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THE EXISTENCE AND UNIQUENESS OF SOLUTION OF ONE COUPLED PLATE THERMOMECHANICS PROBLEM

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Abstract. The thermo-elastic plate system of equations is analysed. The sufficient conditions of existence, uniqueness and continuity dependence on initial data of the Cauchy problem solutions for differential-operational equation of mixed type (a part of the equation of hyperbolic type, and a part of parabolic type) are given in this paper. If the operational coefficients are suitably chosen, the investigated equation can be used to obtain a differential equation describing vibrations of a plate — the modified Germain-Lagrange equation of hyperbolic type. Moreover, in order to define the temperature field, one can use a three-dimensional equation of thermal conductivity (a parabolic equation).

1. Introduction

Let H be a Hilbert space, let L, M be a self-adjoint operators negatively defined in H with the domains $D(M) \subset D(L)$, and let C be an operator bounded in H. It is assumed that the operator L commutes with C and

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with the resolvent M (the definition of commuting is given in reference [5]). Let us consider the following system on the interval [0,T]

$$\begin{aligned} W''(t) + L^2 W(t) + \alpha L C \theta(t) &= q_1(t), \\ \theta'(t) - M\theta(t) - \beta C^* L W'(t) &= q_2(t), \end{aligned} \tag{1}$$

$(\alpha, \beta > 0)$ with unknown functions W and θ.

We take $H = L_2(\Omega)$ as the space of functions defined in the bounded domain with the piecewise smooth boundary $\partial\Omega$ in the space $\Omega = \Omega_1 \times [-h/2, h/2] \subset \mathbb{R}^3$ $(\Omega_1 \subset \mathbb{R}^2,\ h > 0)$ and having the summing up square norm. As M we take the operator which is realization of the formal differential expression

$$\frac{\lambda_0^2 \partial^2}{\partial x^2} + \frac{\lambda_0^2 \partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (\lambda_0 \neq 0),$$

and with the attached boundary condition $\theta \mid_{\partial\Omega} = 0$. As L operator we take the operator being a closure of a tensor product of the operators L_1 and E, where L_1 is generated in $L_2(\Omega_1)$ by the expression

$$\gamma \nabla^2 = \gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad (\gamma > 0)$$

with the boundary condition $W|_{\partial\Omega_1} = 0$. E is the identity operator in $L_2(-h/2, h/2)$, i.e. L is the closure of the operator generated in $L_2(\Omega)$ by the expression

$$\gamma \nabla^2 = \gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and defined on smooth functions from $L_2(\Omega)$ which are equal to zero on $\partial\Omega_1 \times [-h/2, h/2]$. The operator C is defined by the formula

$$C\theta = \int_{-h/2}^{h/2} z\theta(x, y, z)dz.$$

Then

$$C^* g = z \int_{-h/2}^{h/2} g(x, y, z)dz.$$

If L, M, C are chosen in this way, and the numbers α, β, γ are properly chosen, one can get from (1) the known plate equation governing its vibrations (see, for instance, [1]), with the following parameters: $F = 0$, $k_x = k_y = 0$, $P_x = P_y = 0$; it is called the thermal conductivity equation.

Furthermore, in order to simplify the considerations we assume in (1) that $\alpha = \beta = 1$. Observe that practically the introduced considerations do not change other values of α and β.

In order to reduce the system (1) into one equation of the first order we introduce the operator

$$A_0 = \begin{pmatrix} 0 & E & 0 \\ -L^2 & 0 & -LC \\ 0 & C^*L & M \end{pmatrix},$$

acting in the space $\widetilde{H} = H \oplus H \oplus H$. Here and further E means the identity operator in the respective space. We emphasize that the commutativity of the operators C and L yields the commutativity of C^* and $L^* = L$ [5].

Lemma 1. *The operator A_0 allowes of closing of A in the space $\widetilde{H}$. The operator A^{-1} exists, is bounded and defined in the whole space $\widetilde{H}$.*

Proof. The operational matrix

$$\begin{pmatrix} CM^{-1}C^* & -L^{-2} & -L^{-1}CM^{-1} \\ E & 0 & 0 \\ -LM^{-1}C^* & 0 & M^{-1} \end{pmatrix}$$

defines the bounded operator in $\widetilde{H}$. A direct check shows that its product (of any order) with the operational matrix defining the operator A_0 gives the unit operational matrix. All claims of the lemma result from this. Moreover, the operator A^{-1} is defined by the last operational matrix. □

Further on, the operator and its closure will be tagged with one symbol. If we assume $\nu = w'$, and u will denote the column

$$\begin{pmatrix} w \\ \nu \\ \theta \end{pmatrix},$$

then the homogeneous system (1) in the space $\widetilde{H}$ can be written in the following way:

$$u'(t) = Au(t). \tag{2}$$

Furthermore, we denote by $\widetilde{L}^{-2}$ the operator in $\widetilde{H}$, defined by the equation

$$\widetilde{L}^{-2} = \begin{pmatrix} L^{-2} & 0 & 0 \\ 0 & L^{-2} & 0 \\ 0 & 0 & L^{-2} \end{pmatrix}.$$

It results from the commuting properties of the operators that also $\widetilde{L}^{-2}$ and A^{-1} commute.

2. Basic assumption

In order to formulate the basic assumption we need to define the following spaces and the attached operators. The smoothness of a solution to the homogeneous equation (2) depends on that if the initial condition $u(0)$ belongs to the domains of the operators $\widetilde{L}$ and A. Observe that in applications the operator $\widetilde{L}$ is simpler than the operator A and therefore the spaces are constructed with regard to the powers of $\widetilde{L}$. However, we can not completely omit the operator A. Observe also that one can use the powers of A only (without $\widetilde{L}$).

We define the scalar product by the formula $(x,y)_{F^*} = (A^*x, A^*y)$ on the manifold $F^* = D(A^*)$. As the operator A^* possesses the inverse one defined everywhere, then F^* is a complete space and $\| x \|_{F^*} \geq \| x \|_{\widetilde{H}}$. This is why F^* can be considered as the space with a positive norm in relation to $\widetilde{H}$ [2]. We shall denote by $\widetilde{H}^{(-1)}$ the space with a negative norm, constructed using the pair F^*, $\widetilde{H}$.

The operator A^* maps continuously and bijectively F^* into $\widetilde{H}$. This is why the conjugated operator $\widetilde{A}$ acts from $\widetilde{H}$ to $\widetilde{H}^{(-1)}$ continuously and bijectively, and it serves as the extension of A.

We shall define the scalar product on the set $\widetilde{H}^{(+1)} = D(A)$ by the equation $(x,y)_{\widetilde{H}^{(+1)}} = (Ax, Ay)$. $\widetilde{H}^{(+1)}$ appears as the full space and the operator A maps $\widetilde{H}^{(+1)}$ to $\widetilde{H}$ continuously and bijectively (one-to-one).

Because the operator $\widetilde{L}^{-1}$ commutes with A, A^*, and $\widetilde{L}^{-1}$ is self-adjoint and bounded in $\widetilde{H}$, then $\widetilde{L}^{-1}$ is self-adjoint and bounded in the spaces $\widetilde{H}^{(+1)}$ and $\widetilde{H}^{(-1)}$ (the $\widetilde{L}^{-1}$ operator is extended due to continuity in the space $\widetilde{H}^{-1}$). The Hilbert space scales will be denoted by the symbols $\widetilde{H}_\alpha^{(+1)}$, $\widetilde{H}_\alpha = \widetilde{H}_\alpha^0$, $\widetilde{H}_\alpha^{(-1)}$ $(-\infty < \alpha < \infty)$ [2], and they are generated by the operator $\widetilde{L}$ acting in $\widetilde{H}^{(+1)}$, $\widetilde{H}$ and $\widetilde{H}^{(-1)}$, correspondingly. As it is known [2] for $\alpha \geq 0$ the operator $\widetilde{L}$ maps $\widetilde{H}_{\alpha+1}^{(\beta)}$ into $\widetilde{H}_\alpha^{(\beta)}$ $(\beta = -1, 0, 1)$ continuously and bijectively, and $\widetilde{L}$ allows for continuous and bijective extension from $\widetilde{H}_{-\alpha}^{(\beta)}$ to $\widetilde{H}_{-\alpha-1}^{(\beta)}$ $(\alpha \geq 0)$. We shall denote this extension as $\widetilde{L}$ in order not to introduce redundant denotations.

3. Main results

The operators A, $\widetilde{A}$ map $\widetilde{H}_\alpha^{(+1)}$ into $\widetilde{H}_\alpha$, and $\widetilde{H}_\alpha$ into $\widetilde{H}_\alpha^{(-1)}(\alpha \geq 0)$ continuously and bijectively, respectively. The same theorem is true for $\alpha < 0$, and for the operators $\widetilde{L}^\alpha A \widetilde{L}^{-\alpha}$ and $\widetilde{L}^\alpha \widetilde{A} \widetilde{L}^{-\alpha}$, which serve as the extension of A and $\widetilde{A}$, respectively. These extensions are also denoted by A and $\widetilde{A}$.

Theorem 1. *For any element* $u_0 \in \widetilde{H}_\alpha^{(\beta)}$ ($\beta = +1, 0$; $-\infty < \alpha < \infty$) *there exists one function* $u(t)$ *continuous in the closed interval* $[0,T]$ *in the space* $\widetilde{H}_{\alpha-2}^{(\beta-1)}$, *continuous in the half open from the left side interval* $(0,T]$ *in* $\widetilde{H}_{\alpha-4}^{(\beta)}$, *differentiable on* $(0,T]$ *in* $\widetilde{H}_{\alpha-4}^{(\beta-1)}$ *and satisfying both the equation*

$$u'(t) = \widetilde{A}u(t), \tag{3}$$

and the condition $u(0) = u_0$. *(The continuity is understood in the strong sense in a suitable space.)*

Observe that from the formulated theorem one can conclude that, in particular, for $u_0 \in \widetilde{H}_4^{(+1)}$ the function $u(t)$ is a weak solution of the equation (2) (in the sense given in reference [3]).

The proof of Theorem 1 is based on the quoted below estimates of the resolvent of the operator A in a way similar to that presented in [3], and the Laplace inverse transformation.

Let $\lambda = \sigma + i\tau$. We define the operators $\triangle_1 = \triangle_1(\lambda)$, $\triangle_2 = \triangle_2(\lambda)$ in H in the following way:

$$\triangle_1 = (\lambda^2 E + L^2) - \lambda L^2 C(M - \lambda E)^{-1} C^*,$$
$$\triangle_2 = (M - \lambda E) - \lambda L^2 (L^2 + \lambda^2 E)^{-1} C^* C.$$

Lemma 2. *If* $\sigma = \mathrm{Re}\lambda > 0$ *then the operators* $\triangle_1, \triangle_2$ *have bounded inverses:*

$$\| \triangle_1^{-1} \| \leq \frac{1}{\sigma \mid \lambda \mid}, \quad \| \triangle_2^{-1} \| \leq \frac{1}{\sigma}.$$

Proof. Let $S = (1/\lambda)\triangle_1$. Then for any $f \in D(L^2)$ we have:

$$\begin{aligned}(Sf, f) =& (\sigma + i\tau)(f, f) + \frac{1}{\mid \lambda \mid^2}(\sigma - i\tau)(L^2 f, f) \\ & - \big((M - \sigma E + i\tau E) \mid M - \lambda E \mid^{-1} LC^* f, \quad \mid M - \lambda E \mid^{-1} LC^* f\big).\end{aligned}$$

It results from this and from the negative definition of M that $\mathrm{Re}(Sf, f) \geq \sigma \| f \|^2$, and therefore $\| Sf \| \geq \sigma \| f \|$. Analogously we get $\| S^* g \| \geq \sigma \| g \|$. This is why the operator S^{-1} exists, is bounded, defined on the whole H, and $\| S^{-1} \| \leq 1/\sigma$.

The equation

$$-\lambda L^2 (L^2 + \lambda^2 E)^{-1} = -L^2 (\lambda L^2 + \mid \lambda \mid^2 \lambda E) \mid L^2 + \lambda^2 E \mid^{-2}$$

yields $\mathrm{Re}(-\lambda L^2(L^2+\lambda^2 E)^{-1}C^*Cg, g) \leq 0$ $(g \in H)$. From the above and taking into account the negative definition of the operator M we get: $\mathrm{Re}(\triangle_2 f, f) \leq -\sigma(f, f)$ $(f \in D(\triangle_2))$. This is why $|\,\mathrm{Re}(\triangle_2 f, f)\,| \geq \sigma \parallel f \parallel^2$, which yields $\parallel \triangle_2 f \parallel \geq \sigma \parallel f \parallel$. Analogously, we get $\parallel \triangle_2^* f \parallel \geq \sigma \parallel f \parallel$. Two last inequalities prove the lemma on the operator $\triangle_2$. □

Lemma 3. *If* $\mathrm{Re}\lambda \geq 0$ *then the operator* A *possesses the resolvent* $R(\lambda)$ *which is defined by the matrix*

$$R(\lambda) =$$
$$\begin{pmatrix} -\lambda \triangle_1^{-1} + L^2 \triangle_1^{-1} C(M-\lambda E)^{-1}C^* & -\triangle_1^{-1} & -\triangle_1^{-1} LC(M-\lambda E)^{-1} \\ L^2\triangle_1^{-1} & -\lambda\triangle_1^{-1} & -\lambda \triangle_1^{-1} LC(M-\lambda E)^{-1} \\ -L^3(L^2+\lambda^2 E)^{-1} \triangle_2^{-1} C^* & \lambda L(L^2+\lambda^2 E)^{-1} \triangle_2^{-1} C^* & \triangle_2^{-1} \end{pmatrix}.$$

Proof. By direct check and taking into account the commutativity of L and M^{-1} one can verify that the expression $R(\lambda)(A_0 - \lambda E)$ gives a unit operator matrix. Besides, the space of values of the operator $A_0 - \lambda E$ is dense in $\widetilde{H}$. Indeed, let such the element $x = (x_1, x_2, x_3) \in \widetilde{H}$ occurs, that $(A_0 - \lambda E)^* x = 0$. This equation is equivalent to the following three:

$$-\overline{\lambda}x_1 - L^2 x_2 = 0, \quad x_1 - \overline{\lambda}x_2 + CLx_3 = 0, \quad -C^*Lx_2 + (M - \overline{\lambda}E)x_3 = 0.$$

Finding x_1 from the first equation, x_3 from the last one and by substituting in the second one, we get $\triangle_1(\overline{\lambda})x_2 = 0$. It results from Lemma 2 that $x_2 = 0$, and therefore $x_1 = x_3 = 0$. This is why $x = 0$. □

Lemma 4. *If* $\sigma = \mathrm{Re}\lambda > 0$ *are large enough the following estimations hold*

$$\parallel R(\lambda) \parallel \leq k\frac{|\,\lambda\,|}{\sigma}, \tag{4}$$

$$\parallel R(\lambda)\widetilde{L}^{-2} \parallel \leq k\frac{1}{|\,\lambda\,|}. \tag{5}$$

Here and further k denotes constant (independent of λ), but different, generally speaking, for different inequalities.

Proof. First we are going to prove the inequality

$$\parallel L^2\triangle_1^{-1} \parallel \leq \frac{|\,\lambda\,|}{\sigma}. \tag{6}$$

As in the proof of Lemma 2, if $g \in H$, the following holds

$$\begin{aligned}(SL^{-2}g, g) =& (\sigma + i\tau)(L^{-2}g, g) + \frac{1}{|\,\lambda\,|^2}(\sigma - i\tau)(g, g) \\ &- ((M - \sigma E - i\tau E)\,|\,M - \lambda E\,|^{-1} C^*g, |\,M - \lambda E\,|^{-1} C^*g).\end{aligned}$$

Then $\mathrm{Re}(SL^{-2}g,g) \geq \sigma/|\lambda|^2$. Thus (as in Lemma 2) we see that $\| L^2S^{-1} \| \leq |\lambda|^2/\sigma$ and this is why (6) is satisfied.

Let us denote the elements of operator matrix defining $R(\lambda)$ by a_{ij}. Then:

$$\| a_{11} \| \leq \frac{k}{\sigma}, \qquad \| a_{12} \| \leq \frac{1}{|\lambda|\sigma}, \qquad \| a_{13} \| \leq \frac{k}{|\lambda|\sigma},$$

$$\| a_{21} \| \leq \frac{|\lambda|}{\sigma}, \qquad \| a_{22} \| \leq \frac{1}{\sigma}, \qquad \| a_{23} \| \leq \frac{k}{\sigma},$$

$$\| a_{11}L^{-2} \| \leq \frac{k}{|\lambda|}, \qquad \| a_{12}L^{-2} \| \leq \frac{k}{|\lambda|\sigma}, \qquad \| a_{13}L^{-2} \| \leq \frac{k}{|\lambda|^2\sigma},$$

$$\| a_{21}L^{-2} \| \leq \frac{1}{|\lambda|\sigma}, \qquad \| a_{22}L^{-2} \| \leq \frac{k}{|\lambda|}, \qquad \| a_{23}L^{-2} \| \leq \frac{k}{|\lambda|\sigma}.$$

The estimations of elements of the first two rows of the matrix $R(\lambda)$ immediately result from (6) of Lemma 2, the inequalities $\| L(M-\lambda E)^{-1} \| \leq k$, and

$$\| (M-\lambda E)^{-1} \| \leq \frac{1}{|\lambda|}, \tag{7}$$

and also from the following two expressions

$$a_{11} = -\frac{1}{\lambda}E - \frac{1}{\lambda}L^2\Delta_1^{-1}, \quad a_{22}L^{-2} = -\frac{1}{\lambda}L^{-2} + \frac{1}{\lambda}\Delta_1^{-1} + \Delta_1^{-1}C(M-\lambda E)^{-1}C^*,$$

which can be veryfied using the elementary transformations.

In order to evaluate the elements of the third row of the matrix $R(\lambda)$ we shall add some supporting inequalities.

Because for any $f \in D(L^2)$ we have $\mathrm{Re}\big((L^2/\lambda + \lambda E)f, f\big) \geq \sigma(f,f)$, then $\| (L^2/\lambda + \lambda E)^{-1} \| \leq 1/\sigma$. This yields

$$\| (L^2 + \lambda^2 E)^{-1} \| \leq \frac{1}{|\lambda|\sigma}. \tag{8}$$

From this inequality (using the elementary transformations) we get

$$\| L^2(L^2 + \lambda^2 E)^{-1} \| \leq \frac{|\lambda|}{\sigma}. \tag{9}$$

We shall prove that

$$\| L\Delta_2^{-1} \| \leq m, \tag{10}$$

where m does not depend on λ. For any $f \in D(\Delta_2 L^{-1})$ we have

$$(\Delta_2 L^{-1}f, f) = (ML^{-1}f, f) - (\sigma + i\tau)(L^{-1}f, f) - (\sigma - i\tau)\,|\lambda|^2\,(Lg, g) - (\sigma + i\tau)(L^3 g, g),$$

where $g = \sqrt{|L^2 + \lambda^2 E|^{-2}\, C^*Cf}$. It results from the negative definition of L that

$$\mathrm{Re}(\Delta_2 L^{-1}f, f) \geq \mathrm{Re}(ML^{-1}f, f) \geq k \| f \|^2 .$$

The last inequality results from the positive definition of the limited operator LM^{-1}, inversed to ML^{-1}. Therefore, the inequality (10) has been satisfied. The equation

$$\Delta_2^{-1}L^{-1} = (M - \lambda E)^{-1}L^{-1} + (L\Delta_2^{-1})(L^2 + \lambda^2 E)^{-1}C^*C\lambda(M - \lambda E)^{-1}$$

and the inequalities (7), (8), (10) yield

$$\| \Delta_2^{-1}L^{-1} \| \leq \frac{k}{|\lambda|}. \tag{11}$$

Now the inequalities (8), (9), (10), (11) and Lemma 2 allow to obtain the estimation of the elements of the third row of the matrix $R(\lambda)$:

$$\| a_{31} \| \leq k\frac{|\lambda|}{\sigma}, \qquad \| a_{32} \| \leq \frac{k}{\sigma}, \qquad \| a_{33} \| \leq \frac{1}{\sigma},$$
$$\| a_{31}L^{-2} \| \leq \frac{k}{|\lambda|\sigma}, \quad \| a_{32}L^{-2} \| \leq \frac{k}{|\lambda|\sigma}, \quad \| a_{33}L^{-2} \| \leq \frac{k}{|\lambda|}.$$

The proof of Lemma results from the obtained a_{ij} estimates.

Let us notice that some evaluations can be even more improved, for instance $\| a_{12}L^{-2} \| \leq k/|\lambda|^2$. However, they will be not used further.

Now we prove directly the theorem. Following reference [3] we are looking for the solution of the equation (2) in the form

$$u(t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}R(\lambda)u(0)d\lambda,$$

where the integration takes place along the straight line, parallel to imaginary axis and passing through the point $(\sigma, 0)$, where $\sigma > 0$ is sufficiently large.

We assume first that $u_0 \in \widetilde{H}_4^{(+1)}$, i.e. $u_0 = R(0)\widetilde{L}^{-4}z_0$ $(z_0 \in \widetilde{H})$, and we consider the function

$$u(t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}R(\lambda)R(0)\widetilde{L}^{-4}z_0 d\lambda$$
$$= -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\frac{1}{\lambda}R(\lambda)\widetilde{L}^{-4}z_0 - \frac{1}{\lambda}R(0)\widetilde{L}^{-4}z_0)d\lambda. \tag{12}$$

Here, while passing from the first integral to the second one, the resolvent identity has been used. The existence of the integral, the continuity of $u(t)$ on [0, T] in the space $\widetilde{H}_{+2}$ (and correspondingly, in $\widetilde{H}$) and the identity $u(0) = R(0)\widetilde{L}^{-4}z_0$ result directly from the Lemma 4.

Applying the integration by parts to the first integral in (12) and taking into consideration Lemma 4 we get

$$u(t) = \frac{1}{2\pi i}\frac{1}{t}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}R^2(\lambda)R(0)\widetilde{L}^{-4}z_0 d\lambda.$$

It results from the above that $u(t)$ is continuous on (0, T] in the space $\widetilde{H}^{(+1)}$ and it possesses the continuous derivative on (0, T] in $\widetilde{H}$

$$u'(t) = -\frac{1}{2\pi i}\frac{1}{t^2}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}R^2(\lambda)R(0)\widetilde{L}^{-4}z_0 d\lambda$$
$$+\frac{1}{2\pi i}\frac{1}{t}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\lambda R^2(\lambda)R(0)\widetilde{L}^{-4}z_0 d\lambda,$$

and satisfies the equation (2).

The uniqueness of the solution of the Cauchy problem for the equation (2) follows from the estimation (4) and the results presented in reference [3].

Now, let $u_0 \in \widetilde{H}_\alpha^{(\beta)}$ ($\beta = +1, 0;\ -\infty < \alpha < \infty$). Then $x_0 = A^{\beta-1}\widetilde{L}^{\alpha-4}u_0 \in \widetilde{H}_4^{(+1)}$ (here A^0 denotes E). Because the operators $R(\lambda)$ and $\widetilde{L}^\alpha$ commute and from the previous considerations we get the function

$$u(t) = -\frac{1}{2\pi i}\widetilde{A}^{1-\beta}\widetilde{L}^{4-\alpha}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}R(\lambda)x_0 d\lambda$$

with all properties mentioned in the formulated theorem. The uniqueness of such function results from the fact that the function $A^{\beta-1}\widetilde{L}^{\alpha-4}u(t)$ is the solution of the equation (2). Theorem 1 has been proved. □

Remark 1. It results from the above proof that in order to solve the equations (2) and (3), the estimation $\| u(t) \|_{\widetilde{H}_\alpha^{(\beta-1)}} \leq k \| u(0) \|_{\widetilde{H}_{\alpha+2}^{(\beta)}}$ ($\beta = +1, 0;\ -\infty < \alpha < \infty$) holds, where k does not depend on t. It defines the continuous dependence of solution on the initial data in the corresponding spaces.

Let us pass to the consideration of the non-homogeneous equations

$$u'(t) = \widetilde{A}u(t) + f(t). \tag{13}$$

By $U(t)$ we denote the operator

$$U(t)x = -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R(\lambda) x d\lambda, \quad 0 \le t \le T,$$

defined for all x, for which the last integral exists. The Lemma 4 and the equation

$$U(t)R(0)\widetilde{L}^{-2}z = -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \frac{1}{\lambda}(R(\lambda) - R(0))\widetilde{L}^{-2} z d\lambda$$

imply that for every fixed t, the operator $U(t)\colon \widetilde{H}_{\alpha}^{(\beta)} \to \widetilde{H}_{\alpha-2}^{(\beta-1)}$ ($\beta = +1, 0$; $-\infty < \alpha < \infty$) is bounded and the operator function $U(t)$ is strongly continuous on $[0, T]$. For any $u_0 \in \widetilde{H}_{\alpha}^{(\beta)}$, the function $u(t) = U(t)u_0$ has the properties given in Theorem 1.

Theorem 2. *Let for any $t \in [0, T]$, $f(t)$ belongs to the domain of the operator $\widetilde{L}^{\alpha} A^{1+\beta}$ ($\beta = +1, 0$; $-\infty < \alpha < \infty$) and let the function $\widetilde{L}^{\alpha} A^{1+\beta} f(t)$ be continuous in $\widetilde{H}$. Then the function*

$$u(t) = \int_0^t U(t-s) f(s) ds \tag{14}$$

is continuous on $[0, T]$ in the space $\widetilde{H}_{\alpha-2}^{(\beta)}$, is differentiable on $[0, T]$ in the space $\widetilde{H}_{\alpha-2}^{(\beta-1)}$ and satisfies the equation (13).

Proof. Let $g(t) = \widetilde{L}^{\alpha} A^{1+\beta} f(t)$. Following the steps given in the reference [6] one can conclude that the function (14) and its derivative have the form

$$u(t) = \int_0^t U(t-s) A^{-1-\beta} \widetilde{L}^{-\alpha} g(s) ds,$$

$$u'(t) = A^{-1-\beta}\widetilde{L}^{-\alpha} g(t) + \int_0^t U_t'(t-s) A^{-1-\beta}\widetilde{L}^{-\alpha} g(s) ds$$

$$= f(t) + \int_0^t \widetilde{A} U(t-s) A^{-1-\beta} \widetilde{L}^{-\alpha} g(s) ds = f(t) + \widetilde{A} u(t).$$

The existence of integrals in these equations results from the continuity of g and the properties of $U(t)$. In what follows $u(t)$ is the solution of the equation (13). The theorem has been proved. □

Remark 2. The continuity condition of the function $\widetilde{L}^{\alpha} A^{1+\beta}$ can be weakened if the function f is smooth enough.

Finally, we acknowledge the recent developments of a similar problem considered in our work pointed out by one of the reviewers and presented in references [6], [4] (see also the extended bibliography therein). However, in the mentioned works there is no non-local term C but there appears a mixed fourth order term accounting for the rotational forces. In addition, in reference [6] the stability properties are considered using the boundary feedback modifications, which are not considered in our work. In the works of Lasiecka (see [4] and the cited literature therein) the asymptotic behaviour governed by von Kármán equations is analysed which is out of the content of our paper.

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