AWREJCEWICZ, J., KRYSKO, V. A.

Period Doubling Bifurcation and Chaos Exhibited by an Isotropic Plate

Period doubling bifurcations and chaos exhibited by one layer flexible plate are illustrated and analyzed. Using a difference operators method the problem is reduced to that of solving ordinary differential and algebraic equations.

1. Foundations

In this work a dynamics of elastic isotropic plate possessing in the R^3 space the bounded and measured (in the Labesque sense) volume D with the boundary surface ∂D is considered. The R^3 space is parameterized with a use of the Descartes co-ordinate system OXYZ. The co-ordinate lines are attached to the midplane of the plate, whereas the OZ axis has a normal direction with a bottom sense. A surface attached to the given co-ordinate system has the form $z = 0$, and the space $D_0 = D \cup \partial D$ forms a cylinder of the form $D_0 = \Omega_0 \times [+h/2, -h/2]$, $\Omega_0 = \Omega \cup \partial\Omega$, where: $\Omega = \{x, y | \, 0 \le x \le a, \; 0 \le y \le b\}$ denotes a plate projection to the reduced surface, $\partial\Omega$ is the boundary of the reduced surface, and $\pm h/2$ are the front surfaces fixed on the OZ axis. It is assumed that the plate deformations are within an elasticity interval. In addition, in our considerations a pressure of the plate layer parallel to the average surface *(OZ=0)* is neglected. We assume also that the normal stresses σ_z in the thickness direction are much more smaller in comparison with the stresses parallel to the average surface [1].

The problem is reduced to the following dimensionless form of equations

$$\lambda^{-2}\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \lambda^2 \frac{\partial^4 F}{\partial y^4} = -\frac{1}{2}L(w,w), \quad L(w,w) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x^2},$$

$$\frac{\partial^2 w}{\partial t^2} + \varepsilon\frac{\partial w}{\partial t} = -\frac{1}{12(1-\nu^2)}\left(\lambda^{-2}\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \lambda^2\frac{\partial^4 w}{\partial y^4}\right) + \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 F}{\partial y^2} - P_x\right) - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 F}{\partial x^2} - P_y\right), \tag{1}$$

where: ε -damping, ν -Poisson ratio, $\lambda = a/b$ -plate diameters ratio, $w = w(x, y, t)$ -deflection, F -stress function, P_x, P_y - longitudinal loads.

Therefore, a problem of the partial differential equations (PDE) solution is reduced to that of the differential -algebraic equations (DAEs) in regard with the stress function F and the deflection function w, correspondingly

$$C(\ddot{w}_{i\,j} + \varepsilon\dot{w}_{i\,j}) = \{A(w) + B(w,F) - H(w)\}, \qquad D(F) = E(w), \tag{2}$$

where:

$$A(w) = \frac{1}{12(1-\nu^2)}\left(\lambda^{-2}\lambda_x^2 w_{ij} + 2\lambda_{xy}^2 w_{ij} + \lambda^2\lambda_y^2 w_{ij}\right), \quad B(w,F) = \lambda_x w_{ij}\lambda_y F_{ij} + \lambda_y w_{ij}\lambda_x F_{ij} - \lambda_{xy} w_{ij}\lambda_{xy} F_{ij},$$

$$H(w) = \lambda_x w_{ij} P_x + \lambda_y w_{ij} P_y, \quad D(F) = 12(1-\nu^2)A(F), \quad E(w) = -\lambda_x w_{ij}\lambda_y w_{ij} + [\lambda_{x,y} w_{ij}]^2,$$

$$\lambda_x = \frac{1}{h_x^2}[Y(x-h_x) - 2Y(x) + Y(x+h_x)], \tag{3}$$

$$\lambda_x^2 = \frac{1}{h_x^4}[Y(x-2h_x) - 4Y(x-h_x) + 6Y(x) - 4Y(x-h_x) + Y(x+2h_x)],$$

$$\lambda_{xy} = \frac{1}{4h_x h_y}[Y(x+h_x, y+h_y) + Y(x-h_x, y-h_y) - Y(x+h_x, y-h_y) + Y(x-h_x, y+h_y)],$$

$$\lambda_{xy}^2 = \frac{1}{h_x^2 h_y^2}[Y(x-h_x, y-h_y) - 2Y(x-h_x, y) + Y(x+h_x, y-h_y) + Y(x-h_x, y+h_y) -$$

$$- 2Y(x, y-h_y) + 4Y(x,y) - 2Y(x+h_x, y) + Y(x+h_x, y+h_y) - 2Y(x, y+h_y)].$$

In order to solve equations (2) with the initial conditions using the boundary condition (2) the following algorithm is applied. The initial deflection values in each of the finite-difference mesh nodes are substituted to the right-hand side of the AEs, and then the corresponding stress function F_{ij} field is obtained using the Gauss method. The obtained stress function values F_{ij} are substituted to the right-hand side of the ODEs, which are integrated using the Runge-Kutta method. In result a field of deflection w_{ij} is obtained in the next time step.

2. Analysis

For a test purposes a series of the known problems have been considered. The first critical loads for $\lambda = a/b = 1$ for plates subjected to a stretching constant load $P_x(P_y = 0)$ have been calculated. The critical loads are determined using the dynamical excitation. The latter one has been defined by the initial conditions of the form $w|_{t=0} = A_H \sin mx \sin ny$, $\dot{w}|_{t=0} = 0$.

We illustrate and analyze a transition to chaos by period doubling bifurcations on the example of squared free supported plate (boundary condition (2), $\lambda = a/b = 1$). The longitudinal load (P_x) acts along the sides $x = 0$, $x = 1$ $(P_y = 0)$. Before a stationary state loses its stability, a basin of attraction becomes very small and always existing perturbations throw the system from that basin even before full vanishing of attraction property. This behavior is related to the so called stiff stability loss. The mechanical systems leave the stationary state by jumps to a different state. The new state can be a stationary one, characterized by a more complex behavior. Consider a stiff stability loss of the squared plate in relation to the longitudinal load P_x further considered as a control parameter. The Hopf bifurcation sequence is being analyzed. For A_H =0.001 the following sequence of the Hopf bifurcations (in relation to P_x) has been detected: 5.3; 6.7; 7.02; 7.1; 7.1245; 7.129385; 7.129385. For A_H =0.3 the corresponding bifurcation values of P_x are: 5.0; 6.3; 6.431. A small increase of the last P_x values correspond an occurrence of chaos.

A convergence of a sequence of the bifurcation values P_{x_k} and P_{x^*} are characterized by $\delta_k = (P_{x_{k+1}} - P_{x_k})/(P_{x_{k+2}} - P_{x_{k+1}})$. With an increase of k the value of δ_k does not depend on k and converges to the constant $\delta = \lim_{k\to\infty} \delta_k = 4.669201$. This Feigenbaum constant describes a convergence velocity of the bifurcation parameter $P_{x_k} \to P_{x^*}$. After the first period doubling the Poincaré section rotates of 90°, becomes broader and is sloped of about 135° to the horizontal axis. In the phase portrait a sufficient velocities change is observed. The higher harmonics are added to the fundamental one.

For $P_{x2} = 6.7$ in the Poincaré section the strange attractors occur as a result of a weak orbits occurrence having a circle shape in the transversal cross-section. Much more higher velocities are observed. The phase portraits and the Poincaré mapping are similar for every plate's point. Looking for further values of the bifurcation parameter P_{x_k} $(k = 3,4,5,6,7)$ it is seen, that a sequence $\{P_{x_k}\}$ is converged to the critical point $P_{x^*} = 7.129405$.

Beginning from the third Hopf bifurcation a cross-section of the Poincaré map has an ellipse form, the larger axis of which is sloped of 45° to the horizontal axis. After the fourth Hopf bifurcation an orbit cross-section possesses a complex form exhibiting a kneading phenomenon.

For $P_x =$ 7.129405 a regular motion is still observed. A change of P_x for 1×10^{-6} leads to a stiff stability loss ($P_{x^*} = 7.129406$). The system by a jump has achieved another state. In this case the time of a stiff stability loss occurrence belongs to the largest one. It means that with a very slow change of the P_x parameter so called pulling of stability loss occurs. Even for the critical P_{x^*} value a dynamical stability loss plays a secondary role for $t < t_{cr}$, i.e. before an occurrence of a stiff stability loss. With an increase of P_x the time of transition to a new dynamical state decreases ($P_x = 8$) and some of the attracting orbits die.

To conclude, a qualitative picture of dynamical plate behavior is similar to that of its centre. A number of Hopf bifurcations depends on the initial exciting amplitude A_H. An increase of A_H accompanies a decrease of the Hopf bifurcations to three ($A_H = 0.3$) and two ($A_H > 0.3$). For larger A_H a chaotic attractor occurs.

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5. References

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Address: J. AWREJCEWICZ, Division of Automatics and Biomechanics, TU of Łódź, 1/15 Stefanowskiego St., 90-924 Łódź, Poland
V. KRYSKO, Department of Mathematics, Saratov State University, 410054 Saratov, Russia