PERGAMON

Nonlinear Analysis 47 (2001) 2261–2269

Nonlinear Analysis

www.elsevier.nl/locate/na

ASYMPTOTIC APPROACHES TO SIMPLIFIED BOUNDARY VALUE PROBLEMS OF NON-LINEAR DYNAMICS

[a]I. V. Andrianov, [b]J. Awrejcewicz

[b]*Pridneprovye State Academy of Civil Engineering and Architecture, Ukraine*
[a]*Department of Automatics and Biomechanics, Technical University of Lodz, Poland*

Abstract
The composite dynamical equations of the structural-orthotropic cylindrical shell theory are discussed and simplified boundary states are proposed.

Key words: thin shells, stress-strain state, boundary value problems, semi-inextensional theory

1 Introduction

It is needless to say that from engineering point of view we need more accurate results related to thin shells. It means that we need to consider the non-linear effects [2, 3, 5, 8, 10, 21, 22]. The governing equations of motion are written in the form proposed by Sanders [16] (in addition to the terms present in the original equations from reference [16] dynamic terms are added):

$$\begin{aligned}
&T_{1\xi} + S_\eta + H_\eta/(2R) - 0.5[\theta(T_1+T_2)] - \rho R u_{tt} = 0,\\
&S_\xi + T_{2\eta} - N_2 - H_\xi/(2R) + (\theta_1 S + \theta_2 T_2) - 0.5[\theta(T_1+T_2)]_\xi - \rho R v_{tt} = 0,\\
&N_{1\xi} + N_{2\eta} + T_2 - (\theta_1 T_1 + \theta_2 S)_\xi - (\theta_1 S + \theta_2 T_2)_\eta - \rho R w_{tt} = 0,\\
&\theta_1 = -w_\xi/R, \theta_2 = -(w_\eta + v)/R, \theta = (v_\xi - u_\eta)/(2R).
\end{aligned} \tag{1}$$

It is worth noting that Sanders [16] defined the variant of "moderately small rotation" by setting restrictions on the components of the linearized rotation vector, to the effect that magnitude of these components can be at most of the order of magnitude of strains.

The different variants of the non-linear shell theory are discussed in references [4, 6, 7, 9, 13-15, 17, 18].

The stress-strain relations and the expressions for the transverse shear forces have the same character as in the linear case.

The strain-displacement relations are defined as follow [16]:

$$\begin{aligned}
&\varepsilon_{11} = u_\xi/R + 0.5(\theta_1^2 + \theta^2), \quad \varepsilon_{22} = (v_\eta - w)/R + 0.5(\theta_2^2 + \theta^2),\\
&\varepsilon_{12} = (v_\xi + u_\eta)/(2R) + 0.5\theta_1\theta_2, \quad \chi_1 = \theta_{1\xi}/R,\\
&\chi_2 = \theta_{2\eta}/R, \quad \chi_{12} = (\theta_{2\xi} + \theta_{1\eta} - \theta)/(2R).
\end{aligned}$$

The following boundary conditions are assumed for $\xi = 0, d$

$$\begin{aligned}
&v = 0 \quad or \quad S - 1.5R^{-1}H + 0.5(T_1+T_2)\theta = 0,\\
&u = 0 \quad or \quad T_1 = 0,\\
&w = 0 \quad or \quad N_1 + R^{-1}H_\eta - \theta_1 T_1 - \theta_2 S = 0,\\
&\theta_1 = 0 \quad or \quad M_1 = 0.
\end{aligned} \tag{2}$$

0362-546X/01/$ - see front matter
PII: S0362-546X(01)00350-9

The initial conditions for the vector of displacements $\overline{U}(u, v, w)$ have the form

$$\overline{U} = \overline{U}_0, \quad \dot{\overline{U}} = \overline{U}_{00}, \quad for \quad t = 0. \tag{3}$$

In the non-linear shell theory, a full classification of the simplified boundary problems does not exist, although during solutions many particular problems of various approximate relations have been used.

2 A full classification of the structural - orthotropic cylindrical shells

The asymptotic integration parameters α and β characterise longitudinal and circumferential variations. In addition, we introduce the $\gamma, \delta_1, \delta_2$ and δ_3 parameters characterising the variation in time, the non-linearity order and the magnitude of the in-plane displacements in relation to normal displacement, of the following form

$$w_t \sim \varepsilon_1^{\gamma} w, \quad w \sim \varepsilon_1^{\delta_1} R, \quad u \sim \varepsilon_1^{\delta_2} w, \quad v \sim \varepsilon_1^{\delta_3} w.$$

The limiting systems obtained while applying the asymptotic procedure are given below. All of them posses their analogy in the linear case. Thus, the stress-strain relations are defined by the corresponding expressions for analogous linear equations.

- $\beta < 0.5, \quad \alpha = \beta, \quad \gamma = 0, \quad \delta_1 = 0, \quad \delta_1 = 2\beta, \quad \delta_2 = \delta_3 = \beta.$

This case corresponds to the non-linear non-momentous vibrations of a shell. The governing equations have the following form

$$\begin{aligned}
&T_{1\xi} + S_\eta - 0.5\{\theta(T_1 + T_2)\}_\eta = 0,\\
&S_\xi + T_{2,\eta} + 0.5\{\theta(T_1 + T_2)\}_\xi + \{\theta_1 S + \theta_2 T_2\} = 0,\\
&RT_2 - R(\theta_1 T_1 + \theta_2 S)_\xi - R(\theta_1 S + \theta_2 T_2)_\eta - \rho R^2 w_{tt} = 0,\\
&\varepsilon_{11} = u_\xi/R + 0.5\theta_1^2, \quad \varepsilon_{22} = (v_\eta - w)/R + 0.5\theta_2^2,\\
&\varepsilon_{12} = 0.5R^{-1}(u_\eta + v_\xi) + 0.5\theta_1\theta_2, \quad \theta_1 = -w_\xi/R,\\
&\theta_2 = -R^{-1}w_\eta - \{v\}/R, \quad \theta = 0.5(v_\xi - u_\eta)/R.
\end{aligned} \tag{4}$$

The term taken in $\{\ldots\}$ is included only if $\beta = 0$.

- $\beta < 0.5, \quad \alpha = 0.5 + 2\beta, \quad \gamma_1 = -1 + 2\beta, \quad \delta_1 = 2\beta, \quad \delta_2 = 2\beta, \quad \delta_3 = \beta.$

This case corresponds to the non-linear semi-inextensional theory. The governing equations' system has the following form

$$\begin{aligned}
&T_{1\xi} + S_\eta - 0.5\{\theta T_1\}_\eta = 0,\\
&S_\xi + T_{2\eta} - \{M_{2\eta}/R + 0.5(\theta T_1)_\xi + \theta_1 S + \theta_2 T_2\} = 0,\\
&M_{2\eta\eta} + RT_2 - R(\theta_1 T_1 + \theta_2 S)_\xi - R(\theta_1 S + \theta_2 T_2)_\eta - \rho R^2 w_{tt} = 0,\\
&\varepsilon_{11} = u_\xi/R + 0.5\theta_1^2, \quad 0 = (v_\eta - w)/R + 0.5\theta_2^2,\\
&0 = 0.5R^{-1}(u_\eta + v_\xi) + 0.5\theta_1\theta_2, \quad \theta_1 = -R^{-1}w_\xi,\\
&\theta_2 = -R^{-1}w_\eta - \{v/R\}, \quad \theta = 0.5R^{-1}(v_\xi - u_\eta).
\end{aligned} \tag{5}$$

In this case, the non-linear dependencies concerning shear lack and no stretching of the middle surface of a shell are valid. The terms in the square brackets are used only if $\beta = 0$. For $\delta_1 > 2\beta$ the equations (5) can be linearised.

Now we are going to define a boundary layer in order to compensate a disorder in the boundary conditions. We begin with an additive state of semi-inextensional theory equations. This state is characterised by a fast longitudinal variation. Its circumferential variation and variation time should be similar to those in the interior-zone state. Further, an arbitrary component P of the SSS (stress-strain state) can be presented in the form

$$P = P^{(0)} + P^{(K)}. \tag{6}$$

The superscript (0) and (K) correspond to the interior-zone state and the edge effects, respectively.

The relations between the magnitude orders of the interior-zone and edge-zone states are defined by the boundary conditions. The splitting procedure of the boundary conditions for the non-linear case is realised in a way similar to that of the linear one. For all variants of the boundary conditions the following estimation is valid

$$w^{(K)} \sim \varepsilon_1^{(1-2\beta)} w^{(0)}. \tag{7}$$

From the equation (7) we automatically obtain estimation of the non-linearity order of the additive state (for the fundamental case such estimation is already given).

Substituting equation (6) in the governing equations and carrying out the asymptotic splitting (and taking into account the interior-zone and edge-zone states) and the estimation (7), the following limiting system governing the edge effect is obtained:

- $\alpha = 0.5, \quad \beta < 0.5, \quad \delta_2 = 0.5, \quad \delta_3 = 1 - \beta,$

$$\begin{aligned}
&T_{1\xi}^{(K)} + S_{\eta}^{(K)} = 0, \quad S_{\xi}^{(K)} + T_{2\eta}^{(K)} = 0, \quad M_{1\xi\xi}^{(K)} + RT_2^{(K)} + T_1^{(0)} w_{\xi\xi}^{(K)} = 0,\\
&\varepsilon_{11}^{(K)} = u_{\xi}^{(K)}/R + (w_{\xi}^{(K)})^2/(2R^2) + w_{\xi}^{(0)} w_{\xi}^{(K)}/R^2,\\
&\varepsilon_{22}^{(K)} = -w^{(K)}/R + w_{\eta}^{(0)} w_{\eta}^{(K)}/R^2,\\
&\varepsilon_{12}^{(K)} = (v_{\xi}^{(K)} + u_{\eta}^{(K)})/2R + (w_{\xi}^{(K)} w_{\eta}^{(K)} + w_{\xi}^{(0)} w_{\eta}^{(K)} + w_{\eta}^{(0)} w_{\xi}^{(K)})/(2R).
\end{aligned} \tag{8}$$

In the relation (8) the varying coefficient should be "frozen". For instance, when the edge $\xi = 0$ is analysed then instead of $w_{\xi}^{(0)}$ (or $w_{\eta}^{(0)}$) we have to take $w_{\xi}^{(0)} |_{\xi=0}$ (or $w_{\eta}^{(0)} |_{\xi=0}$). Such a frozen procedure is true, because the function $w^{(0)}$ is changed along ξ slower than $w^{(1)}$ [19, 20]. Therefore, the coefficient of equation (8) is changing only along the co-ordinate η.

The boundary effect relations for the non-momentous equations differ from the equation (8) only by the dynamical term $\rho R^2 w_{tt}^{(K)}$.

- $\alpha = \beta = 0.5, \quad \gamma_1 = 0, \quad \delta_1 = 1, \quad \delta_2 = \delta_3 = 0.5.$

It corresponds to the theory of shallow shells. The equations have the form

$$T_{1,\xi} + S_{,\eta} = 0, \qquad S_{,\xi} + T_{2,\eta} = 0, \tag{9}$$

$$M_{1\xi\xi} + 2H_{\xi\eta} + M_{2\eta\eta} + \underline{RT_2} + (w_\xi T_1 + w_\eta S)_\xi + (w_\xi S + w_\eta T_2)_\eta - \rho R^2 w_{tt} = 0,$$

$$\varepsilon_{11} = u_\xi/R + (w_\xi)^2/(2R^2), \qquad \varepsilon_{22} = (v_\eta - w)/R + (w_\eta)^2/(2R^2), \tag{10}$$

$$\begin{aligned}
&\varepsilon_{12} = (u_\eta + v_\xi)/(2R) + (w_\xi w_\eta)/(2R^2),\\
&\chi_1 = \theta_{1\xi}/R, \quad \chi_2 = \theta_{2\eta}/R, \quad \chi_{12} = (\theta_{2\xi} + \theta_{1\eta})/(2R).
\end{aligned}$$

The equations (10) can be linearized for $\delta_1 > 2\beta$. The first two equations of motion are used further in the (9) form.

The relations, describing dynamics of the supported plate, agree with the equations (9), (10), if the underlined term is not used.

The obtained limiting systems have the second order in relation to time t, whereas the initial system has had the order equal to six. Therefore, only a lower part of the free vibration spectrum is represented. In order to analyse high frequency vibrations or the nonstationary process the following limiting system should be used

- $\beta < 1, \quad \alpha = \beta, \quad \delta_1 = 2\beta, \quad \delta_2 = \delta_3 = -\beta$

$$T_{1\xi} + S_\eta - \rho R u_{tt} = 0, \quad S_\xi + T_{2\eta} - \rho R v_{tt} = 0, \tag{11}$$

$$RT_2 + (w_\xi T_1 + w_\eta S)_\xi + (w_\xi S + w_\eta T_2)_\eta - \rho R^2 w_{tt} = 0. \tag{12}$$

The geometrical relations are linear. The equations (11) describe a motion of the structural - orthotropic plate in its plane ($u, v >> w$).

The relations (9) and the following equation govern the complementary state, necessary to satisfy the boundary conditions along the space co-ordinates

$$M_{1\xi\xi} + RT_2 - \rho R^2 w_{tt} = 0. \tag{13}$$

The further asymptotic analysis is carried out for each class of the reinforced shells.

Stringer shells

For the stringer shells the following system describes the dynamical state with the fast variations in circumferential direction

- $\alpha = 0, \quad \beta = 0.5, \quad \gamma = -1, \quad \delta_1 = 1, \quad \delta_2 = 1, \quad \delta_3 = 0.5$

$$\begin{aligned} &M^{(1)}_{1\xi\xi} + 2H^{(1)}_{\xi\eta} + M^{(1)}_{2\eta\eta} + RT^{(1)}_2 - R(\theta^{(1)}_1 T^{(1)}_1 + \theta^{(1)}_2 S^{(1)})_\xi - \\ &-R(\theta^{(1)}_2 S^{(1)} + \theta^{(1)}_2 T^{(1)}_2)_\eta - \rho R^2 w^{(1)}_{tt} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\varepsilon^{(1)}_{11} = u^{(1)}_\xi / R + (w^{(1)}_\xi)^2/(2R^2), \\ &0 = (v^{(1)}_\eta - w^{(1)})/R + (w^{(1)}_\eta)^2/(2R^2), \\ &0 = (u^{(1)}_\eta + v^{(1)}_\xi)/(2R) + w^{(1)}_\xi w^{(1)}_\eta/(2R^2), \\ &\chi^{(1)}_{11} = -w^{(1)}_{\xi\xi}/R^2, \quad \chi^{(1)}_{22} = -w^{(1)}_{\eta\eta}/R^2, \quad \chi^{(1)}_{12} = -w^{(1)}_{\xi\eta}/R^2. \end{aligned} \tag{15}$$

The additive state to that governed by equations (14), (15) is the one localised in the neighbourhood of the shell's edges

- $\alpha = \beta = 0.5, \quad \gamma = -1, \quad \delta_1 = 1 \quad (or \quad \delta_1 = 1.5), \quad \delta_2 = \delta_3 = 0.5$

$$\begin{aligned} &M^{(2)}_{1\xi\xi} + RT^{(2)}_2 - R\theta^{(1)}_{2\eta} T^{(2)}_2 = 0, \\ &\varepsilon^{(2)}_{11} = u^{(2)}_\xi / R, \quad \varepsilon^{(2)}_{22} = (v^{(2)}_\eta - w^{(2)})/R + \theta^{(1)}_2 \theta^{(2)}_2, \\ &\varepsilon^{(2)}_{12} = (v^{(2)}_\xi + u^{(2)}_{,\eta})/(2R) + 0.5\theta^{(1)}_2 \theta^{(2)}_1. \end{aligned} \tag{16}$$

For $\beta = 0.5 + k, \alpha = k, \gamma = -1 + 2k, \delta_1 = 1 + 2k, \delta_2 = 1 + k, \delta_3 = 0.5 + k; k > 0$ the bending vibrations of the stringer plate play a key role.

The limiting equations have the form

$$M_{1\xi\xi} + 2H_{\xi\eta} + M_{2\eta\eta} + \rho R^2 w_{tt} = 0. \tag{17}$$

The geometrical relations are described by the relations (15).

- *For* $\beta > 1/2, \quad \alpha = \beta, \quad \gamma = -1 + 2\beta, \quad \delta_1 = 1, \quad \delta_2 = \delta_3 = 1 - \beta$

the stringer type plate vibrations are described in the direction perpendicular to the supporting ribs' positions (with higher frequencies than in the previous case)

$$\begin{aligned} &M_{1\xi\xi} - R(\theta_1 T_1 + \theta_2 S)_\xi - R(\theta_1 S + \theta_2 T_2)_\eta + \rho R^2 w_{tt} = 0, \\ &\varepsilon_{11} = u_\xi / R + 0.5\theta_1^2, \qquad \varepsilon_{22} = v_\eta / R + 0.5\theta_2^2, \\ &\varepsilon_{12} = (v_\xi + u_\eta)/(2R) + 0.5\theta_1\theta. \end{aligned}$$

Ring - stiffened shells

For the shells with the dominating ring support there are states with fast variations in both longitudinal and circumferential direction. Both of the states are dynamical states (similar to the linear case), and a solution to the problem may begin with a calculation of one of them. It depends on the needs which part of frequency spectrum is analysed. If we begin the calculation from the consideration of the plane stress state, then we obtain the following limiting systems

- $\alpha = \beta = 0.25, \quad \gamma = 0, \quad \delta_1 = 1/2, \quad \delta_2 = \delta_3 = 0.25$

$$\begin{aligned} &M^{(1)}_{2\eta\eta} + RT^{(1)}_2 - R(\theta^{(1)}_1 T^{(1)}_1 + \theta^{(1)}_2 S^{(1)})_\xi - \\ &-R(\theta^{(1)}_2 S^{(1)} + \theta^{(1)}_2 T^{(1)}_2)_\eta - \rho R^2 w^{(1)}_{tt} = 0. \end{aligned} \tag{18}$$

For the additive state we have

- $\alpha = 0.5, \quad \beta = 0.25, \quad \gamma = 0, \quad \delta_1 = 1, \quad \delta_2 = 1/2, \quad \delta_3 = 0.75$

$$\begin{aligned} &M^{(2)}_{1\xi\xi} + M^{(2)}_{2\eta\eta} - RT^{(2)}_2 - R\theta^{(1)}_{2\eta} T^{(2)}_2 - \rho R^2 w^{(2)}_{tt} = 0, \\ &\varepsilon^{(2)}_{11} = u^{(2)}_\xi / R + 0.5(\theta^{(2)}_1)^2 + \theta^{(1)}_1 \theta^{(2)}_1, \\ &\varepsilon^{(2)}_{22} = -w^{(2)}/R + \theta^{(1)}_2 \theta^{(2)}_2, \\ &\varepsilon^{(2)}_{12} = (v^{(2)}_{,\xi} + u^{(2)}_{,\eta})/(2R) + 0.5\theta^{(2)}_1 \theta^{(2)}_2 + 0.5(\theta^{(1)}_1 \theta^{(2)}_2 + \theta^{(1)}_2 \theta^{(2)}_1). \end{aligned} \tag{19}$$

- $\alpha = \beta = 0.25, \quad \gamma = 0, \quad \delta_1 = 0.75, \quad \delta_2 = \delta_3 = 0.25$

$$M^{(1)}_{1\xi\xi} + RT^{(1)}_2 - \rho R^2 w^{(1)}_{tt} = 0. \tag{20}$$

The geometrical relations are here linear.
The complementary state is described by the following relations

- $\beta = 0.25, \quad \alpha = 0.5, \quad \gamma = 0, \quad \delta_1 = 1, \quad \delta_2 = 0.5, \quad \delta_3 = 0.75$

$$\begin{aligned} &M^{(2)}_{1\xi\xi} + M^{(2)}_{2\eta\eta} + RT^{(2)}_2 - RT^{(1)}_1 \theta^{(2)}_{1\xi} - RS^{(1)}(\theta^{(2)}_{2\xi} + \theta^{(2)}_{1\eta}) - \\ &-RT^{(1)}_1 \theta^{(1)}_{1\xi} - RS^{(1)}(\theta^{(1)}_{2\xi} + \theta^{(1)}_{1\eta}) - RT^{(1)}_2 \theta^{(1)}_{2\eta} - \rho R^2 w^{(2)}_{tt} = 0, \\ &\varepsilon^{(2)}_{11} = u^{(2)}_\xi / R + 0.5(\theta^{(2)}_1)^2 + \theta^{(1)}_1 \theta^{(2)}_1 + 0.5(\theta^{(1)}_1)^2, \\ &\varepsilon^{(2)}_{22} = -w^{(2)}/R + 0.5(\theta^{(1)}_2)^2, \\ &\varepsilon^{(2)}_{12} = (v^{(2)}_\xi + u^{(2)}_\eta)/(2R) + 0.5\theta^{(2)}_1 \theta^{(2)}_2 + 0.5(\theta^{(1)}_1 \theta^{(2)}_2 + \theta^{(1)}_2 \theta^{(2)}_1) + 0.5\theta^{(1)}_1 \theta^{(1)}_2. \end{aligned}$$

- $\beta = \alpha = 0.25, \quad \gamma = 0, \quad \delta_1 = 1, \quad \delta_2 = \delta_3 = 0.25.$

For this case the equations for the interior-zone state overlap with the equations (20). To analyse the additive state the following relations are used

- $\beta = 0.25, \quad \alpha = 0.5, \quad \gamma = 0, \quad \delta_1 = 1, \quad \delta_2 = 0.5, \quad \delta_3 = 0.75$

$$M^{(2)}_{1\xi\xi} + M^{(2)}_{2\eta\eta} + RT^{(2)}_2 - RT^{(1)}_1\theta^{(2)}_{1\xi} - \rho R^2 w^{(2)}_{tt} = 0,$$
$$\varepsilon^{(2)}_{11} = u^{(2)}_\xi/R + 0.5(\theta^{(2)}_1)^2, \quad \varepsilon^{(2)}_{22} = -w^{(2)}/R,$$
$$\varepsilon^{(2)}_{12} = (v^{(2)}_\xi + u^{(2)}_\eta)/(2R) + 0.5\theta^{(2)}_1\theta^{(2)}_2 + 0.5\theta^{(1)}_2\theta^{(2)}_1.$$

If we consider (in the beginning) a state with a dominating longitudinal variation then the following limiting systems are obtained

- $\beta = 0.25, \quad \alpha = 0.5, \quad \gamma = 0, \quad \delta_1 = 1, \quad \delta_2 = 0.5, \quad \delta_3 = 0.75$

$$M^{(2)}_{1\xi\xi} + M^{(2)}_{2\eta\eta} + RT^{(2)}_2 - \rho R^2 w^{(2)}_{tt} = 0,$$
$$\varepsilon^{(2)}_{11} = u^{(2)}_\xi/R + \theta^{(2)^2}_1/R, \quad \varepsilon^{(2)}_{22} = -w^{(2)}/R,$$
$$\varepsilon^{(2)}_{12} = (v^{(2)}_\xi + u^{(2)}_\eta)/(2R) + 0.5\theta^{(2)}_1\theta^{(2)}_2.$$

- $\beta = \alpha = 0.25, \quad \gamma = 0, \quad \delta_1 = 0.75, \quad \delta_2 = 0.25, \quad \delta_3 = 0.25$

$$M^{(1)}_{2\eta\eta} + RT^{(1)}_2 - R\theta^{(2)}_{1\xi}T^{(1)}_1 - \rho R^2 w^{(1)}_{tt} = 0.$$

The geometrical relations for this case are linearized $k > 0$.

- $\beta = 0.25 + k, \quad \alpha = 0.5 + k, \quad \gamma = 2k, \quad \delta_1 = 1 + 2k, \quad \delta_2 = 0.5 + k, \quad \delta_3 = 0.75 - k.$

$$M_{1\xi\xi} + M_{2\eta\eta} + \rho R^2 w_{tt} = 0,$$
$$\varepsilon_{11} = u_\xi/R_1 + 0.5\theta^2_1, \quad \varepsilon_{22} = -w/R, \quad \varepsilon_{12} = v_\xi/(2R) \tag{21}$$

The equations (21) govern the dominating bending vibrations of a plate supported in the η direction (parallel to the ribs).

- $\beta > 0.5, \quad \alpha = \beta, \quad \gamma = -0.5 + 2\beta, \quad \delta_1 = 1/2, \quad \delta_2 = \delta_3 = 0.5 - \beta$

$$M_{2\eta\eta} - R(\theta_1 T_1 + \theta_2 S)_\xi - R(\theta_1 S + \theta_2 T_2)_\eta + \rho R^2 w_{tt} = 0. \tag{22}$$

The limiting equation (22) governs the vibrations of the structural - orthotropic plate with higher frequency values than in the previous case.

Waffle shells

For the waffle shells we carry out the splitting in relation to the ε_5 parameter. We introduce the new parameters of asymptotic integration γ_1, δ, and also $\gamma_0, \delta_4, \delta_5$ and δ_6 according to the formulas:

$$\frac{\partial}{\partial\xi} \sim \varepsilon_5^{-\gamma_1}, \quad \frac{\partial}{\partial\eta} \sim \varepsilon_5^{-\delta}, \quad \frac{\partial}{\partial t} \sim \varepsilon_5^{-\gamma_0}, \quad R^{-1}w \sim \varepsilon_5^{\delta_4}, \quad u \sim \varepsilon_5^{\delta_5}w, \quad v \sim \varepsilon_5^{\delta_6}w.$$

As a result of the asymptotic splitting in relation to the equations (9), (10) we obtain the following limiting system

- $\gamma_1 = \delta, \quad \gamma_0 = \delta_4 = 2\delta, \quad \delta_5 = \delta_6 = \delta$

$$M_{1\xi\xi} + 2H_{2\xi\eta} + M_{2\eta\eta} + RT_2 - \rho R^2 w_{tt} = 0,$$
$$\varepsilon_{11} = u_\xi/R + (w_\xi)^2/(2R^2),$$
$$\varepsilon_{22} = (v_\eta - w)/R + w_\eta^2/(2R^2), \tag{23}$$
$$\varepsilon_{12} = (u_\eta + v_\xi)/(2R) + w_\xi w_\eta/(2R^2).$$

The equations (23) can be obtained from the governing system using the hypotheses

$$u_\xi + (w_\xi)^2/(2R^2) - \varepsilon_6 w_{\xi\xi} = 0, \qquad v_\eta - w + (w_\eta)^2/(2R^2) - \varepsilon_7 w_{\eta\eta} = 0. \tag{24}$$

The expressions (2.17) and (2.19) describe the complementary state.

The splitting boundary conditions for the equations of non-linear semi-inextensional theory (5) and the edge effect (8) (which can be used also in the isotropic case) are given in the Table 1.

Table 1: Variants of conditions and boundary conditions

Variants of conditions	Boundary condition for $\xi = 0, d$
A_1	$w^{(1)} = 0, \quad w_\xi^{(1)} = 0, \quad w_\xi^{(2)} = 0,$ $L^{(2)} = v^{(2)} - w^{(2)} + 0.5R{-}1(w_\eta^{(2)})^2 = 0$
A_2	$w^{(1)} = 0, \quad w_\xi^{(1)} = 0, \quad L^{(2)} = 0, \quad w_\xi^{(2)} = -w_\xi^{(1)},$
A_3	$w^{(1)} = 0, \quad w_\xi^{(1)} = 0, \quad L^{(2)} = 0, \quad M_1^{(2)} = -M_1^{(1)},$
A_4	$w^{(1)} = 0, \quad T_1^{(1)} = 0, \quad L^{(2)} = 0, \quad M_1^{(2)} = -M_1^{(1)},$
A_5	$w^{(1)} = 0, \quad w_\xi^{(1)} = 0, \quad w_\xi^{(2)} = 0, \quad S^{(2)} = -S^{(1)},$
A_6	$w^{(1)} = 0, \quad T_1^{(1)} = 0, \quad w_\xi^{(2)} = -w_\xi^{(1)}, \quad S^{(2)} = -S^{(1)},$
A_7	$w^{(1)} = 0, \quad w_\xi^{(1)} = 0, \quad S^{(2)} = -S^{(1)}, \quad M_1^{(2)} = -M_1^{(1)},$
A_8	$w^{(1)} = 0, \quad T_1^{(1)} = 0, \quad S^{(2)} = -S^{(1)}, \quad M_1^{(2)} = -M_1^{(1)},$
A_9	$w^{(1)} = 0, \quad u_1^{(1)} = 0, \quad N_1^{(2)} = 0, \quad w_\xi^{(2)} = -w_\xi^{(1)},$
A_{10}	$w^{(1)} = 0, \quad T_1^{(1)} = 0, \quad N_1^{(2)} = 0, \quad w_\xi^{(2)} = -w_\xi^{(1)},$
A_{11}	$w^{(1)} = 0, \quad w_\xi^{(1)} = 0, \quad N_1^{(2)} = 0, \quad M_1^{(2)} = -M_1^{(1)},$
A_{12}	$w^{(1)} = 0, \quad T_1^{(1)} = 0, \quad N_1^{(2)} = 0, \quad M_1^{(2)} = -M_1^{(1)},$
A_{13}	$u^{(1)} = 0, \quad S^{(1)} = 0, \quad N_1^{(2)} = 0, \quad w_\xi^{(2)} = -w_\xi^{(1)},$
A_{14}	$T_1^{(1)} = 0, \quad S^{(1)} = 0, \quad N_1^{(2)} = 0, \quad w_\xi^{(2)} = -w_\xi^{(1)}$
A_{15}	$u^{(1)} = 0, \quad S^{(1)} = 0, \quad N_1^{(2)} = 0, \quad M_1^{(2)} = -M_1^{(1)},$
A_{16}	$T_1^{(1)} = 0, \quad S^{(1)} = 0, \quad N_1^{(2)} = 0, \quad M_1^{(2)} = -M_1^{(1)}$

The similar like tables can be constructed for the shallow stringer type shells (the equations (14)-(16)), for the waffle shells (the equations (18)-(19)) and the ring-stiffened shells [1].

3 Example

As an example we consider the problem of the simple supported stringer type shells vibrations. Its dynamical stress-state is governed by the equations (14) -(16). The deflection w is presented in the following way

$$w = f_1(t) \sin \overline{m}\xi \cos(n\eta) + f_2(t) \sin^2 \overline{m}\xi, \qquad \overline{m} = m\pi/d.$$

The relations between the time depending functions f_1 and f_2 are obtained using a continuity conditions of the displacement v in the ring direction. Using the Bubnov-Galerkin method along the space co-ordinates and the standard perturbation method [11, 12] (the amplitude of vibration has served as the perturbation parameter) the following amplitude (φ_1) - frequency (ω) relation is obtained.

$$\Omega = \omega/\omega_0 = [1 + 0.125(\gamma_0 + 0.03125\alpha_2(\alpha_1 - 2\gamma_2 - 16\alpha_0) - \\ -6c_1\alpha_1 + 80c_2)\varphi^2]\varphi^2, \quad \varphi = \varphi_1 R^{-1}.$$

Above, ω_0 denotes the frequency of the linear vibrations,

$$c_i = A_i A_1^{-1}, \quad c_2 = A_3 A_2^{-1}; \quad \alpha_1 = c_1 - 2\alpha_0, \quad \alpha_2 = 3c_1 - 2\alpha_0,$$
$$A_1 = \varepsilon_1^2\varepsilon_4 + 2\varepsilon_1^2\varepsilon_3\varepsilon_4 n^2 \overline{m}^{-2} + \varepsilon_1^2\varepsilon_2\varepsilon_4 n^4 \overline{m}^{-4} + n^4(1 - \varepsilon_6 n^2)^2,$$
$$A_2 = 0.063 + 0.5n^4\varepsilon_1^2\varepsilon_4 - 0.15(1 - \varepsilon_6 n^2),$$
$$A_3 = 0.25n^4, \qquad \alpha_0 = 0.094n^4.$$

The calculations have been carried out for the following geometrical - stiffness parameters: $\varepsilon_1 = \varepsilon_6 = 0.02, \varepsilon_2 = 0.0004, \varepsilon_3 = 1, \varepsilon_4 = \varepsilon_5 = 0.4, d = 4$. Results are shown in the Figure 1 ($\varphi^* = h^{-1}\varphi$).

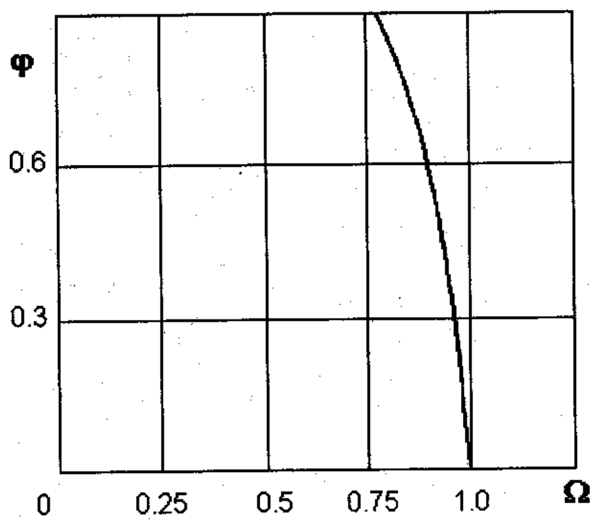

Figure 1: The $\varphi(\Omega)$ relation.

The system's characteristic is soft, which agrees with the known results for the isotropic and reinforced shell [8].

Acknowledgment

This work has been partially supported by the Polish State Committee for Scientific Research.

References

[1] I. V. Andrianov, J. Awrejcewicz, Uniformely suitable approximate solutions in the theory of shells (to appear). [2] I. V. Andrianov, E. G. Kholod, Simplified relations of the non-linear dynamics of smooth shells, Dynamical Systems, **2**, 60-64, (1983).
[3] J. Awrejcewicz , J. Mrozowski, M. Potier-Ferry, Non-linear behaviour of a rectangular plate exposed to airflow, 253-272. In: J. Awrejcewicz (ed.), Bifurcation and chaos, Theory and Applications, Springer-Verlag, Berlin, (1995). [4] E. L. Axelrad, F. A. Emmerling, Vector and tensor form of intrinsic shell-theory relations, International Journal of Non-Linear Mechanics, vol. 23, **1**, 9-23,(1988).
[5] E. L. Axelrad, Theory of Flexible Shells, North Holland, Amsterdam, (1987).
[6] W. T. Koiter, On the nonlinear theory of thin elastic shells, Proc. Koninkl. Nederl. Akademie van Wetenschappen, **B 69(1)**, 1-54, (1966).
[7] W. T. Koiter, J. G. Simmonds, Foundations of shell theory. Theoretical and Applied Mechanics, Proceedings of the 13th IC TAM, Moscow University, (1973), ed. E. Becker and G. K. Mikhailov. Springer Verlag, Berlin, 150-176.
[8] V. D. Kubenko, P. S. Kovalchuk, N. P. Podtchasov, Nonlinear Vibrations of Cylindrical Shells, Vissha Shkola, Kiev, (1989).
[9] A. Libai, J. G. Simmonds, Nonlinear elastic shell theory. In: Advances in Applied Mechanics, ed. J. W. Hutchinson, T. W. Wu, Academic Press, New York, Vol. 23, 271-371, (1983).
[10] Kh. M. Mushtari, K. Z. Galimov, Nonlinear Theory of Elastic Shells. The Israel Program for Sci., Monson, Jerusalem, (1961).
[11] A. H. Nayfeh, Perturbation Methods, John Wiley & Sons, New York, (1973).
[12] A. Nayfeh, Introduction to Perturbation Techniques, John Wiley & Sons, New York, (1981).
[13] W. Pietraszkiewicz, Introduction to the Non-Linear Theory of Shells, Ruhr Universitt, Mitt.-Inst. für Mech. **10**, Bochum, (1977).
[14] W. Pietraszkiewicz, Finite Rotations and Lagrangean Description in Non-Linear Theory of Shells, Polish Scientific Publishers, Warszawa-Poznań, (1979).
[15] W. Pietraszkiewicz, Geometrically nonlinear theories of thin elastic shells, Advances in Mechanics, vol. 12, **1**, 51-130, (1989).
[16] J. L. Sanders, Nonlinear theories for thin shells, Quarterly of Applied Mathematics **21**, 21-36, (1963).
[17] R. Schmidt, A current trend in shell theory: constrained geometrically nonlinear Kirchhoff-Love type theories based on polar decomposition of strains and rotations, Computers and Structures, vol. 20, 265-275, (1985).
[18] R. Valid, The Nonlinear Theory of Shells Through Variational Principles, Wiley, Chichester, (1995).
[19] A. B. Vasil'eva, V. F. Butuzov, L. V. Kalachev, The Boundary Function Method for Singular Perturbation Problems, SIAM, Philadelphia, (1995).
[20] M. I. Vischik, L. A. Lyusternik, Regular singularity and boundary layer for the linear differential equations with a small parameter bý a higher derivative, Advanced in Mathematical Sciences, vol. 5 (77), 3-122, (1957).
[21] A. S. Volmir, Biegsame Platten und Schalen, Verlag für Bauwesen, Berlin, (1962).
[22] T. Wah, The normal modes of vibration of certain nonlinear continuous systems, Trans. ASME, J. Appl. Mechs, vol. 31, **1**, (1964).