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Methods of Small and Large δ in the Nonlinear Dynamics – A Comparative Analysis *

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Abstract. New asymptotic approaches for dynamical systems containing a power nonlinear term x^n are proposed and analyzed. Two natural limiting cases are studied: $n \approx 1 + \delta$, $\delta \ll 1$ and $n \to \infty$. In the first case, the 'small δ method' (SδM) is used and its applicability for dynamical problems with the nonlinear term $\sin\alpha$ as well as the usefulness of the SδM for the problem with small denominators are outlined. For $n \to \infty$, a new asymptotic approach is proposed (conditionally we call it the 'large δ method' – LδM). Error estimations lead to the following conclusion: the LδM may be used, even for small n, whereas the SδM has a narrow application area. Both of the discussed approaches overlap all values of the parameter n.

Keywords: Dynamical systems, asymptotic approaches, small δ method.

1. Introduction

Choosing a perturbation parameter value during the asymptotic integration of the differential equations governing nonlinear dynamics certainly does not belong to trivial tasks. The quasi-linear decomposition, which possesses a wide range of applications, does not allow for achieving the desired results in many cases [1, 2]. Therefore, recently more attention has been paid to alternative methods of perturbation parameter introduction. Particularly, in the case of the occurrence of x^n-type nonlinearity, the parameter n can be used and the analysis can be split into two parts: $n = 1 + \delta$, $\delta \ll 1$ [3, 4] ('small δ method' – SδM) and $n \to \infty$ [5, 6] ('large δ method' – LδM).[1]

In this paper, we analyze the above-mentioned approaches on the basis of some problems of nonlinear dynamics. In particular, we show that the SδM can be reduced to the LδM.

2. Algebraic Equation – A Simple Example

In [3], an excellent choice of types of algebraic equations has been made to illustrate the applied method. Here, we briefly describe the results given in the introduction to [3] and compare them with an additional approach (see also [1]).

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* Contributed by Professor A. F. Vakakis.

[1] The last term is proposed by us.

Consider the equation

$$x^5 + x = 1. \tag{1}$$

A solution of Equation (1) can be found using, for instance, Newton's method. The real root $x \approx 0.7549$. Because a perturbation parameter does not explicitly occur in (1), here we show that there are many ways of introducing it.

2.1. Quasi-Linear Approach [3]

$$\varepsilon x^5 + x = 1. \tag{2}$$

Further, x is developed into a series of ε and, finally, $\varepsilon = 1$ is taken. One obtains

$$x = \sum_{i=0}^{\infty} a_i \varepsilon^i, \tag{3}$$

$$a_n = [(-1)^n (5n)!]/[n!(4n+1)!]. \tag{4}$$

The convergence radius of series (3) $R = 4^4/5^5 \approx 0.082$. The series is characterized by a quick divergence. This poor result can be improved by the use of Padé approximants (AP). The AP [3/3] for $\varepsilon = 1$ gives $x \approx 0.7637$ (the error related to the exact solution reaches 1.2%).

2.2. Strong Nonlinear Asymptotics [3]

$$x^5 + \varepsilon x = 1. \tag{5}$$

A solution of Equation (5) can be presented by series (3), where

$$a_n = -\{\Gamma[(4n-1)/5]/(2\Gamma[4-n)/5]n!)\}. \tag{6}$$

In this case, the radius of convergence $R = 5/4^{4/5} \approx 1.65$ and series (3), (5) can be summed. Taking the first six terms into account, we obtain $x \approx 0.07543$ (the error related to the exact solution reaches 0.07%).

2.3. The Method of Small δ (SδM) [3]

Bender et al. [3] suggest taking the perturbation parameter in the following way:

$$x^{1+\delta} + x = 1. \tag{7}$$

The solution to Equation (7) is sought in the form of

$$x = \sum_{i=1}^{\infty} b_i \delta^i, \tag{8}$$

and, in the final form, we have to take $\delta = 4$.

The introduction of series (8) into Equation (7) allows us to obtain the coefficients b_i:

$$b_0 = 0.5; \quad b_1 = 0.25 \ln 2; \quad b_2 = -0.125 \ln 2 \ldots$$

and the radius of convergence of the series (8) is $R = 1$. An application of AP [3/3] allows one to find $x \approx 0.07545$ (the error is 0.05%).

2.4. The Method of Large δ (LδM)

An asymptotic mathematician says [3]: if for $\varepsilon \to 0$ an asymptotic behavior occurs, then for $\varepsilon \to \infty$ an asymptotic process should also be evident. We try to find a solution to Equation (1) for $n \to \infty$. We change the variables according to the rule $x = y^{1/n}$ and we use the following approximation:

$$y^{1/n} = 1 + \frac{1}{n} \ln y + \cdots.$$

In the first approximation we get

$$x(n) \approx 1 - \ln n/n,$$

which leads to the value $x \approx 0.6781$ (the error is 10.6%). The application of the AP gives $x(n) \approx 1/(1 + \ln n/n)$, $x(5) \approx 0.7564$ (the error is only 0.2%!).

An occurrence of the series term

$$x(n) \approx \left(\frac{\ln n - \ln(\ln n)}{n}\right)^{1/n}$$

allows one to improve the result $x(5) \approx 0.07558$. The error related to the required solution is 0.1%.

The above elementary considerations lead to the following conclusions. The introduction of a small (large) parameter into the series exponent can lead to more effective results than the traditional methods of a weak or strong link (coupling). For the considered example, the LδM is more effective than the SδM.

Finally, applying the SδM leads to the occurrence (in a solution) of the 'ln'-type singularities. Besides, in the case of the LδM, the bifurcation points occur because of the roots of n powers.

3. The Small δ Method for Homogeneous Nonlinear Dynamical Equations

We consider the homogeneous nonlinear differential equation

$$\ddot{x} + x^n = 0, \tag{9}$$

and we are going to find a periodic solution with the following initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0. \tag{10}$$

Using the SδM, Equation (9) is transformed into

$$\ddot{x} + x^{1+2\delta} = 0. \tag{11}$$

Taking into account the series

$$(x^2)^\delta = 1 + \delta \ln(x^2) + \frac{\delta^2}{2}[\ln(x^2)]^2 + \cdots, \tag{12}$$

a solution to Equation (11) is sought in the form

$$x = \sum_{n=0}^{\infty} x_n \delta^n. \tag{13}$$

Besides, we introduce a standard change of time following the formula

$$t = \tau/\omega, \tag{14}$$

and the introduced parameter ω is defined by

$$\omega^2 = 1 + \alpha_1\delta + \alpha_2\delta^2 + \cdots. \tag{15}$$

Substituting expressions (13–15) into Equation (11) and taking into account (12) (after splitting with respect to δ), we get the following recurrent system of equations

$$\ddot{x}_0 = x_0 = 0, \tag{16}$$

$$x_0(0) = 1, \quad \dot{x}_0(0) = 0, \tag{17}$$

$$\ddot{x}_1 + x_1 = -x_0 \ln(x_0)^2 - \alpha_1 \ddot{x}_0, \tag{18}$$

$$x_1(0) = \dot{x}_1(0) = 0, \tag{19}$$

$$\ddot{x}_2 + x_2 = -[x_1 \ln(x_0^2) + 2x_1] - x_0[\ln(x_0^2)]^2 - \alpha_2\ddot{x}_0 - \alpha_1\ddot{x}_1, \tag{20}$$

$$x_2(0) = \dot{x}_2(0) = 0. \tag{21}$$

From Equations (16, 17), one obtains

$$x_0 = \cos t.$$

Then from Equation (18) we get

$$\ddot{x}_1 + x_1 = -\cos \ln(\cos^2 t) + \alpha_1 \cos t. \tag{22}$$

The most popular method of constructing the solution to Equation (22) is related to the development of the Fourier series of the first term of the right-hand side equations in order to cancel the resonance behavior by a proper choice of α_1. The averaging procedure applied to the right-hand side of Equation (22) is related to the so-called Lobatczevski function

$$L(x) = -\int_0^x \ln|\cos t|\, dt,$$

which has the following properties:

1. Symmetry and periodicity

$$L(x) = -L(-x) \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

2. Pseudoperiodicity

$$L(x-\pi)=L(x)-\pi\ln 2,\quad L(x+\pi)=L(x)+\pi\ln 2,$$

3. Series approximation

$$L(x)=x\ln 2-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin 2k_x}{k^2}$$

Taking into account the two first properties, one obtains:

$$L(2\pi)=2\pi\ln 2,\quad \text{and thus}\quad \alpha_1=4\pi\ln 2.$$

Therefore, a period of the solution can be defined as

$$T\approx 2\pi[1-2\delta\ln 2]. \tag{23}$$

For $n=3(\delta=1)$, formula (23) yields $T=6.8070$, where the exact period value $T=7.4164$ (the error does not exceed 8%). The next approximation gives an almost exact value ($T=7.5111$). The approximation formula (23) does not represent a real period value for $n=5$.

4. Mathematical Pendulum

The SδM allows for some rather simple investigations of many problems for which an application of the quasi-linear approach is difficult.

As an example, we consider the mathematical pendulum governed by the equation

$$\ddot{x}+\sin x=0. \tag{24}$$

The quasi-linear approach does not guarantee the required accuracy of the solution to boundary-value problem (24).

Equation (24) can be transformed into the form

$$\ddot{x}+x-\frac{1}{3!}x^{1+2\delta}+\frac{1}{5!}x^{1+4\delta}+\cdots=0, \tag{25}$$

which can be solved using the SδM with the boundary conditions of (10).

Let us consider the expression

$$\Omega(x,\delta)=x-\frac{1}{3!}x^{1+2\delta}+\frac{1}{5!}x^{1+4\delta}-\cdots,$$

which reads as

$$\Omega(x,0)=x\left(1-\frac{1}{3!}+\frac{1}{5!}-\cdots\right)=x\sin 1.$$

Defining $\omega^2=\sin 1$, we apply the series (12) to each of the term of the function $\Omega(x,\delta)$.

Therefore,

$$\Omega(x,\delta)=x(\omega^2+\delta\ln(x^2)\omega_1^2+\delta^2[\ln(x^2)]^2+\cdots],$$

where

$$\omega_1^2 = \frac{1}{3!} - \frac{2}{5!} + \frac{3}{7!} - \frac{4}{9!} + \cdots,$$

$$\omega_2^2 = 0.5\left(\frac{1}{3!} - \frac{2^2}{5!} + \frac{3^2}{7!} - \frac{4^2}{9!} + \cdots\right).$$

After splitting with respect to δ, we get the following recurrent equations:

$$\ddot{x}_0 + \omega^2 x_0 = 0, \tag{26}$$

$$x_0(0) = 1, \quad \dot{x}_0(0) = 0, \tag{27}$$

$$\ddot{x}_1 + \omega^2 x_1 = \omega_1^2 x_0 \ln(x_0)^2 - \alpha_1 \ddot{x}_0, \tag{28}$$

$$x_1(0) = \dot{x}_1(0) = 0, \tag{29}$$

$$\ddot{x}_2 + \omega^2 x_2 = \omega_1^2 x_1 \ln(x_0^2) + \omega_2^2 x_0 [\ln(x_0^2)]^2 - \alpha_2 \ddot{x}_0 - \alpha_1 \ddot{x}_1, \tag{30}$$

$$x_2(0) = \dot{x}_2(0) = 0. \tag{31}$$

The zero order solution (initial-value problem (26), (27)) has the form

$$x_0 = \cos \omega t.$$

From the first approximation equation (28), using a method for avoiding the secular terms, we get

$$\alpha_1 = -2(\omega_1/\omega)^2 \ln 2.$$

Taking $\delta = 1$, we get the formula for the period of oscillations

$$T = 2\pi \left[1 - (\omega_1/\omega)^2 \frac{\ln 2}{\pi}\right]. \tag{32}$$

The next successive approximations can be obtained in a similar way.

The numerical computation of the period of Equation (24) gives the value $T = 6.6$. Approximation of zero order gives $T = 6.8$ and this approximation is better than usually using approximation of the order $T = 2\pi$. In the first approximation, one obtains an almost exact value $T = 6.57$ (error consists approximately of 0.5%).

Therefore, the SδM can be treated as an adequate one for the approximate integration of equations which do not explicitly include small parameters.

5. Large δ Method (LδM)

Consider a construction of the periodic solution to the equation

$$\ddot{\mathbf{x}} + \mathbf{x}^n = 0, \tag{33}$$

with the boundary conditions

$$x(0) = 0, \tag{34}$$

for $x = 1$ one has $\dot{x} = 0$. (35)

Integration of Equation (33) yields

$$\left(\frac{2}{n+1}\right)^{1/(n+1)} t = \int_0^{0 \le \xi \le 1} \frac{d\xi}{\sqrt{1 - \xi^{n+1}}}. \tag{36}$$

A change of variables $\xi = \sin^{\varepsilon}\Theta$, $\varepsilon = 2/(n+1)$ applied to expression (36) gives

$$\varepsilon^{2\varepsilon^{-1}} t = \varepsilon \int^{0 \le \Theta \le \pi/2} \sin^{-1+\varepsilon}\Theta \, d\Theta. \tag{37}$$

In order to obtain the first approximation of the right-hand side asymptotics of (37), the following relations are used:

$$\sin^{-1+\varepsilon}\theta = \theta^{-1+\varepsilon}\left(\frac{\theta}{\sin\theta}\right)^{1-\varepsilon} = \theta^{-1+\varepsilon}\left[\frac{\theta}{\sin\theta} - \varepsilon \ln \frac{\theta}{\sin\theta} + \cdots\right],$$

$$\sin^{-1+\varepsilon}\theta = \theta^{-1+\varepsilon}\frac{\theta^{1+\varepsilon}}{3!} + \cdots + O(\varepsilon).$$

Coming back to the initial variables, we finally get

$$x \approx \varepsilon^{2\varepsilon} \sin^{\varepsilon}(t^{\varepsilon^{-1}} \varepsilon^{0.5}). \tag{38}$$

Solution (38) is valid in the interval of $0.25T$, and then it can be periodically extended. For period T, we get

$$T(n) = 4\left(\frac{\pi\sqrt{n+1}}{2\sqrt{2}}\right)^{(2/n+1)}. \tag{39}$$

If we develop expression (38) with respect to ε, then in the first approximation we get the solution obtained by Pilipchuk [5, 6]. Formula (39) for $n = 1$ (a linear problem) gives the exact value of 2π; for $n \to \infty$ we have $T(n) \to 4$, which should be expected. For particular n values, we get $T(3) = 5.0$ (the exact value $T(3) = 5.2$, the error is 3.8%); $T(5) = 4.64$ (the exact value $T(5) = 4.8$, which causes an occurrence of the error of 3%).

In Figures 1a–1d, the numerical integration results of Equation (33) with initial conditions (34) and for different values of n (dashed curve) and calculations due to formulae (35) (solid curve) in the $0.25T$ interval are given. (We have the exact solutions for $n = 1$ and $n \to \infty$.) The maxima values of the presented curves correspond to the following abscissa of time: $n = 3$, $t_n = 1.559$, $t_a = 1.490$; $n = 5$, $t_n = 1.458$, $t_a = 1.396$; $n = 11$, $t_n = 1.290$, $t_a = 1.252$; $n = 21$, $t_n = 1.184$, $t_a = 1.162$.

Therefore, we are able to conclude that solutions (38) and (39) give an efficiently accurate (for arbitrary values of n) approximate formula describing the inversion of the incomplete Beta-function [7] (construction of the Ateb-functions [8]).

The obtained formulas (38) and (39) are elegant, essentially generalizing the trigonometric function (they represent an inverse of the incomplete Beta-function in an elementary sense) for obituary n. This result can also be read in the following way. The asymptotic inversion of

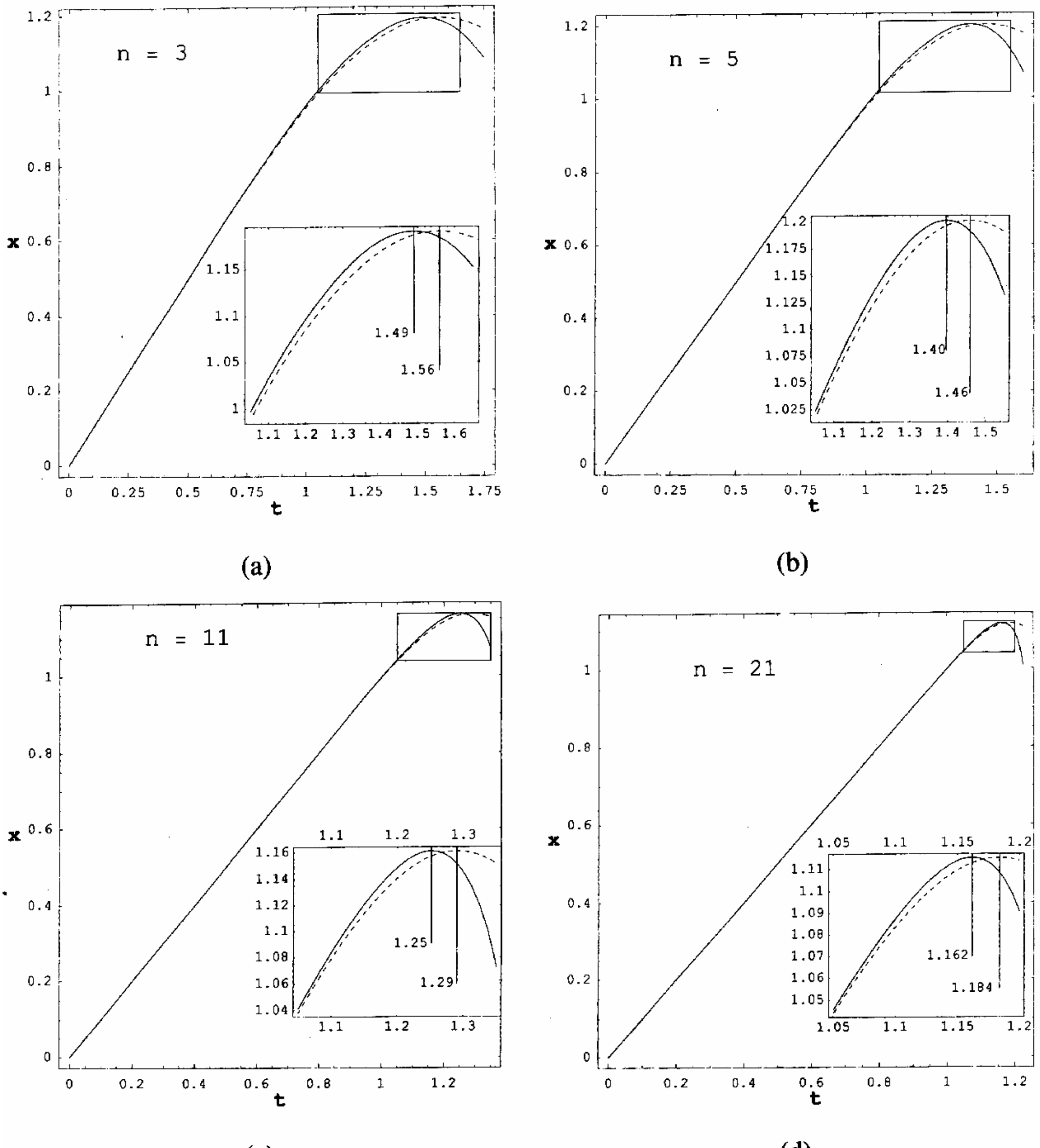

Figure 1. **Comparison of numerical and analytical solution results to the initial-value problem equations (33), (34) for different values of n: (a) $n = 3$; (b) $n = 5$; (c) $n = 11$; (d) $n = 21$ (the solid line corresponds to the analytical calculations, whereas the dashed line corresponds to the analytical calculations).**

the incomplete Beta-function, realized for large n, occurred exactly as for arbitrary values of n.

6. A Link between Large and Small δ Methods

Consider an elementary nonlinear ordinary differential equation

$$y' = y^n, \quad y(0) = 1. \tag{40}$$

The exact solution has the form

$$y = [1 - (n-1)x]^{-(1/(n-1))}. \tag{41}$$

In order to solve Equation (40), as has been shown in [3], the SδM for average values of n (for instance, $n = 7$) can be applied.

Let us assume that we are going to find a solution to Equation (40) for $n \to \infty$. Changing the variables $z = y^{1/n}$, $x_1 = nx$, we get a problem with small δ

$$\frac{dz}{dx_1} = z^{2-\delta} \quad \text{where} \quad \delta = \frac{1}{n}.$$

For zero asymptotic approximation in the initial variables, we obtain

$$y = (1 - nx)^{-1/n},$$

which results from the exact solution (41).

Now we come back to the analysis of the initial problem (33), (34) with $n \to \infty$.

It should be noted that a solution to a similar equation has been analyzed using the method of generalized function series [2, 9, 10].

Changing the variables $x = y^{1/n}$, $\tau = nt$, we get the following nonlinear ordinary differential equation:

$$\tau^{2(1-\varepsilon)} y \frac{d^2y}{d\tau^2} + \tau^{1-2\varepsilon} y \frac{dy}{d\tau} - (1-\varepsilon)\tau^{2(1-\varepsilon)} \left(\frac{dy}{d\tau}\right)^2 + \varepsilon y^{3-\varepsilon} = 0, \tag{42}$$

$$y(0) = 0, \quad \tau^{1-\varepsilon} y^{-1+\varepsilon} \frac{dy}{d\tau} = 1, \tag{43}$$

where $\varepsilon = 1/n$. For small ε, we have

$$\begin{aligned} \tau^{1-\varepsilon} &= \tau - \varepsilon\tau \ln + \cdots, \\ y^{3-\varepsilon} &= y^3 - \varepsilon y^3 \ln y + \cdots. \end{aligned} \tag{44}$$

We are going to find a solution in the form of

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{45}$$

Substituting (44) and (45) into the initial boundary-value problem and, after splitting in relation to ε, we get

$$\tau y_0 \frac{d^2y}{d\tau^2} + y_0 \frac{dy_0}{d\tau} - \tau \left(\frac{dy_0}{d\tau}\right)^2 = 0, \tag{46}$$

$$y_0(0) = 0, \quad \tau \frac{dy_0}{d\tau} = y_0, \tag{47}$$

$$\tau^2 \frac{d^2y_1}{d\tau^2} - \tau \frac{dy_1}{d\tau} + y_1 = -\tau^3, \tag{48}$$

$$y_1(0) = \frac{dy_1}{d\tau} = 0. \tag{49}$$

A boundary-value problem solution of zero-order approximation (46), (47) and the first-order approximation (48), (49) have the form

$$y_0 = \tau, \quad y_1 = -\frac{1}{4}\tau^3.$$

Then this process may be continued further.

7. Conclusion

An application of small and large δ methods leads to the solution of a new complicated series of problems [3–6]. In this work, a link between the two approaches has been established which gives a possibility of extending the class of the considered problems.

It should be noted that the solution determined using the small and large δ methods allows for application of two-points Padé approximants.

We hope that the obtained results will initiate a new approach, within the framework of both small and large δ methods, to the problems of small denominators.

Finally, the LδM can be successfully applied to complex problems of quantum mechanics [11]. However, especially in nonlinear mechanics, for the case of solving at least two-degrees-of-freedom systems, the discussed approach of applying small and large δ methods still needs to be developed.

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