Machine Dynamics Problems
2002, Vol. 26, No. 1, 5–10

Averaging in Theory of Waffle Plates and Shells

Igor V. Andrianov[1)], Jan Awrejcewicz[2)]

[1)] Institute of General Mechanics RWTH Aachen
[2)] Technical University of Łódź
awrejcew@ck-sg.p.lodz.pl

(Received: 12 January 2001; accepted: 9 October 2002)

Abstract

A use of the homogenization procedure to solve boundary value problems of waffle plates and shells is discussed and illustrated.

Key words: averaging, homogenization, waffle plates and shells.

1. Introduction

There exists number of papers devoted to the analysis and calculation of ribbed plates and shells (Amiro and Zarutsky, 1981; 1998; Awrejcewicz et al., 1998; Andrianov and Awrejcewicz, 2000; Manevitch, 1972). The averaging method is applied (Awrejcewicz et al., 1998; Andrianov and Awrejcewicz, 2000; Manevitch, 1972), but in general to stringer shells. In contrast, in this work we consider waffle constructions and some problems not treated in the above-mentioned references, for instance, the calculations of constructions with concentrated loads.

2. Rectangular cells

The applied method is illustrated on the classical model of membrane deformations, which is reinforced in two main directions by the fibres (Fig. 1).

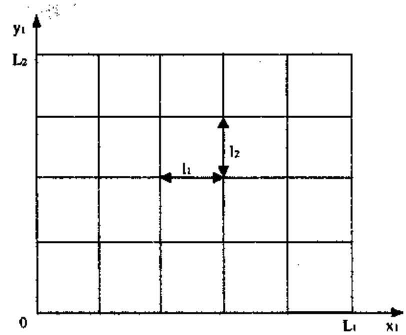

Fig. 1. Scheme of a two-periodic medium

The following equilibrium equation hols between the fibres

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial y_1^2}=q_1(x_1,y_1) \tag{1}$$

During transition through the fibres the displacements remain continuous, whereas the first derivatives are discontinuous:

$$\left(\frac{\partial u}{\partial x_1}\right)^+-\left(\frac{\partial u}{\partial x_1}\right)^-=g_1\frac{\partial^2 u}{\partial y_1^2} \tag{2}$$

$$\left(\frac{\partial u}{\partial y_1}\right)^+-\left(\frac{\partial u}{\partial y_1}\right)^-=g_2\frac{\partial^2 u}{\partial x_1^2} \tag{3}$$

where: g_1, g_2 are the parameters characterising the fibre stiffness, the following notation is also used

$$(...)^{(\pm)}=\lim_{x_1\to kl_1(\pm)}(...),\qquad k=1,2,...,n \quad \text{for equation (2),}$$

$$(...)^{(\pm)}=\lim_{y_1\to kl_2(\pm)}(...),\qquad k=1,2,...,m \quad \text{for equation (3).}$$

The following boundary conditions are applied:

$$u=0 \quad \text{for} \quad x=0,\ L_1, \quad y=0,\ L_2\,(4) \tag{4}$$

Assuming that $l_1 \sim l_2$ we introduce the following small parameters $\varepsilon = l_1/L_1$. In addition, we introduce the so called "fast" (ξ, η) and "slow" (x, y) variables via the relations: $\xi = x_1/l_1$; $\eta = y_1/l_2$; $x = x_1/L_1$; $y = y_1/L_2$.

The following new relations hold for the derivatives

$$\frac{\partial}{\partial x_1}=\frac{1}{L_1}\left(\frac{\partial}{\partial x}+\varepsilon^{-1}\frac{\partial}{\partial \xi}\right),\qquad \frac{\partial}{\partial y_1}=\frac{1}{L_2}\left(\frac{\partial}{\partial y}+\varepsilon^{-1}1L\frac{\partial}{\partial \eta}\right) \tag{5}$$

where: $l=l_2/l_1$, $L=L_1/L_2$.

The function being sought is approximated by the following asymptotic series

$$u=u_0(x,y)+\varepsilon^2[u_{10}(x,y)+u_1(x,y,\xi,\eta)]+\varepsilon^2[u_{20}(x,y)+u_2(x,y,\xi,\eta)]+.... \tag{6}$$

Substituting (6) into (1)-(3) and taking into account (5) one obtains

$$\frac{\partial^2 u_0}{\partial x^2}+L^2\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 u_1}{\partial \xi^2}+l^2L^2\frac{\partial^2 u_1}{\partial \eta^2}=q(x,y) \tag{7}$$

....

$$\left(\frac{\partial u_1}{\partial \xi}\right)^+-\left(\frac{\partial u_1}{\partial \xi}\right)^-=d_1\frac{\partial^2 u_0}{\partial y^2} \tag{8}$$

$$\left(\frac{\partial u_1}{\partial \eta}\right)^+-\left(\frac{\partial u_1}{\partial \eta}\right)^-=d_2\frac{\partial^2 u_0}{\partial x^2} \tag{9}$$

where:

$q=L_1^2q$, $d_1=g_1l_1/L_1^2$, $d_2=g_2l_2/L_2^2$,

$(...)^{(\pm)}=\lim\limits_{\xi_1\to k(\pm)0}(...)$, $k=1,2,...,n$ for equation (8),

$(...)^{(\pm)}=\lim\limits_{\eta_1\to k(\pm)0}(...)$, $k=1,2,...,m$ for equation (9).

Furthermore, we assume $d_1\sim d_2\sim 1$. The averaging procedure applied to (7) with the attached relations (8) and (9), yields

$$(1+d_1)\frac{\partial^2 u_0}{\partial x^2}+L^2(1+l^2d_2)\frac{\partial^2 u_0}{\partial y^2}=q(x,y) \tag{10}$$

The boundary conditions for (10) are yielded by (4) and they have the form $u_0=0$ for $x=0,\ 1;\quad y=0,\ 1$.

The "fast" part of a solution is defined by the equations

$$\frac{\partial^2 u_1}{\partial \xi^2}+l^2L^2\frac{\partial^2 u_1}{\partial \eta^2}=-d_1\frac{\partial^2 u_0}{\partial x^2}-L^2l^2d_2\frac{\partial^2 u_0}{\partial y^2} \tag{11}$$

$$u_1\big|_{\xi=l-0}=u_1\big|_{\xi=+0} \tag{12}$$

$$\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=1-0}-\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=+0}=d_1\frac{\partial^2 u_0}{\partial y^2} \tag{13}$$

$$\left.\frac{\partial u_1}{\partial \eta}\right|_{\eta=1-0}-\left.\frac{\partial u_1}{\partial \eta}\right|_{\eta=+0}=d_2\frac{\partial^2 u_0}{\partial x^2} \tag{14}$$

This solution to the boundary value problem (11)-(14) does not satisfy the assumed boundary conditions on the boundaries $x = 0, l$ and $y = 0, l$, and we must to attach a boundary layer. The procedure, for constructing a boundary layer, is schematically illustrated in Fig. 2.

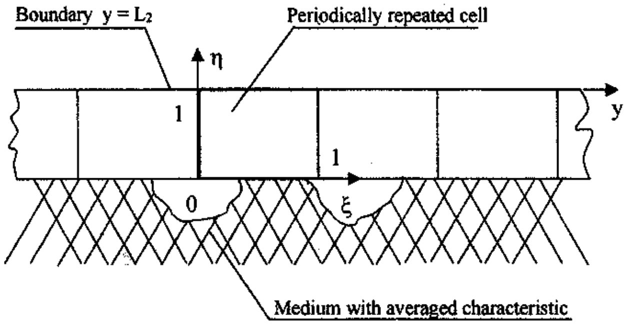

Fig. 2. For $\eta = 1$ the boundary conditions yielded by the boundary layer problem must be introduced, whereas for $\eta = 0.1$ – the conditions related to a periodical extension; for $\eta = 0$ joint conditions of a medium with averaged characteristics must be applied

A generalization for the angle cells is obvious.

3. Non-rectangular cells

It happens that in engineering practice a non-rectangular mesh of ribs is often also applied (Christensen, 1979). In this case, the application of the homogenisation method results in the known averaged relations (Kantorovitch, Krylov, 1958) and a formulation of the periodical problem of a cell.

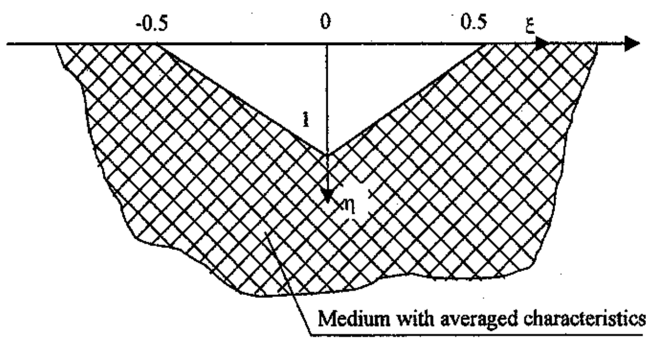

Fig. 3. For $\eta = 0$ the boundary layer conditions must be introduced; for $\eta = (\pm 2\xi + 1)1$, $0 \le \eta \le 1$ the joint conditions of a medium with averaged characteristics must be applied

In the case of a boundary layer problem formulation one has to separate the cell from the boundary, and the remaining part of the construction averaged medium with the defined properties (Fig. 3).

4. Solutions to local problems

In order to solve the problems related to periodic cell organization either analytically or numerically and the boundary layer problems for the non-rectangular contour the so called R-function methods can be efficiently applied (Prokopov, 1957; Ryabov, 1963; Guz, Nemish, 1989). The numerical experiments indicate an efficiency of the Ritz, Bubnov-Galerkin and Vlasov-Kantorovitch methods combined with an application of the Padé approximants (Andrianov, Awrejcewicz, 2000; Manevitch, 1972). Very often the following polynomials $\sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij}x^i y^j$ can be applied for low values of m and n.

5. Concentrated loads, localized oscillations and local stability loss

It seems that a 3D model can be effectively applied for the calculation of waffle plates and shells subjected to concentrated load actions.

It is clear that the loads acting on the supported constructions are transmitted to the supporting elements. Therefore, a combination of the Prokopov's skeleton method (Amiro, Zarutsky, 1981) with the Riabov's successive approximation method can be applied. First, a ribbed mesh is considered (here the constructive – orthotropic theory can be applied), and then the deformation of the stiff ribs joint is analysed. If a load is transmitted to a joint (for instance, via local stresses), then a 3D phase model can be applied (Fig. 2), by the separation of the shell bounded by the ribs with the concentrated load. The remaining part of the plate (shell) is substituted by a smooth plate (shell) with approximate characteristics.

The link between the cell and the surrounding medium is realized via the ribs. In order to solve the problem either analytical or numerical methods can be applied. For example, if we consider a squared cell with the same rib characteristics, a perturbation method (Awrejcewicz et al., 1998) can then be applied together with the introduction of a circle cell instead of a squared one. A similar approach can be applied during the calculation of a cell's localized oscillations or its local stability loss.

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