$ is the relative velocity, $\mu \le \mu_0$. The relation between static and dynamic coefficients is introduced in the following way:

$$\mu = \frac{\mu_0}{1 + \delta \left| \upsilon_{w,i} \right|} \tag{3}$$

Above, δ coefficient characterizes a way of dynamic coefficient decrease, which accompanies an increase of the relative velocity. The static forces occur when the relative velocity is equal to zero. Therefore, one gets

$$\begin{cases} |F| \le F_s & \upsilon_{\text{rel}} = 0 \\ F = -\operatorname{sgn} \upsilon_{rel} \dfrac{F_s}{1+\delta|\upsilon_{rel}|} & \upsilon_{\text{rel}} \ne 0 \end{cases} \tag{4}$$

3. Approximation to the sign function

The following approximation to the sign function is applied (Awrejcewicz and Olejnik, 2001a, 2001b)

$$\operatorname{sgn} \upsilon_{rel} = \frac{2}{\pi} \arctan(\varepsilon \upsilon_{rel}) \qquad 5)$$

where $\varepsilon > 0$ is a large number. The estimation of the above approximation is shown in Fig. 2.

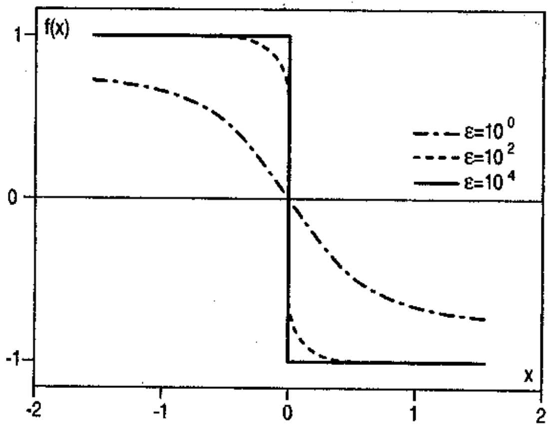

Fig. 2. Influence of ε for the approximation governed by equation (5)

4. Two cases of the Coulomb friction law

For a simulation the Coulomb friction law was applied.

1. The first case

In this case the friction coefficient was approximated by an exponential Stribeck function (see Fig. 3).

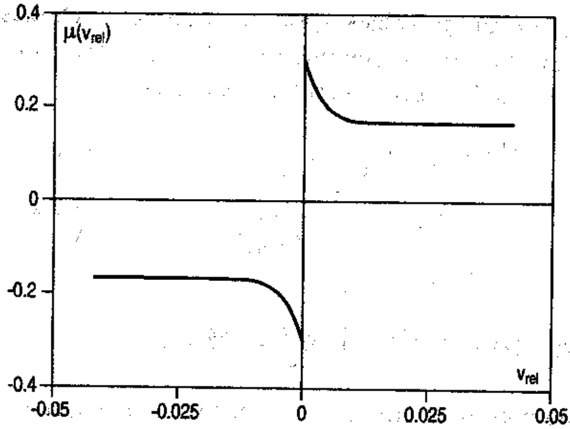

Fig. 3. Stribeck curve for the first case

$$\mu(v_{rel})=\left((\mu_0-\mu_G)\exp\left(\frac{-c}{\mu_0-\mu_G}|v_{rel}|\right)+\mu_G\right)\operatorname{sgn}(v_{rel}) \qquad (6)$$

where μ_G is representing the coefficient of sliding friction for $v_{rel}\to\infty$, μ_0 is the sticking friction coefficient, and $\mu(v_{rel})$ is the sliding friction coefficient.

2. The second case

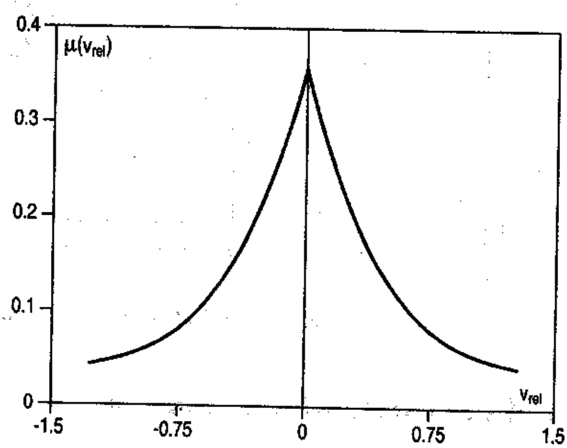

Fig. 4. Pick type of characteristics for approximation to the dynamic friction coefficient

In this case the friction coefficient was approximated by pick function (see Fig. 4) of the form

$$\mu(v_{rel})=\frac{\mu_0}{1+\delta|v_{rel}|} \qquad (7)$$

The approximations in both cases are usually used in numerical investigations (Brandl and Pfeiffer, 1999; Van de Vrande et al., 1999).

5. An error estimation during approximation to the sgn function

In the next step we estimate errors of the approximation given in equation (5). The curves illustrated in Fig. 5 show good convergence of the arctan approximation when ε control parameter increases. In the second case, an accuracy of this approximation is more exact and will be shown in further considerations.

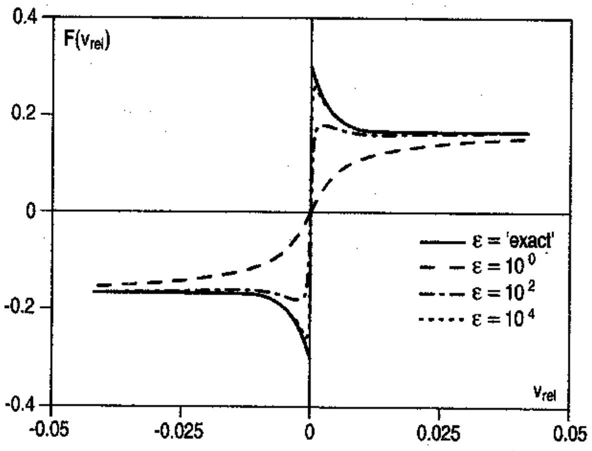

Fig. 5. Stribeck curve approximated by arctan ($\varepsilon = 10^0$, 10^2, 10^4, and 'exact', respectively)

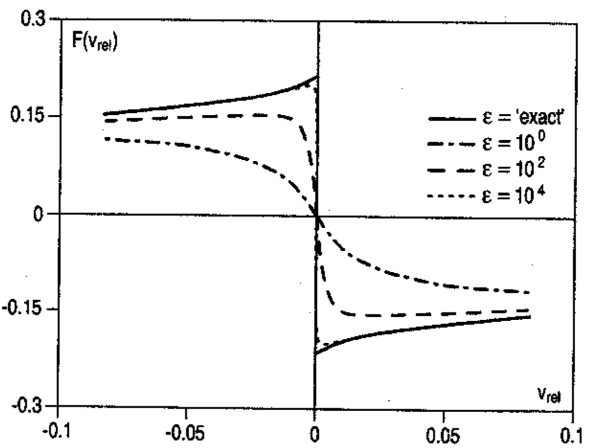

Fig 6. The second pick type of characteristics using arctan approximation ($\varepsilon = 10^0$, 10^2, 10^4, and 'exact', respectively)

It is easy to see approximations in both cases, when we plot the example trajectories in the corresponding phase space. The phase portraits corresponding to the friction force shown in Figs 5 and 6 are reported in Figs 8 and 7, respectively. Again it is seen that $\varepsilon = 10^4$ gives a good approximation to the friction.

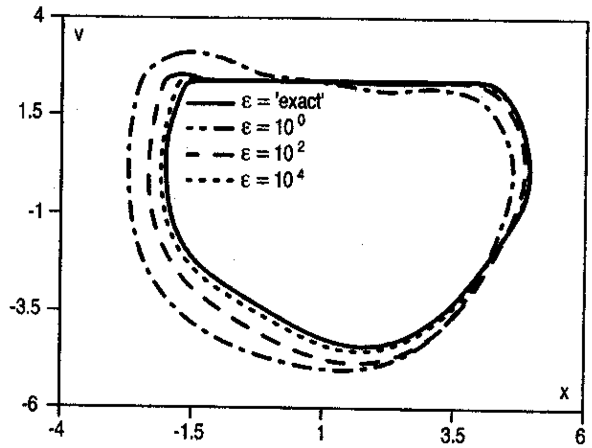

Fig. 7. The pick type characteristics corresponding to Figure 6 (a simple example of the stick-slip motion)

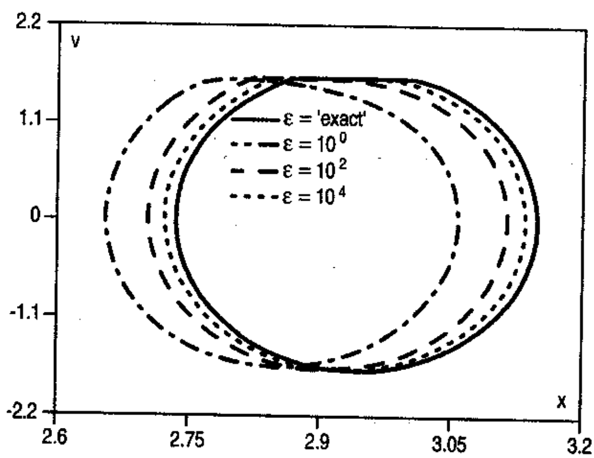

Fig. 8. Stribeck type of approximation corresponding to Fig. 5 (a simple example of the stick-slip motion)

6. Equations of motion of the analysed system

The considered model illustrated in Fig. 1 have the following equations of motion:

$$\begin{aligned} & m\ddot{x} + k_1 x - k_1 \varphi a = F_{friction} \\ & \frac{m\left(a^2+b^2\right)}{3}\ddot{\varphi} + k_1\varphi a^2 + k_2(a-b)^2\varphi - k_1 x a = 0 \end{aligned} \tag{7}$$

The following non-dimensional differential equations govern dynamics of our two bodies dynamical system:

$$\dot{y}_1 = y_2, \quad \dot{y}_3 = y_4, \quad \dot{y}_4 = \frac{1}{e_3}\left(y_1 - (1+e_4)y_3\right)$$

$$\dot{y}_2 = \frac{1}{e_1}\left(y_3 - y_1 - \underbrace{\begin{Bmatrix} \operatorname{sgn} v_{rel}\dfrac{1+e_2 y_3}{1+\gamma|v_{rel}|} \\ 1-e_2 y_3 \end{Bmatrix}}_{\text{THE SECOND CASE}} \quad \text{or} \quad \underbrace{\begin{Bmatrix} \operatorname{sgn}(v_{rel})\left(\dfrac{\mu_0-\mu_G}{\mu_0}\exp\left(\dfrac{-c}{\mu_0-\mu_G}|v_{rel}|\right)+\dfrac{\mu_G}{\mu_0}\right) & v_{rel}\neq 0 \\ 1-e_2 y_3 & v_{rel}=0 \end{Bmatrix}}_{\text{THE FIRST CASE}} \right) \tag{8}$$

The relations between physical values of parameters, coordinates and time, and their non-dimensional quantities are:

$$e_1 = \frac{m\omega^2}{k_1},\; e_2 = \frac{k_2(a-b)\mu_0}{k_1 a},\; e_3 = \frac{m\omega^2\left(a^2+b^2\right)}{3k_1 a^2},\; e_4 = \frac{k_2(a-b)^2}{k_1 a^2}\; \gamma = \frac{F_s}{\sqrt{mk_1}}.$$

7. Exact solution by Hénon method

The first two equations of the system (Awrejcewicz and Olejnik, 2001b) can be presented by the state equation

$$\dot{y} = f(y) \tag{9}$$

where a prime denotes a differentiation with respect to τ, $y=[x_1, x_2]$, and assuming $V_{rel} \le 0$, f is given by

$$f_1 = \left[x_2, \frac{1}{e_1}\left(\varphi_1 - x_1 + \frac{1+e_2\varphi_1}{1-\gamma V_{rel}} \right) \right] \quad \text{and} \quad f_2 = [x_2, 0] \tag{10}$$

in the slip and stick phases, respectively. Equation (10) is integrated by the fourth-order Runge-Kutta scheme

$$\begin{aligned} k_1 &= f(y_i) \\ k_2 &= f\left(y_i + \frac{hk_1}{2} \right) \\ k_3 &= f\left(y_i + \frac{hk_2}{2} \right) \\ k_4 &= f(y_i + hk_3) \\ y_{i+1} &= y_i + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4) \end{aligned} \tag{11}$$

where h is the time step and y_i represents y at $\tau = ih$. When during the slip phase, $x_2 > v_b$, the last integration step is repeated with the 'inverted' slip equation

$$f_1^{invert}(x_2, z_1) = \left[\frac{e_1 x_2}{\varphi_1 - x_1 + \frac{1+e_2\varphi_1}{1-\gamma V_{rel}}}, \frac{e_1}{\varphi_1 - x_1 + \frac{1+e_2\varphi_1}{1-\gamma V_{rel}}} \right] \tag{12}$$

where $z_1 = [x_1, \tau]$ and x_2 is the independent variable. In one step of the integration

$$\begin{aligned} k_1 &= f_1^{invert}(x_{2,i}, z_{1,i}) \\ k_2 &= f_1^{invert}\left(x_{2,i} + \frac{h_1}{2}, z_{1,i} + \frac{h_1 k_1}{2} \right) \\ k_3 &= f_1^{invert}\left(x_{2,i} + \frac{h_1}{2}, z_{1,i} + \frac{h_1 k_2}{2} \right) \\ k_4 &= f_1^{invert}(x_{2,i} + h_1, z_{1,i} + h_1 k_3) \\ z_{1,(i+1)} &= z_{1,i} + \frac{h_1}{6}(k_1 + 2(k_2 + k_3) + k_4) \end{aligned} \tag{13}$$

where: $h_1 = v_b - x_{2,i}$, the slip-to-stick transition is found and the time integration of equation (10) can be continued. When during the stick phase, $x_{1,(i+1)} > 1$, the last integration step is repeated with the 'inverted' stick equation

$$f_2^{invert}(x_1, z_2) = \left[\frac{1}{x_2}, 0\right] \tag{14}$$

where:

$z_2 = [\tau, x_2]$ and x_1 is the independent variable. In one step of integration scheme

$$\begin{aligned}
k_1 &= f_2^{invert}(x_{1,i}, z_{2,i}) \\
k_2 &= f_2^{invert}\left(x_{1,i} + \frac{h_2}{2}, z_{2,i} + \frac{h_2 k_1}{2}\right) \\
k_3 &= f_2^{invert}\left(x_{1,i} + \frac{h_2}{2}, z_{2,i} + \frac{h_2 k_2}{2}\right) \\
k_4 &= f_2^{invert}(x_{1,i} + h_2, z_{2,i} + h_2 k_3) \\
z_{2,(i+1)} &= z_{2,i} + \frac{h_2}{6}(k_1 + 2(k_2 + k_3) + k_4)
\end{aligned} \tag{15}$$

where:

$h_2 = 1 - x_{1,i}$, the stick-to-slip transition is found and the time integration of equations (11) can be continued.

8. The relative errors of the approximated solutions

The relative error e is defined by:

$$e = \sqrt{\frac{\sum_i \left(y_{1,i} - y_{1,i}^{exact}\right)^2 + \sum_i \left(y_{2,i} - y_{2,i}^{exact}\right)^2}{\sum_i {y_{1,i}^{exact}}^2 + \sum_i {y_{2,i}^{exact}}^2}} \tag{16}$$

where an 'exact' solution is derived from the Hénon exact method. The differences between two demonstrated cases of approximation to the friction coefficient are shown in Table 1.

Table 1. The two cases comparison of the introduced approximation

ε-parameter		10^6	10^5	**10^4**	10^3	10^2	50	10
e	CASE 1- *pick type*	0.0009	0.0006	**0.0040**	0.0231	0.0968	0.1291	0.2065
	CASE 2- *Stribeck*	0.0005	0.0011	**0.0042**	0.0133	0.0337	0.0430	0.7641

ε serves as the control parameter and the different periodic orbits for different ε values are shown in Fig. 9.

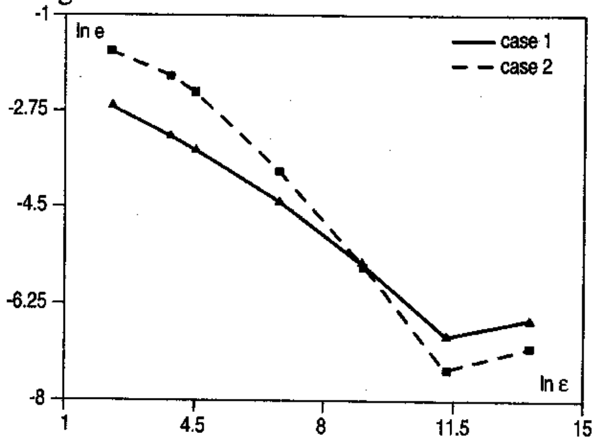

Fig. 9. The accuracy of approximations to the sgn function used in both cases

There is one interesting observation in our error investigations. Namely, with respect to arctan approximation for $\varepsilon = 10^4$ both of the presented models have the same accuracy.

9. Examples of stick-slip motions

Some computational results are presented in Figs 10 and 11. A typical projection of a trajectory associated with our system with friction is shown in Fig. 10.

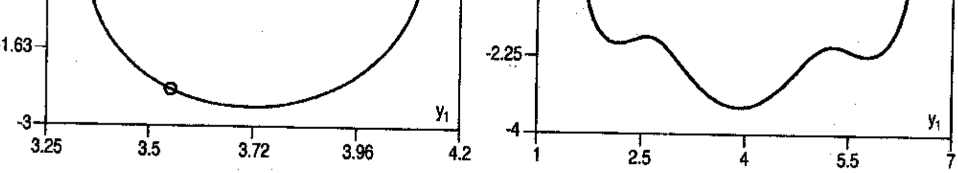

Fig. 10. Periodic trajectories for the cases (a) and (b) for the parameters: $t_0 = 1300$, $t_k = 2300$, $h = 0.001$, $\gamma = 0.2$, $v_b = 2$, $e_1 = 0.11$, $e_2 = 0.06$, $e_3 = 0.04$, $e_4 = 0.23$, $\varepsilon = 10^4$ (in (b) $\mu_0 = 0.25$, $\mu_G = 0.15$, $c = 10$); open cirlcles indicate the Poincaré map

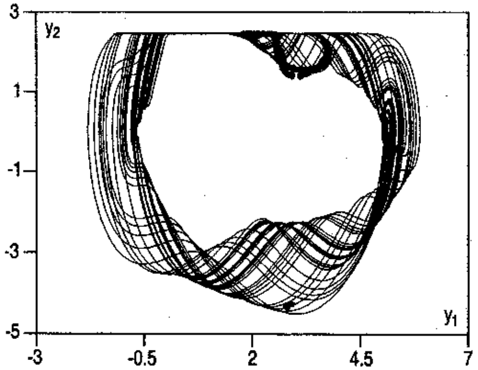

Fig. 11. Chaotic trajectory and its Poincaré map (black points) for the parameters: $t_0 = 1000$, $t_k = 5500$, $h = 0.002$, $\gamma = 0.2$, $v_b = 2$, $e_1 = 0.27$, $e_2 = 0.12$, $e_3 = 0.09$, $e_4 = 0.48$, $\varepsilon = 10^4$

10. Conclusions

In this report, two different numerical approaches have been proposed to solve a classical problem with friction modelled by a mechanical system with two-degrees-of-freedom. Since one of two bodies lies on a moving tape, its dynamics has been tracked using the Hénon's method, whereas the second body (pendulum) oscillations have been followed by the standard Runge-Kutta method.

The attention has been focused on approximation of the friction term sgn by arctan function with control parameter ε. Then, an efficiency of both approaches has been investigated. It has been observed that with an increase of ε, when a 'shape' of arctan highly approximates friction, the more reliable is the Hénon's method. In spite of its relative simplicity, the computational time required to hold the assumed accuracy is a few times lower than the standard Runge-Kutta method. Hence, the Hénon's method is expected to give more reliable and computationally efficient results.

Since the authors mainly have been focused on developing the Hénon's procedure, only a few examples are devoted to investigate dynamics of the analysed two-degrees-of-freedom system. It has been shown that the system can exhibit either periodic (Fig. 10) or chaotic (Fig. 11) motion.

Acknowledgment

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References

Awrejcewicz, J., 1996, *Deterministic Oscillations of Lumped Systems* (original title in Polish: Drgania Deterministyczne Układów Dyskretnych), WNT, Warsaw.

Awrejcewicz, J., Delfs, J., 1990, Dynamics of a self-excited stick-slip oscillator with two degrees of freedom, Part I, Investigation of equilibria and Slip-Stick, Slip-Slip and Stick-Slip Transition, *European J. Mech.*, A/Sol., 9(4) 269-282.

Awrejcewicz, J., Delfs, J., 1990, Dynamics of a self-excited stick-slip oscillator with two degrees of freedom, Part II, Periodic and chaotic orbits, *European J. Mech.*, A/Sol. 9(5), 397-418.

Awrejcewicz, J., Olejnik, P., 2001, Stick-slip phases in a system with friction, *Euro Conference on Computational Mechanics and Engineering Practice*, Szczyrk, 114-119.

Awrejcewicz, J., Olejnik, P., 2001, Stick-slip dynamics of a two-degree-of freedom system, *Proceedings of DETC' 01, ASME Design Technical Conferences*, Pittsburgh, PA, USA, 9 pages.

Brandl, M., Pfeiffer, F, 1999, Tribometer for dry friction measurement, *Proceedings of the ASME Design Engineering Technical Conference*, Las Vegas, Nevada.

Hénon, M., 1982, On the numerical computation of Poincaré maps, *Physica*, D **5**, 412-414.

Van de Vrande, B.L., Van Campen, D.H., De Kraker, A., 1999, An Approximate Analysis of Dry-Friction-Induced Stick-Slip Vibrations by a Smoothing Procedure, *Nonlinear Dynamics and Chaos* **19**, 157-169.